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CENTRAL DERIVATIONS OF LIE ALGEBRAS

1. Preliminaries and notations

Let L be a finite-dimensional Lie algebra over a field F. A derivation of L is a linear mapping of L into itself such that

$$(1)\qquad D([l,\bar{l}]) = [D(l),\bar{l}] + [l,D(\bar{l})]$$

for all $l,\bar{l} \in L$. We shall employ the following notations:

D(L): The derivation algebra of L, that is, the Lie algebra of all the derivations of L.

J(L): The ideal of D(L) consisting of all the inner derivations of L, that is $D(l) = [q,l]$, where q is an element of L.

Out(L): The set of all outer derivations of L, that is, those not being inner.

C(L): The ideal of D(L) consisting of all the central derivations of L, that is, such that $D : L \to Z(L)$, where Z(L) denotes the center of L.

L_* = The set of all the Lie algebra homomorphisms of L into the abelian Lie algebra F.

P r o p o s i t i o n 1.1. Let $m = \dim Z(L) > 0$, there is a 1-1 correspondence between the C(L) and L_*^m given by

$$(2)\qquad L_*^m \ni (\alpha_1,\dots,\alpha_m) \mapsto D = \alpha_1 z_1 + \dots + \alpha_m z_m \in C(L),$$

where $z_1,\dots,z_m$ is a basis of Z(L).

P r o o f . The mapping (2) is evidently injective and well defined. To prove that it is also surjective, consider an arbitrary central derivation D. Then D(1) decomposes into $\alpha_1(l)z_1 + \alpha_2(l)z_2 + \dots + \alpha_m(l)z_m$, where α_i are linear forms on L.

Then $D([l,\bar{l}]) = \alpha_1([l,\bar{l}])z_1 + \dots + \alpha_m([l,\bar{l}])z_m$. But $D([l,\bar{l}]) = [D(l),\bar{l}] + [l,D(\bar{l})] = 0$, for every $l,\bar{l} \in L$, because $D(l)$, $D(\bar{l}) \in Z(L)$.

It follows

$$\alpha_1([l,\bar{l}]) = 0,\dots,\alpha_m([l,\bar{l}]) = 0,$$

because $z_1,z_2,\dots,z_m$ are linearly independent. Thus $\alpha_i \in L_*$.

C o r o l l a r y 1.1. The problem of determination of the ideal C(L) reduces to that of L_*, and every central derivation has the form defined in (2).

In this note we shall try to determine the space L_* for a large class of Lie algebras and especially of linear Lie algebras.

2. General properties of L_*

Let L^* be the dual space to L, and $L^2 = [L,L]$ the derived ideal of L. Evidently L_* is the annihilator of L^2 in L . Therefore $\dim L_* = \operatorname{codim} L^2 = \dim L - \dim L^2$.

If L is a semisimple Lie algebra, then $L = [L,L]$ which implies $L_* = \{0\}$ and also $C(L) = \{0\}$.

Let $L = S + R$ be the Levi decomposition of L into the semidirect sum of a semisimple subalgebra S and the radical R of L. Since $S = [S,S] \subset L^2$, we obtain $\alpha(l) = \alpha(l_S) + \alpha(l_R)$, $\alpha(l_S) = 0$ because S is semisimple, then we have

$$\alpha(l) = \alpha(l_R). \tag{3}$$

P r o p o s i t i o n 2.1. L_* is the subspace of R^* which annihilates $[L,R]$.

P r o o f . R^* is the dual space of the radical R, hence $L_* \subset R^*$ by (3). Now

$$[L,L] = [S+R, S+R] = [S,S] + [S,R] + [R,R] = S + [L,R].$$

Let $\alpha \in L_*, \alpha/_{[L,L]} = \alpha/_S + \alpha/_{[L,R]}, \alpha/_S = 0$ because S is semisimple Lie algebra. Then $\alpha/_{[L,R]} = 0$.

In particular if L is reductive then $L = S + Z(L)$, that is, $R = Z(L)$ and $[L,R] = \{0\}$. So we get

P r o p o s i t i o n 2.2. For a reductive Lie algebra L, L_* is the whole dual space of the center $Z(L)$ and $C(L) \cong End(Z(L))$.

P r o o f . From Proposition 2.1, L_* is the whole dual space of $Z(L)$, because $[L,R] = \{0\}$ implies $L_* = R^* = Z(L)^*$. Therefore D may be non zero only on $Z(L)$. On the other hand every linear mapping $h : Z(L) \longrightarrow Z(L)$ defines a central derivation by: $D/_S = 0$, $D_{Z(L)} = h$.

Consider the Killing form on L

$$K(l,\bar{l}) = tr(ad_l \circ ad_{\bar{l}}).$$

For $\bar{l} = l_o$ fixed $\alpha = K(-, l_o)$ is a linear form on L and $\alpha([l,l_o]) \equiv 0$ if and only if l_o belongs to the radical of L. (The radical is the orthogonal complement of $[L,L]$ relative to the Killing form, see ([1], p.73). Thus we have the mapping

$$R \longmapsto L_*; \; l_o \longmapsto K(-, l_o),$$

which in general is not surjective. For instance in the case of a nilpotent Lie algebra L, the Killing form $K = 0$, ([2], p.151) but $\dim L_* > 0$.

3. <u>The case of linear Lie algebra</u>

Let $L \subset gl(n)$ be a Lie algebra of $n \times n$ matrices over a field F of characteristic 0. Assume that L is irreducible or completely reducible as a set of linear transformations of R^n. We may employ the following well known theorem:

A Lie algebra of linear transformations of a finite dimensional vector space over a field of characteristic 0 is completely reducible iff:

i) $L = S + Z$, where S is a semisimple ideal, Z the center of L,

ii) The elements of Z are semisimple ([1], p.81).

Combining this with Proposition 2.2, we get

P r o p o s i t i o n 3.1. Let L be a completely reducible linear Lie algebra. Then $\mathcal{C}(L)$ is isomorphic with the space of all the linear endomorphism of Z, and for every $D \in \mathcal{C}(L)$, we have $D(l) = h(l_z)$, where l_z is the Z-component of L, and h is a linear mapping $Z \mapsto Z$.

If F is algebraically closed and L is irreducible then $\dim Z \leq 1$, hence $\dim \mathcal{C}(L) \leq 1$.

If $F = R$ then $\dim Z(L) \leq 2$, hence $\dim \mathcal{C}(L) \leq 4$, see ([1], p.66).

Let K be the Killing form on the full linear Lie algebra $gl(n)$, then

$$K(x,y) = 2[n\,tr(xy) - (tr x)(tr y)]$$

([3], doctorate thesis), we put $[c,x]$ instead of x, then we have

$$K([c,x],y) = 2[n\,tr([c,x]y) - (tr[c,x])(tr y)] = 2n\,tr([c,x]y)$$

since $tr[c,x] = 0$. On the other hand

$$tr([c,x]y) = tr(cxy - xcy) = tr(cxy - cyx) = tr(c[x,y]).$$

Therefore

$$K([c,x],y) = 2n\,tr(c[x,y]). \tag{4}$$

Let $\alpha \in L_*$, α being a linear mapping. We have

$$\alpha(x) = \sum_{ij} c_{ji}x_{ij} = tr(cx).$$

Also every element α of L_* can be written in the form

(5) $$\alpha(x) = tr(cx),$$

where c is a constant matrix from gl(n).

In view of (4), we obtain

P r o p o s i t i o n 3.2. The element α defined by (5) belongs to L_* if and only if $[c,L]$ is contained in the orthogonal complement of L in gl(n) relative to the Killing form K in gl(n). All the elements of L_* can be defined in this form.

P r o o f . We have

$$\alpha([x,y]) = tr(c[x,y]) = \frac{1}{2n} K([c,x],y)$$

and this is identically zero for $x,y \in L$ iff, for $\forall x \in L$, $[c,x]$ is in the orthogonal complement of L in gl(n) relative to the Killing form in gl(n).

In particular, there follows

C o r o l l a r y 3.1. If L^c belongs to the centralizer of L in gl(n), then α given by (5) belongs to L_*.

In fact, we have $[c,L] = 0$.

P r o p o s i t i o n 3.3. Let L be a solvable linear Lie algebra over a field algebraically closed. We may assume that L consists of some upper triangular matrices (see [4], p.10). Then for any upper triangular matrix c the linear form α defined by (5) for $x \in L$ belongs to L_*.

P r o o f . L is solvable, hence every element of $[L,L]$ has zeros on the main diagonal. Hence if c is upper triangular matrix, then $c[x,y]$ has also zeros on the main diagonal and consequently,

$$tr(c[x,y]) = 0 \quad \text{for all} \quad x,y \in L.$$

4. Outer central derivations

L e m m a 4.1. If an inner derivation $D = ad_q$ is a central derivation, then $[q,L^2] = 0$, that is, q belongs to the centralizer of L^2 in L.

P r o o f . From the Jacobi identity, it follows that for $l, \bar{l} \in L$

$$[q,[l,\bar{l}]] = [[q,l],\bar{l}] + [l,[q,\bar{l}]],$$

and we have $ad_q \in Z(L)$ for $[q,l] \in Z(L)$. Therefore $[q,[l,\bar{l}]] = 0$ whence $[q,L^2] = 0$.

P r o p o s i t i o n 4.1. If $Z(L) \not\subset L^2$, then there exist outer central derivations of L.

P r o o f . Let $z_0 \in Z(L)$ but $z_0 \notin L^2$. Take $\alpha \in L_*$ and consider the central derivation $D = \alpha \cdot z_0$. Assume that D is inner; then $D(l) = \alpha(l)z_0 = [q,l]$ implies $z_0 \in L^2$, which is in contradiction with the choice of z_0.

Since every inner derivation is L^2-valued, and every central derivation is $Z(L)$-valued, there follows

P r o p o s i t i o n 4.2. If $Z(L) \cap L^2 = \{0\}$, then every non trivial central derivation is an outer derivation.

C o r o l l a r y 4.1. All non trivial central derivations of a reductive but not semisimple Lie algebra are outer derivations.

In fact, in this case $L = S + Z(L)$, S-semisimple, it follows that $L^2 = [S,S] = S$ but $Z(L) \cap S = \{0\}$, hence $Z(L) \cap L^2 = \{0\}$.

R e m a r k . If $Z(L) \subset L^2$, then it may occur that every central derivation is inner. For instance take L to be the triangular matrix algebra

$$L = \left\{ \begin{pmatrix} 0 & a & c \\ & 0 & b \\ & & 0 \end{pmatrix}, \ a,b,c \in F \right\}. \text{ Then } L^2 = Z(L) = \left\{ \begin{pmatrix} 0 & 0 & c \\ & 0 & 0 \\ & & 0 \end{pmatrix} \right\},$$

$$\dim Z(L) = 1. \quad \text{Take} \quad z_0 = \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \in Z(L).$$

Since $L_* = \{\lambda a + \mu b\}$, then $D(l) = \alpha(l)z_0$ has the form

$$D(l) = (\lambda a + \mu b) \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix} = \left[\begin{pmatrix} 0 & \mu & 0 \\ & 0 & -\lambda \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & a & c \\ & 0 & b \\ & & 0 \end{pmatrix} \right].$$

Also, D is an inner derivation.

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