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ON THE APPLICATION OF THE LEGENDRE POLYNOMIALS TO THE DIRICHLET PROBLEM FOR THE LAPLACE EQUATION

1. Let r,s,t be the spherical coordinates of points, $Q = \{(r,s,t): 0 < r < 1,\ 0 \leq s \leq \pi,\ 0 \leq t \leq 2\pi\}$, $S = \{(r,s,t): r=1,\ 0 \leq s \leq \pi,\ 0 \leq t \leq 2\pi\}$ and $\bar{Q} = Q + S$. Let R be the rectangle defined by $0 \leq s \leq \pi$, $0 \leq t \leq 2\pi$.

Denote by $C^m(Q)$ (m is a non-negative integer, i.e. $m \in N$) the class of all real-valued functions defined in Q and having the partial derivatives of the order $\leq m$ continuous in Q. Analogously will be interpreted the symbols $C^m(\bar{Q})$ and $C^\infty(Q)$.

The symbol $C^m(R)$ will denote the class of functions having the properties as above and such that $f(s+\pi,\ t+2\pi) = f(s,t)$.

Let Δ be the Laplace operator, i.e. $\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial s^2} + \frac{\cos s}{r^2 \sin s}\frac{\partial}{\partial s} + \frac{1}{r^2 \sin^2 s}\frac{\partial^2}{\partial t^2}$, and $\Delta^n u = \Delta(\Delta^{n-1}u)$ for $n = 2,3,\ldots$ and $u \in C^\infty(Q)$, $(\Delta^1 \equiv \Delta)$.

In this paper we shall give a solution of the Dirichlet problem for the equation $\Delta^n u(r,s,t) = 0$ in $\bar{Q}$. We shall construct the function u of the class $C^n(\bar{Q})$, $n \geq 1$, such that $\Delta^{n+1} u(r,s,t) = 0$ in Q, $u(r,s,t)\big|_S = f(s,t)$, where f is a fixed function of the class $C(R)$ $(C(R) = C^0(R))$, and $\left.\frac{\partial^p u}{\partial r^p}\right|_S = 0$ for $p=1,\ldots,n$.

The solution of Dirichlet's problem for $\Delta u(r,s,t) = 0$ in $\bar{Q}$ was given in [3] (p.472).

The similar problem was considered in [2].

2. Using the mathematical induction, we can prove

L e m m a 1 . If $u \in C^{\infty}(Q)$ and $n=1,2,\ldots$, then

$$\Delta^n \left(r \frac{\partial u}{\partial r}\right) = r \frac{\partial}{\partial r} \Delta^n u(r,s,t) + 2n\Delta^n u(r,s,t);$$

$$\Delta^n \left(r^3 \frac{\partial u}{\partial r}\right) = r \frac{\partial}{\partial r} \Delta^{n-1} \left(6u(r,s,t) + r^2\Delta u(r,s,t) + 4r \frac{\partial u}{\partial r}\right) + $$
$$+ 2(n-1) \Delta^{n-1}\left(6u(r,s,t) + r^2\Delta u(r,s,t) + 4r \frac{\partial u}{\partial r}\right)$$

and

$$\Delta^n\left(r^2\Delta u(r,s,t)\right) = r^2\Delta^{n+1} u(r,s,t) + 4nr \frac{\partial}{\partial r} \Delta^n u(r,s,t) +$$
$$+ (6n+4n(n-1)) \Delta^n u(r,s,t) \quad \text{for } (r,s,t) \in Q.$$

From Lemma 1 and by the linearity of the operator Δ^n we obtain

L e m m a 2. If $u \in C^{2n+2}(Q)$ $(n \geq 1)$ and $\Delta^n u(r,s,t) = 0$ in Q, then the function v,

$$(1) \qquad v(r,s,t) = u(r,s,t) + \frac{r-r^3}{2(n+1)} \frac{\partial}{\partial r} u(r,s,t),$$

satisfies the equation $\Delta^{n+1}v(r,s,t) = 0$ in Q.

Moreover, the following result can be easily obtained.

L e m m a 3. If $u \in C^{n+2}(\bar{Q})$, $n \geq 1$, and $\left.\frac{\partial^p u}{\partial r^p}\right|_S = 0$ for $p = 1,\ldots,n$, then v defined by (1) satisfies the condition $\left.\frac{\partial^p v}{\partial r^p}\right|_S = 0$, with $p = 1,\ldots,n+1$.

3. Let $a_{k,l}(f)$, $b_{k,l}(f)$, $c_{k,l}(f)$ and $d_{k,l}(f)$ be the coefficients of double trigonometric Fourier series of $f \in C(R)$.

L e m m a 4. (cf. [4], [5]). If $f \in C^{2n+2}(R)$ $(n \geq 0)$, then, for every $p,q \in N$ and $p+q = 2n$, the series

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (k+1)^p (l+1)^q \left(|a_{k,l}(f)| + |b_{k,l}(f)| + |c_{k,l}(f)| + |d_{k,l}(f)| \right)$$

is convergent *).

4. Let P_n be the Legendre polynomial

$$P_n(x) = \frac{1}{2^n\, n!} \frac{d^n}{dx^n} (x^2 - 1)^n \quad (n=0,1,\ldots;\ x \in \langle -1,1 \rangle),$$

and let

$$P_{n,m}(x) = \left(1 - x^2\right)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x)$$

for $x \in \langle -1,1 \rangle$, $n=1,2,\ldots$ and $m=1,\ldots,n$ ([3], p.453).

Denote by $A_{k,l}(f)$ and $B_{k,l}(f)$ the coefficients of the Fourier series of $f \in C(R)$ with the orthogonal system

$$P_{k,l}(\cos s) \begin{pmatrix} \cos lt \\ \sin lt \end{pmatrix} \quad ((s,t) \in R),$$

i.e.

$$A_{k,0}(f) = \frac{2k+1}{4\pi} \iint_R f(s,t) P_k(\cos s) \sin s \, ds\, dt,$$

$$A_{k,l}(f) = \frac{(2k+1)(k-l)!}{2\pi(k+l)!} \iint_R f(s,t)\, P_{k,l}(\cos s) \cos lt \sin s \, ds\, dt,$$

$$B_{k,l}(f) = \frac{(2k+1)(k-l)!}{2\pi(k+l)!} \iint_R f(s,t) P_{k,l}(\cos s) \sin lt \sin s \, ds\, dt,$$

([3], p.455). Let, as in [3],

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*) We shall say that a series $\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} y_{k,l}$ is convergent if there exists $\lim_{m,n\to\infty} \sum_{k=0}^{m} \sum_{l=0}^{n} y_{k,l} = \alpha$ and $|\alpha| < +\infty$.

$$Y_0(s,t;f) = A_{0,0}(f),$$

$$Y_k(s,t;f) = A_{k,0}(f)P_k(\cos s) +$$

$$+ \sum_{l=1}^{k} (A_{k,l}(f)\cos lt + B_{k,l}(f)\sin lt)\, P_{k,l}(\cos s)$$

for $k=1,2,\ldots$; $(s,t)\in R$ and $f\in C(R)$. In [3] (p.467) there was given a sufficient condition for the convergence to f of the Fourier series

$$\sum_{k=0}^{\infty} Y_k(s,t;f) \quad ((s,t)\in R). \tag{2}$$

It is clear that $P_n(\cos s)$ and $P_{n,m}(\cos s)$ are some trigonometric polynomials of the order n. The series (2) can be written in the form of double trigonometric Fourier series for the function f if it is absolutely convergent in R. Lemma 4 and the result given in [3] (p.467) imply the following lemmas:

L e m m a 5. If $f\in C^2(R)$, then the series (2) is convergent to f uniformly. Moreover, this series is absolutely convergent for every $(s,t)\in R$.

L e m m a 6. Suppose that $f\in C^{2n+2}(R)$ $(n\geq 1)$, $p,q\in N$ and $p+q = 2n$. Then the series

$$\sum_{k=0}^{\infty} (k+1)^{2n}\, Y_k(s,t;f) \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{\partial^{2n}}{\partial s^p\, \partial t^q} Y_k(s,t;f)$$

are absolutely convergent for every $(s,t)\in R$.

From Lemmas 5, 6 and by the fundamental theorems on power series there follows:

C o r o l l a r y 1. If $f\in C^{2n+2}(R)\geq (n\geq 0)$, then $U_0(f)$, defined by formula

$$U_0(r,s,t;f) = \sum_{k=0}^{\infty} r^k \, Y_k(s,t;f),$$

is a function of the class $C^{2n}(\bar{Q})$. Moreover, $U_0(f) \in C^{\infty}(Q)$ and

$$(3) \qquad U_0(1,s,t;f) = f(s,t) \quad ((s,t) \in R).$$

<u>5.</u> Let, as in [1]:

$$(4) \quad D^n(r^k) = \begin{cases} r^k & \text{if } n = 0, \\ D^{n-1}(r^k) + \dfrac{r - r^3}{2n} \dfrac{d}{dr} D^{n-1}(r^k) & \text{if } n=1,2,\ldots \end{cases}$$

for $k=0,1,\ldots$ and $r \in \langle 0,1 \rangle$.

By mathematical induction we can prove

L e m m a 7. If $n=1,2,\ldots$, then

$$D^n(r^k) = r^k + \sum_{q=1}^{n} W_q(r;n) \frac{d^q}{dr^q} r^k \quad (k = 0,1,\ldots;\ r \in \langle 0,1 \rangle),$$

where W_q are some algebraic polynomials with coefficients depending on n only and such that

$$\left(\frac{d^p}{dr^p} W_q(r;n) \right)_{r=1} = \begin{cases} 0 & \text{if } p \neq q, \\ (-1)^p & \text{if } p = q \end{cases}$$

for $p = 0,1,\ldots,n$ and $q = 1,\ldots,n$ (see [1]).

Arguing similarly as in [1], [2], we shall prove

T h e o r e m . Suppose that $f \in C^{2n+2}(R)$, $n \geq 0$, and $Y_k(f)$, $D^n(r^k)$ are defined as in (2) and (4). Then the function

$$(5) \qquad U_n(f) = U_n(r,s,t;f) = \sum_{k=0}^{\infty} D^n(r^k) \, Y_k(s,t;f)$$

has the properties:

1° $U_n(f) \in C^n(\bar{Q})$,

2° $\Delta^{n+1} U_n(r,s,t;f) = 0$ for $(r,s,t) \in Q$,

3° $U_n(r,s,t;f)\big|_S = f(s,t)$

and, if $n \geq 1$,

4° $\frac{\partial^p}{\partial r^p} U_n(r,s,t;f)\big|_S = 0$, for $p=1,\dots,n$.

P r o o f . The conditions 1° - 3° for $U_0(f)$ are given in [3] (p.455-472).

By Corollary 1, Lemma 7 and (4), (5), we obtain

$$(6)\quad U_m(r,s,t;f) = U_0(r,s,t;f) + \sum_{q=1}^{m} W_q(r;m) \frac{\partial^q}{\partial r^q} U_0(r,s,t;f)$$

for $(r,s,t) \in \bar{Q}$ and $1 \leq m \leq n$. Moreover,

$$(7)\quad U_m(r,s,t;f) = U_{m-1}(r,s,t;f) + \frac{r - r^3}{2n} \frac{\partial}{\partial r} U_{m-1}(r,s,t;f)$$

for $(r,s,t) \in \bar{Q}$ and $1 \leq m \leq n$.

Hence, by (6) and Corollary 1, we get

$$(8)\qquad U_m(f) \in C^\infty(Q) \quad \text{and} \quad U_m(f) \in C^{2n-m}(\bar{Q})$$

for $0 \leq m \leq n$.

The condition 2° for $U_0(f)$, (7)-(8) and Lemma 2 imply 2° for $U_n(f)$. The condition 3° holds by (3)-(5).

If $n \geq 1$, then, by (7) and (8),

$$(9)\qquad \frac{\partial}{\partial r} U_1(r,s,t;f)\Big|_S = 0.$$

Applying (7)-(9) and Lemma 3, we obtain 4° for $U_n(f)$. Thus the proof is completed.

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