DEMONSTRATIO MATHEMATICA
Vol. XVIII No 2 1985

Sherzed Al-Talabani

IDEALS IN $FSJ^{(r)}$

1. Introduction

In the present paper we shall investigate the free special quadratic Jordan algebras $FSJ^{(r)}$. We shall give a Galois correspondence between special ideals in $FSJ^{(r)}$ and ideals in the associative algebra $\Phi\{x_1,x_2,\ldots,x_r\}$. We prove that for every quadratic ideal in $FSJ^{(r)}$ there exists the smallest special ideal containing it. We shall describe the quadratic ideals in $FSJ^{(1)}$, in the case of char $\Phi = 2$.

In particular we show that in this case every ideal is generated by two elements and a quadratic ideal is prime if and only if it is a prime ideal of $\Phi[x]$.

This paper is a part of my Thesis, presented to Department of Mathematics, Informatics and Mechanics, Warsaw University 1979.

We adopt the definitions and notations from [2].

D e f i n i t i o n . A Φ-submodule I of a quadratic Jordan algebra QJA, J is said to be an inner (outer) ideal of J if for $a \in I$, $b \in J$, $U_b(a) \in I$ ($U_a(b) \in I$). We say that I is a quadratic ideal of J if I is an inner and an outer ideal.

For every subset $B \subseteq J$, the smallest quadratic ideal of J which contains B is called the quadratic ideal generated by B and is denoted by $(B)_{QJA}$.

D e f i n i t i o n . A quadratic Jordan algebra J is said to be special if there exists an associative algebra A and a subalgebra J' of A^+ such that J is isomorphic with J', otherwise J is said to be exceptional.

D e f i n i t i o n . Let $\mathfrak{A} = \Phi\{x_1,\ldots,x_r\}$ be a free associative algebra with 1 generated by $x_1,\ldots,x_r$ over the field Φ. The quadratic Jordan subalgebra of the algebra $\mathfrak{A}^+$, generated by $1,x_1,\ldots,x_r$ is called the free special Jordan algebra with free generators $x_1,\ldots,x_r$ and is denoted by $FSJ^{(r)}$.

D e f i n i t i o n . A quadratic ideal I of an algebra J is said to be a special ideal if the algebra J/I is a special QJA. Otherwise, we call I an exceptional ideal.

L e m m a 1. (see [2]). An ideal I of $FSJ^{(r)}$ is special if and only if $(I) \cap FSJ^{(r)} = I$, where (I) is the ideal generated by I of the associative algebra $\Phi\{x_1,\ldots,x_r\}$.

T h e o r e m 1. A quadratic ideal K of $FSJ^{(1)}$ is special if and only if for all $a \in K$ we have $xa \in K$.

P r o o f . Evidently, if $r = 1$, then $\Phi[x] = \Phi\{x\}$ and $FSJ^{(1)} = \Phi[x]^+$. Then by Lemma 1 the ideal K is special if and only if $(K) = K$. Consequently if K is special in $FSJ^{(1)}$ and if $a \in K$, then $xa \in K$. If now for every $a \in K$ we have $xa \in K$, then for $i \geq 1$ we have $x^i a \in K$ and from the equality: $a_0+a_1x+\ldots)a =$ $= \sum_{i\geq 0} a_i(x^i a) \in K$, we have that K is an ideal of $\Phi[x]$, and hence a special ideal of $FSJ^{(1)}$.

L e m m a 2. If $f \in FSJ^{(1)} = \Phi[x]^+$, then $(f)_{QJA} =$ $+ f^2\Phi[x] + f\Phi[x^2]$.

P r o o f . If $g,h \in f^2\Phi[x] + f\Phi[x^2]$, then $U_g(h) =$ $= g^2h \in f^2\Phi[x]$, hence $f^2\Phi[x] + f\Phi[x^2]$ is a quadratic ideal in $FSJ^{(1)}$ containing f. Since $fx^{2i} = x^i f x^i = U_{x^i}(f)$ and $f^2x^i = U_f(x^i)$, $i \geq 0$, and these elements generate the linear space $f^2\Phi[x] + f\Phi[x^2]$, it follows that $f^2\Phi[x] + f\Phi[x^2]$ is contained in every quadratic ideal of $FSJ^{(1)}$ which contains f. Then $(f)_{QJA} = f^2\Phi[x] + f\Phi[x^2]$.

C o r o l l a r y . If $f \in FSJ^{(1)}$, then $(f^2) \subseteq (f)_{QJA} \subseteq (f)$.

E x a m p l e . Now we show that the quadratic ideals of $FSJ^{(1)} = \Phi[x]^+$ need not be determined by the non-zero polynomials with the smallest degree which are contained in them, unlike to the ideals of $\Phi[x]$. In the ideals $(x^6)_{QJA}$ of $FSJ^{(1)}$, the polynomial x^6 is of minimum degree among the non zero elements of $(x^6)_{QJA}$. Moreover $(x^6)_{QJA} = x^{12}\Phi[x] + x^6\Phi[x^2] \subset \subset \Phi x^6 + x^8\Phi[x]$, and hence $(x^6) \not\subset (x^6)_{QJA}$. We remark also that $x^7 = x^6 x \in \Phi x^6 + x^8\Phi[x] \supseteq (x^6)_{QJA}$, hence $(x^6)_{QJA}$ is not a special ideal in $FSJ^{(1)}$.

T h e o r e m 2. Every increasing sequence of quadratic ideals of $FSJ^{(1)} = \Phi[x]^+$ is finite.

P r o o f . Let $K_1 \subset K_2 \subset K_3 \subset \dots \subset K_r \subset \dots$ be an increasing sequence of quadratic ideals of $FSJ^{(1)}$, such that $K_1 \neq 0$. Let f be the non-zero polynomial belonging to K_1. From corollary of Lemma 2, we have $(f^2) \subseteq K_1$. Because $FSJ^{(1)}/(f^2)^+ \simeq (\Phi[x]/(f^2))^+$ is a finite-dimensional algebra, then by the correspondence between the ideals of $FSJ^{(1)}/(f^2)^+$ and ideals of $FSJ^{(1)}$ which contain (f^2), it follows that the sequence of ideals $K_1 \subset K_2 \subset \dots$ is finite.

2. A Galois correspondence

Now let us consider the set A of all quadratic ideals of $FSJ^{(r)}$, and the set B of all ideals of the associative algebra $\Phi\{x_1, x_2, \dots, x_r\}$. We now define the mappings $\alpha : A \to B$, $\beta : B \to A$ by:

(1) $\alpha(K) = (K)$ where $K \in A$ and (K) is the ideal of $\Phi\{x_1, \dots, x_r\}$ generated by K.

(2) $\beta(\mathfrak{A}) = \mathfrak{A} \cap FSJ^{(r)}$ when $\mathfrak{A} \in B$.

From the definition of α and β we have: $\alpha(K) \subseteq \alpha(K')$ if $K \subseteq K'$ for every $K, K' \in A, \mathfrak{A}, \mathfrak{A}' \in B$; and $\beta(\mathfrak{A}) \subseteq \beta(\mathfrak{A}')$ if $\mathfrak{A} \subseteq \mathfrak{A}'$ and moreover $\alpha\beta(K) \supseteq K$, hence (α, β) is a Galois correspondence (see [1]) and $\alpha\beta\alpha = \alpha, \beta\alpha\beta = \beta$.

L e m m a 3. An ideal K of $FSJ^{(r)}$ is special if and only if $K \in \beta(B)$.

P r o o f . Using the definition of α and β we can write the lemma in the following way: the ideal K of $FSJ^{(r)}$ is special if and only if $K = \beta\alpha(K)$. If K is of the form $\beta(\mathfrak{A})$, $\mathfrak{A} \in B$, then $K = \beta(\mathfrak{A}) = \beta\alpha\beta(\mathfrak{A}) = \beta\alpha(K)$, and hence K is a special ideal. Now, if K is a special ideal, then $K = \beta(\alpha(K))$ and hence $\alpha(K) \in B$.

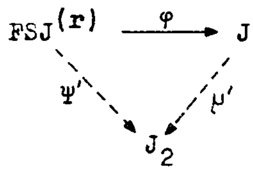
L e m m a 4. The ideal $\mathfrak{A}$ of the algebra $\Phi\{x_1, \ldots, x_r\}$ is of the form $\alpha(K)$ for $K \in A$ if and only if there is a set of generators of $\mathfrak{A}$ contained in $FSJ^{(r)}$.

P r o o f . If $\mathfrak{A} = \alpha(K)$, then from the definition of α we have that $\mathfrak{A} = (K)$ and $\mathfrak{A}$ is generated by the set $K \subseteq FSJ^{(r)}$. If $\mathfrak{A} = ((a_\xi)_{\xi \in \mathfrak{X}})$ where $a_\xi \in FSJ^{(r)}$, then define $K = (a_\xi)_{QJA}$. Since $(K) \subseteq \mathfrak{A} = ((a_\xi)_{\xi \in \mathfrak{X}}) \subseteq (K)$ we have then $\alpha(K) = (K) = \mathfrak{A}$.

T h e o r e m 3. For every quadratic ideal K of $FSJ^{(r)}$ there exists the smallest special ideal which contains it; this is the ideal $\bar{K} = \beta\alpha(K) = (K) \cap FSJ^{(r)}$.

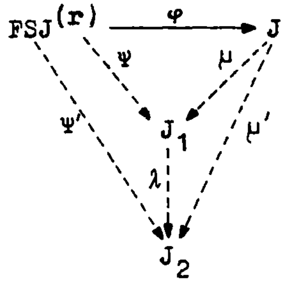
P r o o f . We have that the ideal I of $FSJ^{(r)}$ is special if and only if $\beta\alpha(I) = I$. Consequently, since α and β preserve inclusion, we have that if I is a special ideal which contains K, then $\beta\alpha(K) \subseteq \beta\alpha(I) = I$. We have $\beta\alpha(\beta\alpha(K)) =$ $= \beta(\alpha\beta\alpha(K)) = \beta\alpha(K)$, hence $\beta\alpha(K)$ is a special ideal.

C o r o l l a r y . For every homomorphism φ of $FSJ^{(r)}$ on QJA, J, there exists an homomorphism Ψ from $FSJ^{(r)}$ on a special QJA, J_1, and there exists a homomorphism $\mu : J \to J_1$ such that $\Psi = \mu\varphi$. Moreover, for every commutative triangle:



in which J_2 is a special QJA, there exists exactly one homomorphism $\lambda : J_1 \to J_2$ such that $\lambda\Psi = \Psi'$, $\lambda\mu = \mu'$.

P r o o f . Let $\mathrm{Ker}\,\Psi = K$ and $\bar{K}$ be the smallest special ideal containing K and let $J_1 = FSJ^{(r)}/\bar{K}$ and $\Psi : FSJ^{(r)} \to J_1$ be natural homomorphisms:



The existence of μ follows from the theorem about isomorphism. Because Ker $\Psi' = (\mu'\varphi)^{-1}(0) \supseteq \varphi^{-1}(0)$ = ker φ, from the theorem about isomorphism there exists λ such that $\lambda\Psi = \Psi'$ and is unique. Since $\lambda\mu\varphi = \Psi\lambda = \Psi' = \varphi\mu'$ and φ is onto J, hence $\lambda u = \mu'$.

3. Exceptional ideals in $FSJ^{(1)}$

We know that special ideals of $FSJ^{(1)}$ are ideals of $\Phi[x]$, hence we shall consider only the exceptional ideals. For char $\Phi \neq 2$, every quadratic ideal of $FSJ^{(1)}$ is special, hence we suppose from now on that char $\Phi = 2$.

Let K be an exceptional ideal of $FSJ^{(1)}$ and let $\bar{K} = FSJ^{(1)} \cap (K) = (K)$. Then $\bar{K}$ is an ideal of $\Phi[x]$, hence there exists a polynomial $f \in \Phi[x]$, such that $\bar{K} = (f)$.

Let $G = \{g; fg \in K\}$, then we have the following statements:

1. G does not depend on the choice of f.
2. G is a subspace of the linear space $\Phi[x]$.
3. If $g \in G$, $h \in \Phi[x]$, then $h^2 g \in G$.
   Since for $g \in G$, $gf \in K$, then $hghf \in K$, hence $h^2 g \in G$.
4. If $g \in G$, $h \in \Phi[x]$, then $fhg^2 \in G$. If $g \in G$, then $gf \in K$, hence $(gf)h(gf) \in K$ and consequently $g^2 hf \in G$.
5. $(G) = (1) = \Phi[x]$. If $G \neq (1)$, then $\bar{K} = (K) = (fG) = f(G) \neq$ $\neq f = \bar{K}$.

L e m m a 5. If $0 \neq f \in \Phi[x]$ and G is any set of polynomials which satisfies (1-5) then the set $K = \{fg, g \in G\}$ is a quadratic ideal of $FSJ^{(1)}$ and $\bar{K} = (f)$.

P r o o f . From (2) we obtain that K is a subspace. If $fg \in K$, $h \in \Phi[x]$, then by (3) $hfgh \in K$ and by (4) $fghfg \in K$.

Hence K is a quadratic ideal. By the definition of K we have $\bar{K} = (fG) = f(G) = (f)$ by (5).

L e m m a 6. If f,G,K are as above then $(f) \subseteq G$, $(f) \neq G$.

P r o o f . By (5) there exist $k_\alpha \in \Phi[x]$, $g_\alpha \in G$, $\alpha \in I$, such that $1 = \sum k_\alpha g_\alpha$. Since char $\Phi \neq 2$, we have $1 = \sum k_\alpha^2 g_\alpha^2$. Then for $h \in \Phi[x]$ we get from (4) that $fh = \sum fhk_\alpha^2 g_\alpha^2 \in G$, hence $(f) \subseteq G$. There are two cases: either $(f) = G$ or $(f) \nsubseteq G$. Since K is not an ideal of $\Phi[x]$, and in the first case $K = fG = f(f) \neq (f^2)$, the second case holds.

We now describe the linear space G. Let g be a polynomial of minimal degree in G. From (2) we get that $gx^{2n} \in G$, $n \geqslant 0$. If the set $\{g^{2n}, n \geqslant 0\}$ is a base of G, then $G = (g)$. By (5), $(G) = \Phi[x]$, hence $\deg(g) = 0$ which means that $G = \text{lin}\{1, x^2, x^4 \dots\} = \Phi[x^2]$. From (4) we have that $f \in G$, $xf \in G$, and this means that also some polynomials of odd degrees belong to G, which is not posible, thus $\{g, x^2 g, \dots\}$ is not a base of G.

Let k be a polynomial of smallest degree in G such that $\deg(g) \not\equiv \deg(k) \pmod 2$; such a k exists since $f, xf \in G$. Now we show that the set $\{g, k, x^2g, x^2k, \dots\}$ is a base of the linear space G. Let $L \in G$, then either $\deg(L) \equiv \deg(g) \pmod 2$ or $\deg(L) \equiv \deg(k) \pmod 2$. Hence there exists m such that either $\deg(L) = \deg(gx^{2m})$ or $\deg(L) = \deg(kx^{2m})$. Now we apply the following easy lemma.

L e m m a 7. If B is a linear space contained in $\Phi[x]$ which contains only polynomials of degree $n_1 < n_2 < \dots$, then taking a set consisting of one polynomial of degree n_i for every $i = 1, 2, \dots$, we get a base of B".

Hence the set $\{g, k, x^2g, x^2k, \dots\}$ is a base of G. Thus we have proved the following theorem.

T h e o r e m 4. If K is an exceptional ideal of $FSJ^{(1)}$, then K is generated by two elements.

The polynomials k and g considered above satisfy the following conditions:

(1) $(g,k) = 1.$

This follows from $(g,k) = (G) = \Phi[x] = (1)$.

(2) There exist polynomials $\lambda(x^2)$, $\mu(x^2)$, $\delta(x^2)$, $\tau(x^2) \in \Phi[x]$ such that $f(x) = g(x)\lambda(x^2) + k(x)\mu(x^2)$ and $xf(x) = g(x)\delta(x^2) + k(x)\tau(x^2)$.

This follows from above

(3) $\deg(f) \geq \max(\deg(g), \deg(k))$.

Clearly $f = g(x)\lambda(x^2) + k(x)\mu(x^2)$ and $\deg(g(x)\lambda(x^2)) \neq \deg(k(x)\mu(x^2)$, hence $\deg(f) = \max(\deg(g(x)\lambda(x^2), k(x), \mu(x^2)) \geq \max(\deg(g), \deg(k))$. Hence the following is true.

L e m m a 8. Let $f \neq 0$, $f \in \Phi[x]$ and the polynomials $g, k \in \Phi[x]$ satisfy:

a) One of the numbers $\deg(g)$ and $\deg(k)$ is odd and the other is even,

b) $(g,k) = 1$,

c) $f(x) = g(x)\lambda(x^2) + k(x)\mu(x^2)$ and $xf(x) = g(x)\delta(k^2) + k(x)\tau(x^2)$, where $\lambda(x)$, $\mu(x)$, $\delta(x)$, $\tau(x) \in \Phi[x]$.

Then the set $G = g(x)\Phi[x^2] + k(x)\Phi[x^2]$ satisfies (1-5).

Hence we have shown that quadratic ideals K of $FSJ^{(1)}$ such that $\bar{K} = (f)$ are in the one-to-one correspondence with the sets G satisfying the following conditions:

1) G is a $\Phi[x^2]$-module,

2) $(f) \subset G \subset \Phi[x]$,

3) $(G) = \Phi[x]$.

4. Prime quadratic ideals in $FSJ^{(1)}$

D e f i n i t i o n . An element $a \neq 0$ of a QJA, J, is said to be a zero divisor if there exists $0 \neq b \in J$, such that $U_a(b) = 0$ or $U_b(a) = 0$.

D e f i n i t i o n . The quadratic ideal K of QJA, J, is said to be prime if and only if $K \neq J$ and for $a, b \in J$, if $a, b \notin K$, then $U_a(b) \notin K$ and $U_b(a) \notin K$.

It is clear that K is prime if and only if the algebra J/K has no zero divisors.

L e m m a 9. If K is a proper ideal of $FSJ^{(1)}$ and $K = \bar{K}$ and $\bar{K}$ is a prime ideal of $\Phi[x]$, then K is a prime quadratic ideal.

P r o o f . The result follows from the isomorphism $FSJ^{(1)}/K \approx (\Phi[x]/_{\bar{K}})^+$, since $\Phi[x]/_K$ is a field.

T h e o r e m 5. If K is an ideal of $FSJ^{(1)}$ and $\bar{K}$ is not a prime ideal in $\Phi[x]$, then K is not prime.

P r o o f . If $K = \bar{K}$, then the theorem is true by Lemma 9. Let $K \neq \bar{K}$, then we know that $K = (fg, fk)_{QJA}$ for some $g, k \in \Phi[x]$, where $\bar{K} = (f)$. Suppose K is a prime ideal and $\bar{K}$ is not prime. Then the polynomial $f \in G$ is reducible, i.e. $f = f_1 f_2$, $\deg(f_1) \geq \deg(f_2) > 0$. Since $U_{f_1}(f^2) = f_1^2 f_2^2 = f^2 \in K$, it follows that $f_1 \in K$ or $f_2^2 \in K$, because K is a prime quadratic ideal. Hence $f | f_1$ of $f | f_2^2$, where $K \subseteq \bar{K} \subseteq (f)$. Since $\deg(f_2) > 0$, it is imposible that $f | f_1$. Consequently $f | f_2^2$, i.e. $f_1 f_2 | f_2^2$, hence $f_1 | f_2$.

Since $\deg(f_1) \geq \deg(f_2)$, we can take $f_1 = f_2$. We have $U_{f_1}(g) = f_1^2 g = fg \in K$, and hence $f_1 \in K$ or $g \in K$, since K is prime. Since $K \subset (f_1^2)$, $\deg(f_1) > 0$, we have $f_1 \notin K$. Consequently $g \in K \subseteq (f)$ i.e. $f | g$. In the same way we show that $f | k$. Therefore $\deg(f) \leq \min(\deg(g), \deg(k)) < \max(\deg(g), \deg(k)) \leq \deg(f)$, a contradiction, so that K is not a prime ideal.

L e m m a 10. If K is an exceptional quadratic ideal of $FSJ^{(1)}$ such that $K \neq FSJ^{(1)}$ and $\bar{K}$ is a prime ideal, then K is not prime.

P r o o f . Let $\bar{K} = (f)$, then f is irreducible and $K \subsetneq \bar{K}$. Let $h \in \bar{K} \setminus K$. Since $\bar{K} = (f)$, there exists $u \in FSJ^{(1)}$ such that $h = fu$. Hence $h^2 = f^2 u^2 = f(fu^2) \in K$, since $fu^2 \in (f) \subseteq G$.

If K is a prime ideal, then from the equality $U_h(1) = = h^2 \in K$, we get that $h \in K$ or $1 \in K$, because $K \neq FSJ^{(1)}$, hence $1 \notin K$, and $h \notin K$ by the choice of h, thus K is not a prime ideal.

It follows from Lemmas 9, 10 and Theorem 5 that:

T h e o r e m 6. A quadratic ideal K of $FSJ^{(1)}$ is a prime ideal if and only if K is a prime ideal in $\Phi[x]$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CONSTANTINE,
CONSTANTINE, ALGERIA
Received February 10, 1984.