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ON SOME CONDITION OF THE EQUALITY OF CLASSES OF CONTINUOUS FUNCTIONS WITH DIFFERENT TOPOLOGIES

In last years several mathematicians of Łódź studied the classes $C(X,T)$ of real continuous functions with some set X and different topologies T ([5]). The first important result was obtained by Kocela ([3]):

Let (X,T_0) be a compact metrizable space and let T be a topology stronger than T_0 ($T \supset T_0$). Then the necessary and sufficient condition for $C(X,T_0) = C(X,T)$ is that every T-continuous function be bounded.

Moreover, in [3] it was proved that if X is an interval $\langle a,b\rangle$ with the natural topology T_0, T is a topology stronger than T_0 and $C(\langle a,b\rangle,T_0) = C(\langle a,b\rangle,T)$, then the two conditions must be satisfied:

(1) Every interval $\langle c,d\rangle \subset \langle a,b\rangle$ is T-connected

(2) Every T-neighbourhood of an arbitrary point $x \in \langle a,b\rangle$ is a T_0-dense set in some T_0-neighbourhood of the point x.

It is natural to ask if these two conditions are sufficient for the equality $C(\langle a,b\rangle,T_0) = C(\langle a,b\rangle,T)$. The answer is negative ([7]). What is the structure of T-open sets for the condition $C(\langle a,b\rangle,T_0) = C(\langle a,b\rangle,T)$?

In [3] a condition is given, unfortunately, not very useful. It depends on the notion of a totally open set.

Now, let (X,T_0) be an arbitrary topological space and $T \supset T_0$. In [2] the conditions analogous to (1) and (2) were considered:

(I) Topology T does not restrict the class of T_0-connected sets (it is easy to observe that (I) is the necessary condition of $C(\langle a,b\rangle,T_0) = C(\langle a,b\rangle,T)$, too).

(II) Every T-neighbourhood of an arbitrary point $x \in X$ is a T_0-dense set in some T_0-neighbourhood of the point x.

In [2] it was shown that (I) and (II) are not necessary conditions for the equality $C(X,T_0) = C(X,T)$. We shall show that for a $T_{3\frac{1}{2}}$-space (X,T_0) satisfying the first countability axiom (II) is the necessary condition of $C(X,T_0) = C(X,T)$.

We adopt the terminology of Engelking ([1]).

T h e o r e m 1. Let (X,T_0) be a $T_{3\frac{1}{2}}$-space satisfying the first countability axiom and let $C(X,T_0) = C(X,T)$, where $T_0 \subset T$. Then every T-neighbourhood of an arbitrary point $x \in X$ is a T_0-dense set in some T_0-neighbourhood of the point x.

P r o o f . First, let us observe that the condition (II) is equivalent to the following implication

$$x \in U \in T \Rightarrow \bigvee_{U_0 \in T_0} x \in U_0 \subset Cl_{T_0} U,$$

where Cl_{T_0} denote the closure operator with respect to the topology T_0. Suppose that it is not the case. This means that there exists U T such that for some $x_0 \in U$ there is no $U_0 \in T_0$ with $x_0 \in U_0 \subset Cl_{T_0} U$. Let $\{B_k\}_{k=1}^{\infty}$ be a T_0-base for X at the point x_0 such that

$$Cl_{T_0} B_{k+1} \subset B_k \in T_0 \quad \text{for every} \quad k.$$

Then there is $G_1 \in T_0$ such that $G_1 \subset B_1$ and $G_1 \cap U = \emptyset$. Let $m = \min_{G_1 \not\subset Cl_{T_0} B_k} k$ and $V_1 = B_m$.

Let us choose $U_1 \in T_0$ such that $G_1 \setminus Cl_{T_0} V_1 \supset Cl_{T_0} U_1 \neq \emptyset$. Next, if V_n and U_n are defined, there is $G_{n+1} \in T_0$ such that $G_{n+1} \subset V_n$ and $G_{n+1} \cap U = \emptyset$. Let $m = \min_{G_{n+1} \not\subset Cl_{T_0} B_k} k$ and $V_{n+1} = B_m$. Let us choose

$U_{n+1} \in T_0$ such that $G_{n+1} \setminus Cl_{T_0} V_{n+1} \supset Cl_{T_0} U_{n+1} \neq \emptyset$.
Thus we can define inductively a sequence $\{U_n\}_{n=1}^{\infty}$ of T_0-open sets. $\{U_n\}_{n=1}^{\infty}$ is a discrete family in the space $X \setminus \{x_0\}$ in the topology T_0. Let $x_n \in U_n$. There exists a T_0-continuous function $f_n \colon X \to \langle 0,1 \rangle$ such that $f_n(x_n) = 1$ and $f_n(X \setminus U_n) = 0$. The formula

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

defines a function f which is T_0-continuous for every $x \neq x_0$, but is T_0-discontinuous at x_0. On the other hand f is T-continuous on X. Therefore $C(X,T_0) \neq C(X,T)$, which concludes the proof of the theorem.

In [2] it was proved that if $U \subset Int_{T_0} Cl_T U$ for every $U \in T$, then $C(X,T_0) = C(X,T)$, where $T \supset T_0$ and Int_{T_0}, Cl_T denote the interior operator and the closure operator with respect to the topology T_0, T respectively. On the other hand, if the condition (II) is satisfied then the inverse theorem is also true ([2]). From the above and Theorem 1 the following theorem follows.

T h e o r e m 2. Let (X,T_0) be a $T_{3\frac{1}{2}}$-space with the first countability axiom. Then the necessary and sufficient condition for $C(X,T_0) = C(X,T)$, where $T \supset T_0$, is that $U \subset Int_{T_0} Cl_T U$ for every $U \in T$.

C o r o l l a r y . Let T_0 be the natural topology and let $T \supset T_0$. Then $C(\langle a,b \rangle, T_0) = C(\langle a,b \rangle, T)$ if and only if $U \subset Int_{T_0} Cl_T U$ for every $U \in T$.

This corollary shows what is the structure of T-open sets satisfying the condition $C(\langle a,b,T_0 \rangle) = C(\langle a,b \rangle, T)$.

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