DEMONSTRATIO MATHEMATICA
Vol. XVIII No 2 1985

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THE HAHN-BANACH THEOREM FOR TOTALLY CONVEX SPACES

1. Introduction

In [8] the category of Eilenberg-Moore algebras was determined for the category $\mathcal{B}an_1$ of real or complex Banach spaces with linear contractions as morphisms with respect to the unit ball functor, assigning to any Banach space its closed unit ball. It was proved that the category of Eilenberg-Moore algebras of $\mathcal{B}an_1$, the smallest algebraic category "generated" by $\mathcal{B}an_1$, so to speak, consists of an equationally definable category of Ω-algebras over the category of sets, where Ω is the set of all real resp. complex sequences $(\alpha_n|n\in\mathbf{N})$ with $\sum_{i=1}^{\infty}|\alpha_i|\leqslant 1$. For an Ω-algebra X the operation of $\alpha\in\Omega$, $\alpha = (\alpha_n|n\in\mathbf{N})$, is written as $\sum_{i=1}^{\infty}\alpha_i x_i$ for each $x\in X^{\mathbf{N}}$, $x = (x_n|n\in\mathbf{N})$. The defining equations for the Eilenberg-Moore algebras are then (see [3]):

(TC 1) $$\sum_i \delta_{ik}x_i = x_k,$$

(TC 2) $$\sum_i \alpha_i\Big(\sum_k \beta_{ik}x_k\Big) = \sum_k\Big(\sum_i \alpha_i\beta_{ik}\Big)x_k,$$

if $\alpha,\beta_i\in\Omega$, $\beta_i = (\beta_{ik}|k\in\mathbf{N})$ and $x\in X^{\mathbf{N}}$. An Ω-algebra X is called a real resp. complex totally convex space, if X is non-empty and (TC 1) and (TC 2) are valid. A morphism between two totally convex spaces is just an Ω-algebra morphism. This category of totally convex spaces is denoted by $\mathcal{TC}$ and is the Eilenberg--Moore category of $\mathcal{B}an_1$.

If B is a Banach-space, the closed unit ball $O(B) = \{x \mid x \in B \text{ and } \|x\| \leq 1\}$ is in a natural way a totally convex space and is as such denoted by $\hat{O}(B)$. This induces a full and faithful functor $\hat{O} : \mathcal{B}an_1 \to \mathcal{TC}$ (see [8], (3.5)) the so called comparison functor. $\hat{O}$ has a left adjoint $S : \mathcal{TC} \to \mathcal{B}an_1$ (cp. [8], §7) assigning a Banach space S(C) to every totally convex space C in a universal way. As $\mathcal{TC}$ is "generated" by $\mathcal{B}an_1$ in a canonical way (cp. e.g. [5]) it seems natural to ask the question if one can prove a Hahn-Banach theorem for totally convex spaces. Of course, the Hahn-Banach extension theorem in $\mathcal{B}an_1$ carries over to $\mathcal{TC}$ via the left adjoint S of $\hat{O}$ to some extent and this fact is used in [8] (see e.g. [8], (10.8) and (11.2)). Nevertheless, it should be interesting to have an independent version of the Hahn-Banach theorem for totally convex spaces for the following reasons: (i) $\mathcal{TC}$ consists of algebras of rank $\aleph_1$ and to have a Hahn-Banach theorem for such a category should give further insight into the algebraic structure of $\mathcal{B}an_1$; (ii) $\mathcal{TC}$ contains many objects having a structure entirely different from the structure of subsets of the unit ball of a Banach space (see [8], Linton's example (4.4), (4.5), and §11). Thus, one could hope that a Hahn-Banach theorem for totally convex spaces would provide new and interesting examples; (iii) Because of the close connection between $\mathcal{B}an_1$ and $\mathcal{TC}$ it should be possible to retrieve the usual Hahn-Banach theorem of $\mathcal{B}an_1$ from the Hahn-Banach theorem of $\mathcal{TC}$ as a special case.

Rodé proved in [9] an abstract version of the Hahn-Banach theorem containing all known Hahn-Banach type theorems as special cases. The setting in which he formulates and proves his result suggests a close connection between the Hahn-Banach theorem and algebras of a certain type, namely finitary (universal) algebras with a commuting set of operations (cp. e.g. [3], p.127). Totally convex spaces are of this type as has been proved in [8], with one important difference: The operations are infinitary. In the following we will prove a Hahn-Banach theorem for totally convex spaces. For this, Rode's method has been modified in two ways. First, for totally convex spaces, the canonical object for a functional to take its values in is $\hat{O}(\mathbb{R})$ and not, as in [9], the half open interval $[-\infty,+\infty[$. Secondly, we are dealing with infinitary operations in totally convex spaces and we make full use of the algebraic structure of totally convex spaces as developped in [8].

H. König has generalized Rodé's theorem to the case of universal algebras with infinitary operations in [6]. This generalization is proved under very general assumptions: As in [9] the boundary functions p and q (cp. (2.2)) are permitted to take their values in the half-open interval $[-\infty,\infty[$ and they are only assumed to be bounded from above on a so-called boundary system in the domain of definition. This, together with the fact that the connection between an operation and the admissible elements $\alpha \in \ell^1_+$ (cp. [6]) is not necessarily unique, requires ingenious arguments in the proof.

Section 2 is devoted to the proof of the general Hahn-Banach theorem for real totally convex spaces. In section 3 extension theorems in the real case are derived. It turns out that a morphism $f : C_0 \to \hat{O}(\mathbb{R})$ from a subspace C_0 of a totally convex space C can be extended to C if and only if it is norm-bounded (see (3.4)). The appropriate type of totally convex spaces for extension theorems are the normed spaces (see [8], (12.1)); actually they are characterized by the fact that for any non-zero element $x \in C$ there is a morphism $\varphi : C \to \hat{O}(\mathbb{R})$ with $\varphi(x) = \|x\|$ and $\|\varphi\| = 1$ (see (3.7)). This is remarkable, as there are normed totally convex spaces which are not isomorphic to totally convex subsets of Banach spaces (see (3.9)).

In section 4 the results of section 3 are extended to complex totally convex spaces in a way analogous to the method in the classical case. Then it is briefly sketched how the classical Hahn-Banach theorems can be easily retrieved from the corresponding results for totally convex spaces. Finally, we take a look at some other categories of algebras corresponding to the examples of Rodé [9], which, surprisingly, are defined in a way closely related to the definition of totally convex spaces ((4.3)-(4.5)).

2. The Hahn-Banach theorem

First we need some notions generalizing the classical notions of convex and sublinear for totally convex spaces. In this section, totally convex space always means real totally convex space, i.e. the operations are the elements of $\Omega = \left\{(\alpha_i \mid i \in \mathbb{N}) \mid \alpha_i \in \mathbb{R} \text{ and } \sum_i |\alpha_i| \leq 1\right\}$; the elements of Ω are denoted by α, $\alpha = (\alpha_i \mid i \in \mathbb{N})$. $\Omega_+ := \{\alpha \mid \alpha \in \Omega$ and $\alpha_i \geq 0$ for all $i \in \mathbb{N}\}$ is the subset of non-negative operators.

Definition 2.1. Let C be a totally convex space. Then, for a mapping $p : C \to \hat{O}(\mathbb{R})$, we say that:

(i) p is Ω_+-convex, if, for any $\alpha \in \Omega_+$ and $x \in C^{\mathbb{N}}$,

$$p\left(\sum_i \alpha_i x_i\right) \leq \sum_i \alpha_i p(x_i).$$

(ii) p is Ω_+-concave, if, for any $\alpha \in \Omega_+$ and $x \in C^{\mathbb{N}}$,

$$p\left(\sum_i \alpha_i x_i\right) \geq \sum_i \alpha_i p(x_i).$$

(iii) p is positive homogeneous, if $p(\alpha x) = \alpha p(x)$ for any $\alpha \in \mathbb{R}$ with $0 \leq \alpha \leq 1$ and any $x \in C$.

(iv) p is sublinear, if p is positive homogeneous and Ω_+-convex.

If $p,q : C \to \hat{O}(\mathbb{R})$ are mappings, we write $p \leq q$ if and only if $p(x) \leq q(x)$ for every $x \in C$. Obviously the set of all mappings from C to $\hat{O}(\mathbb{R})$ is ordered by "$\leq$", i.e. "$\leq$" is reflexive, transitive and antisymmetric.

Theorem 2.2. Let C be a totally convex space and $p,q : C \to \hat{O}(\mathbb{R})$ be two mappings, such that p is Ω_+-convex,

q is Ω_+-concave and $q \leq p$ holds. Then there is a morphism $\varphi : C \to \hat{O}(R)$ of totally convex spaces with $q \leq \varphi \leq p$.

The proof is carried out in several steps (see [9]).

L e m m a 2.3. $Z = \{\pi \mid \pi : C \to \hat{O}(R),\ \pi$ is Ω_+-convex and $q \leq \pi \leq p\}$ is a non-empty set, inductively ordered from below by "$\leq$".

P r o o f . $Z \neq \Phi$ as $p \in Z$ holds. If $W \subset Z$ is a linearly ordered subset, put $\pi(x) := \inf\{w(x) \mid w \in W\}$ for $x \in C$. Then $\pi : C \to \hat{O}(R)$ is well defined and it is an easy exercise to verify that π is Ω_+-convex and, of course, it is a lower bound for W.

Now, for the following, let φ be a minimal element of Z, which exists by Zorn s lemma because of (2.3).

L e m m a 2.4. φ is positive homogeneous.

P r o o f . One has to prove $\varphi(\alpha x) = \alpha\varphi(x)$, for $0 \leq \alpha \leq 1$, $x \in C$. The assertion is trivial for $\alpha = 0$. For $\alpha > 0$ put $\pi(x) = \alpha^{-1}\varphi(\alpha x)$ and verify $\pi \in Z$. Besides, $\pi(x) = \alpha^{-1}\varphi(\alpha x) \leq \varphi(x)$, i.e. $\pi \leq \varphi$, which implies $\pi = \varphi$ and hence our assertion.

L e m m a 2.5. For every $\varepsilon > 0$ there exists $x_\varepsilon \in C$ such that $\varphi(x_\varepsilon) - q(x_\varepsilon) < \varepsilon$.

P r o o f . This follows immediately from the minimality of φ.

L e m m a 2.6. Let $\alpha \in \Omega_+$. Then for any $n \in \mathbb{N}$ and for every $x \in C^{\mathbb{N}}$

$$\varphi\left(\sum_{i=1}^{\infty} \alpha_i x_i\right) \geq \sum_{i=1}^{n} \alpha_i \varphi(x_i) + \sum_{i=n+1}^{\infty} \alpha_i q(x_i).$$

P r o o f . This inequality is proved by induction. The case $n = 0$ is trivial, so assume that the inequality is true for n. If $\alpha_{n+1} = 0$, we are finished, as the assertion for $n+1$ is equivalent with the induction hypothesis. Hence we can restrict ourselves to the case $0 < \alpha_{n+1} \leq 1$. For $z \in C$ put

$$\pi(z) := \inf\Big\{\alpha_{n+1}^{-1}\Big(\varphi\Big(\sum_{i=1}^{\infty}\alpha_i t_i\Big) - \sum_{i=1}^{n}\alpha_i\varphi(t_i) - \sum_{i=n+2}^{\infty}\alpha_i q(t_i)\Big)\Big|$$

$t_i \in C$ for $i \in \mathbf{N}$ with $t_{n+1} = z$ and $t_i = \beta_i x_i$ with $0 \leq \beta_i \leq 1$ for $1 \leq i \leq n\Big\}$.

The induction hypothesis at once implies $q \leq \pi$. Now take $\varepsilon > 0$ and put $t_i = x_\varepsilon$, for $i \geq n+2$, then (2.5) implies

$$\pi(z) \leq \alpha_{n+1}^{-1}\Big(\varphi\Big(\sum_{i=1}^{\infty}\alpha_i t_i\Big) - \sum_{i=1}^{n}\alpha_i\varphi(t_i) - \sum_{i=n+2}^{\infty}\alpha_i q(t_i)\Big) \leq$$

$$\leq \alpha_{n+1}^{-1}\Big(\alpha_{n+1}\varphi(z) + \sum_{i=n+2}^{\infty}\alpha_i\Big(\varphi(x_\varepsilon) - q(x_\varepsilon)\Big)\Big) \leq \varphi(z) +$$

$$+ \alpha_{n+1}^{-1}\Big(\sum_{i=n+2}^{\infty}\alpha_i\Big)\varepsilon$$

and hence $q \leq \pi \leq \varphi$.

In order to prove that π is Ω_+-convex, we take $\gamma \in \Omega_+$ and $u \in C^{\mathbf{N}}$. Take $\varkappa_\nu \in \mathbf{R}$ with $\pi(u_\nu) < \varkappa_\nu \leq 2$, $\nu \in \mathbf{N}$. Then the definition of $\pi(u_\nu)$ yields the existence of $t_{i\nu} \in C$ and $\beta_{i\nu}$ with $0 \leq \beta_{i\nu} \leq 1$, $\nu \in \mathbf{N}$, $1 \leq i \leq n$, with $t_{i\nu} = \beta_{i\nu} x_i$ for $1 \leq i \leq n$, $\nu \in \mathbf{N}$ and $t_{n+1,\nu} = u_\nu$, $\nu \in \mathbf{N}$, such that

$$\pi(u_\nu)\quad \alpha_{n+1}^{-1}\Big(\varphi\Big(\sum_{i=1}^{\infty}\alpha_i t_{i\nu}\Big) - \sum_{i=1}^{n}\alpha_i\varphi(t_{i\nu}) - \sum_{i=n+2}^{\infty}\alpha_i q(t_{i\nu})\Big) <$$

$$< \varkappa_\nu,\ \nu \in \mathbf{N}.$$

One puts $t_i := \sum_{\nu=1}^{\infty}\gamma_\nu t_{i\nu}$, $i \in \mathbf{N}$, in the totally convex space C (see [8]), and gets

$$t_i = \Big(\sum_{\nu=1}^{\infty}\gamma_\nu\beta_i\Big)x_i, \quad 1 \leq i \leq n,$$

$$t_{n+1} = \sum_{\nu=1}^{\infty}\gamma_\nu u_\nu,$$

because of [8], (2.12), [8], (2.4), (ix), yields

$$\sum_{i=1}^{\infty} \alpha_i t_i = \sum_{i=1}^{\infty} \alpha_i \Big(\sum_{\nu=1}^{\infty} \gamma_\nu t_{i\nu} \Big) = \sum_{\nu=1}^{\infty} \gamma_\nu \Big(\sum_{i=1}^{\infty} \alpha_i t_{i\nu} \Big) .$$

Due to (2.4), we have

$$\sum_{i=1}^{n} \alpha_i \varphi(t_i) = \sum_{i=1}^{n} \alpha_i \varphi\Big(\Big(\sum_{\nu=1}^{\infty} \gamma_\nu \beta_{i\nu}\Big) x_i\Big) =$$

$$= \sum_{i=1}^{n} \alpha_i \Big(\sum_{\nu=1}^{\infty} \gamma_\nu \beta_{i\nu}\Big) \varphi(x_i) = \sum_{\nu=1}^{\infty} \gamma_\nu \Big(\sum_{i=1}^{n} \alpha_i \varphi(t_{i\nu}) \Big)$$

and, as q is Ω_+-concave,

$$\sum_{i=n+2}^{\infty} \alpha_i q(t_i) = \sum_{i=n+2}^{\infty} \alpha_i q\Big(\sum_{\nu=1}^{\infty} \gamma_\nu t_{i\nu} \Big) \geqslant$$

$$\geqslant \sum_{i=n+2}^{\infty} \alpha_i \sum_{\nu=1}^{\infty} \gamma_\nu q(t_{i\nu}) = \sum_{\nu=1}^{\infty} \gamma_\nu \Big(\sum_{i=n+2}^{\infty} \alpha_i q(t_{i\nu}) \Big) .$$

Putting these results together one obtains the following estimate

$$\pi\Big(\sum_{\nu=1}^{\infty} \gamma_\nu u_\nu\Big) \leqslant \alpha_{n+1}^{-1}\Big(\varphi\Big(\sum_{i=1}^{\infty} \alpha_i t_i\Big) - \sum_{i=1}^{n} \alpha_i \varphi(t_i) - \sum_{i=n+2}^{\infty} \alpha_i q(t_i)\Big) \leqslant$$

$$\leqslant \alpha_{n+1}^{-1}\Big(\sum_{\nu=1}^{\infty} \gamma_\nu \Big(\varphi\Big(\sum_{i=1}^{\infty} \alpha_i t_{i\nu}\Big) - \sum_{i=1}^{n} \alpha_i \varphi(t_{i\nu}) - \sum_{i=n+2}^{\infty} \alpha_i q(t_{i\nu})\Big)\Big) \leqslant$$

$$\leqslant \sum_{\nu=1}^{\infty} \gamma_\nu \varkappa_\nu .$$

As $\varkappa_\nu$ was arbitrary, subject to $\pi(u_\nu) < \varkappa_\nu \leq 2$, $\nu \in \mathbb{N}$, this implies the Ω_+-convexity of π. Thus we get $\pi \in Z$ and $\pi \leq \varphi$ and hence $\pi = \varphi$. Choosing $z = x_{n+1}$, $t_i = x_i$, $i \neq n+1$, in the definition of π, one has

$$\varphi(x_{n+1}) \leq \alpha_{n+1}^{-1}\left(\varphi\left(\sum_{i=1}^{\infty} \alpha_i x_i\right) - \sum_{i=1}^{n} \alpha_i \varphi(x_i) - \sum_{i=n+2}^{\infty} \alpha_i q(t_i)\right)$$

and hence our assertion.

P r o o f of Theorem 2.2. For $\varepsilon > 0$ there is a $n_0(\varepsilon)$, such that for $n > n_0(\varepsilon)$

$$\left|\sum_{i=n+1}^{\infty} \alpha_i q(x_i)\right| < \varepsilon ,$$

hence Lemma 2.6 implies, for $n > n_0(\varepsilon)$, that

$$\varphi\left(\sum_{i=1}^{\infty} \alpha_i x_i\right) \geq \sum_{i=1}^{\infty} \alpha_i \varphi(x_i) - \varepsilon$$

and one gets

$$\varphi\left(\sum_{i=1}^{\infty} \alpha_i x_i\right) \geq \sum_{i=1}^{\infty} \alpha_i \varphi(x_i).$$

This, together with the fact that φ is Ω_+-convex, proves that

$$(*) \qquad \varphi\left(\sum_{i=1}^{\infty} \alpha_i x_i\right) = \sum_{i=1}^{\infty} \alpha_i \varphi(x_i)$$

for arbitrary $\alpha \in \Omega_+$, $x \in C^{\mathbb{N}}$.

In order to see that φ is a morphism of totally convex spaces one has to show the validity of $(*)$ for all $\alpha \in \Omega$ (cp. [8], (2.6)). Now, for any $x \in C$, [8], (2.12), shows that $0 = \frac{1}{2} x + \frac{1}{2} (-x)$, hence, as $\varphi(0) = 0$, $0 = \varphi(0) = \frac{1}{2} \varphi(x) + \frac{1}{2} \varphi(-x)$ and thus $\varphi(-x) = -\varphi(x)$. Taking any $\alpha \in \Omega$ and $x \in C^{\mathbb{N}}$ one gets (see [8], (2.12), (iii), (v))

$$\varphi\left(\sum_{i=1}^{\infty}\alpha_i x_i\right)=\varphi\left(\sum_{i=1}^{\infty}|\alpha_i|\ (sg(\alpha_i)x_i\ \right)=$$

$$=\sum_{i=1}^{\infty}|\alpha_i|\varphi(sg(\alpha_i)x_i)=\sum_{i=1}^{\infty}\alpha_i\varphi(x_i),$$

$sg(\alpha_i)$ denoting the sign of α_i. Hence Theorem 2.2 is proved.

Remark 2.7. With the seminorm introduced for a totally convex space C in [8], (6.1), one has for the mappings p,q in Theorem 2.2

$$|p(x)|\leqslant\|x\|\quad\text{and}\quad|q(x)|\leqslant\|x\|\quad\text{for any } x\in C.$$

3. Extension theorems

We will now apply (2.2) to get analogoues of the classical extension theorems for totally convex spaces. In this section "totally convex space" always means real totally convex space.

Proposition 3.1. Let C be a totally convex space and $p : C \to \hat{O}(R)$ be an Ω_+-convex mapping with $p(0) = 0$. Then there exists a morphism $\varphi : C \to \hat{O}(R)$ of totally convex spaces with

$$-p(-x)\leqslant\varphi(x)\leqslant p(x),\quad\text{for}\quad x\in C.$$

Proof. As usual (cp. e.g. [10], p.103) one defines

$$q(x) := -p(-x),\quad\text{for}\quad x\in C.$$

Because $\frac{1}{2}x+\frac{1}{2}(-x)=0$ holds for $x\in C$ ([5], (2.12)), we get $0=p(0)\leqslant\frac{1}{2}p(x)+\frac{1}{2}p(-x)$ implying $q(x)\leqslant p(x)$. q is trivially Ω_+-concave, hence our assertion follows from Theorem 2.2.

Theorem 3.2. (The Hahn-Banach extension theorem). Let $C_0 < C$ be a subspace of the totally convex space C and let $p : C \to \hat{O}(R)$ be sublinear. Then, for any morphism $f : C_0 \to \hat{O}(R)$ of totally convex spaces, such that $f(y)\leqslant p(y)$ for every $y\in C_0$, there exists a morphism $\varphi : C \to \hat{O}(R)$ extending f, i.e. with $\varphi | C_0 = f$, and

$$-p(-x) \leqslant \varphi(x) \leqslant p(x), \quad \text{for} \quad x \in C.$$

P r o o f . We define, for $x \in C$ (cp. e.g. [4], p.31),

$$q(x) = 2 \inf\left\{p\left(\tfrac{1}{2} x - \tfrac{1}{2} y\right) + \tfrac{1}{2} f(y) \,\middle|\, y \in C_0\right\}.$$

As

$$p\left(\tfrac{1}{2} x - \tfrac{1}{2} y\right) + \tfrac{1}{2} f(y) \leqslant \tfrac{1}{2} p(x) + \tfrac{1}{2} p(-y) + \tfrac{1}{2} f(y)$$

for $y \in C_0$, one gets $q(x) \leqslant p(x) + p(-y) + f(y)$ and hence, $q \leqslant p$ for $y = 0$.

Besides, we have, due to [8], (2.12), for $x \in C$, $y \in C_0$

$$-\tfrac{1}{4} y = \tfrac{1}{2}\left(\tfrac{1}{2} x - \tfrac{1}{2} y\right) - \tfrac{1}{4} x,$$

which implies

$$\tfrac{1}{4} p(-y) \leqslant \tfrac{1}{2} p\left(\tfrac{1}{2} x - \tfrac{1}{2} y\right) + \tfrac{1}{4} p(-x).$$

This, as $f(-y) \leqslant p(-y)$, $y \in C_0$, leads to

$$p\left(\tfrac{1}{2} x - \tfrac{1}{2} y\right) + \tfrac{1}{2} f(y) \geqslant -\tfrac{1}{2} p(-x).$$

From this we conclude $q(x) \geqslant -p(-x)$ and so $q : C \to \hat{O}(R)$ is well defined with $-p(-x) \leqslant q(x) \leqslant p(x)$, $x \in C$. The last inequality implies $q(0) = 0$.

In order to be able to apply (3.1) we are going to prove that q is Ω_+-convex. For this, take $\alpha \in \Omega_+$, $x, y \in C^N$. Due to [8], (2.4), (ii), (which is the defining equation (TC 2) in the introduction) we get

$$\sum_i \alpha_i\left(\tfrac{1}{2} x_i - \tfrac{1}{2} y_i\right) = \tfrac{1}{2}\sum_i \alpha_i x_i - \tfrac{1}{2}\sum_i \alpha_i y_i.$$

If $\varepsilon > 0$ is given, then, according to the definition of $q(x_i)$, there is a $y_i \in C_0$ with

$$q(x_i) \leqslant 2\left(p\left(\frac{1}{2} x_i - \frac{1}{2} y_i\right) + \frac{1}{2} f(y_i)\right) < q(x_i) + \varepsilon .$$

This, together with

$$p\left(\frac{1}{2}\sum_i \alpha_i x_i - \frac{1}{2}\sum_i \alpha_i y_i\right) \leqslant \sum_i \alpha_i p\left(\frac{1}{2} x_i - \frac{1}{2} y_i\right)$$

implies

$$\sum_i \alpha_i q(x_i) \geqslant 2\left(\sum_i \alpha_i p\left(\frac{1}{2} x_i - \frac{1}{2} y_i\right) + \frac{1}{2}\sum_i \alpha_i f(y_i)\right) - \left(\sum_i \alpha_i\right) \varepsilon \geqslant$$

$$\geqslant 2\left(p\left(\frac{1}{2}\sum_i \alpha_i x_i - \frac{1}{2}\sum_i \alpha_i y_i\right) + \frac{1}{2} f\left(\sum_i \alpha_i y_i\right)\right) - \varepsilon \geqslant q\left(\sum_i \alpha_i x_i\right) - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary q is Ω_+-convex and, due to (3.1), we get a morphism $\varphi : C \to \hat{O}(R)$ with

$$-p(-x) \leqslant -q(-x) \leqslant \varphi(x) \leqslant q(x) \leqslant p(x), \quad x \in C.$$

Now, choosing $x = y \in C_0$, one has

$$q(y) \leqslant 2\left(p(0) + \frac{1}{2} f(y)\right) = f(y),$$

hence $q(-y) \leqslant f(-y) = -f(y)$ or $-q(-y) \geqslant f(y)$, which implies

$$\varphi(y) = f(y) = q(y), \quad \text{for } y \in C_0,$$

and hence our assertion.

A natural choice for an Ω_+-convex mapping on a totally convex space C is the norm on C introduced in [8], (6.1), (2.7) shows that it is the greatest Ω_+-convex mapping. Interesting applications can be expected mainly from (3.2), because the assertion of (3.1) can always be fulfilled by the zero-morphism $\varphi = 0$. Contrary to the classical case, however, two things have to be taken into account when applying (3.2). First, the norm on a totally convex space C is sublinear if and only if C is normed in the sense of [8], (12.1). Secondly, if $C_0 < C$ is a subspace of a normed totally convex space C, the norm $\|-\|_0$ on C_0 is in general different from the restriction of the norm $\|-\|$ on C to C_0. Due to [8], (6.3), one has $\|y\| \leq \|y\|_0$ for any $y \in C_0$. Thus, if one wants to extend a morphism $f : C_0 \to \hat{O}(R)$ to C, a necessary condition is that $|f(y)| \leq \|y\|$, $y \in C_0$, while in general one only knows $|f(x)| \leq \|y\|_0$. This leads to the

D e f i n i t i o n 3.3. Let $C_0 < C$ be a subspace of the totally convex space C. For a morphism $f : C_0 \to \hat{O}(R)$ one defines

$$|f| := \sup \left\{ \frac{|f(y)|}{\|y\|} \;\middle|\; y \in C_0 \setminus \{0\} \right\}$$

(cp. [8], (6.7), (6.9)). f is called norm-bounded in C, iff $|f| \leq 1$. Obviously, if C_0 is an isometrical subspace, then $|f| = \|f\|$.

P r o p o s i t i o n 3.4. Let $C_0 < C$ be a subspace of a normed, totally convex space. Then for any morphism $f : C_0 \to \hat{O}(R)$ the following are equivalent:

(i) f can be extended to a morphism on C.

(ii) f is norm-bounded in C.

Moreover, if f is norm-bounded, it can be extended to a morphism φ with $\|\varphi\| = |f|$.

P r o o f . (i) $\Rightarrow$ (ii): Let $\varphi : C \to \hat{O}(R)$ be given with $\varphi|C_0 = f$. Then, for $y \in C_0$, $y \neq 0$,

$$\frac{|f(y)|}{\|y\|} = \frac{|\varphi(y)|}{\|y\|} \leq \|\varphi\| \leq 1,$$

hence $|f| \leq \|\varphi\| \leq 1$.

(ii) $\Rightarrow$ (i): Put $p(x) := |f| \, \|x\|$; then p is sublinear, due to [8], (6.2) and (12.1). Moreover, for $y \in C$, one has $|f(y)| \leqslant |f| \, \|y\| = p(y)$. Hence, from (3.2) one gets an extension $\varphi : C \to \hat{O}(R)$ with $\varphi|C_0 = f$ and $|\varphi(x)| \leqslant p(x) = |f| \, \|x\|$, $x \in C$.

The comparison of the norms, on the other hand, leads to the

Definition 3.5. Let $C_0 < C$ be a subspace of the totally convex space C. One defines the norm quotient of C_0 in C as

$$\gamma_{C_0} := \inf \left\{ \frac{\|y\|}{\|y\|_0} \,\middle|\, y \in C_0 \setminus \{0\} \right\},$$

$\|-\|$ denoting the norm in C and $\|-\|_0$ the norm in C_0. C_0 is called a norm-equivalent subspace of C, if $\gamma_{C_0} > 0$ holds. This notion is a natural generalization of the equivalence of norms in vector spaces. C_0 is an isometrical subspace of C, if and only if $\gamma_{C_0} = 1$. Obviously

$$\gamma_{C_0} = \sup \left\{ \varkappa \,\middle|\, 0 \leqslant \kappa \leqslant 1 \text{ and } \varkappa \|y\|_0 \leqslant \|y\| \text{ for } y \in C_0 \right\}.$$

Proposition 3.6. Let $C_0 < C$ be a norm-equivalent subspace of a normed totally convex space. Then, for any morphism $f : C_0 \to \hat{O}(R)$ the following are equivalent:

(i) f can be extended to a morphism on C. .

(ii) $\|f\| < \gamma_{C_0}$.

Proof. As we have $\gamma_{C_0} > 0$, an easy computation shows $\|f\| = \gamma_{C_0} |f|$, hence our assertion follows from (3.4).

Corollary 3.7. For a totally convex space C the following are equivalent:

(i) C is normed.

(ii) For every $x_0 \in C$ with $x_0 \neq 0$, there is a morphism $\varphi : C \to \hat{O}(R)$ with $\varphi(x_0) = \|x_0\|$ and $\|\varphi\| = 1$.

P r o o f . (i) $\Rightarrow$ (ii): Take for C_0 the subspace of C generated by x_0, $C_0 = \{\alpha x_0 \mid \alpha \in \mathbb{R} \text{ and } |\alpha| \leq 1\}$ ([8], (2.12)). As C is aspherical (see [8], (12.2)) and $x_0 \neq 0$, $\alpha x_0 = \beta x_0$ implies $\alpha = \beta$. Hence, by $f(\alpha x_0) := \alpha \|x_0\|$ we have a well defined morphism $f : C_0 \to \hat{O}(\mathbb{R})$. Besides, $|f| = 1$, hence (ii) follows from (3.4). By the way, we have $\gamma_{C_0} = \|x_0\|$.

(ii) $\Rightarrow$ (i): Let $x_0 \in C$, $x_0 \neq 0$, be given. Then with the morphism φ in (ii) one has for any $\alpha \in \mathbb{R}$, with $0 < |\alpha| \leq 1$, $|\alpha| \|x_0\| = |\alpha| \varphi(x_0) = |\alpha\varphi(x_0)| = |\varphi(\alpha x_0)| \leq \|\alpha x_0\| \leq |\alpha| \|x_0\|$. Hence, we get $\|\alpha x_0\| = |\alpha| \|x_0\|$ and (i) is proved.

C o r o l l a r y 3.8. For a totally convex space, the following are equivalent:

(i) C is separable.

(ii) For any $x_0, y_0 \in C$ with $x_0 \neq y_0$ there is a morphism $\varphi : C \to \hat{O}(\mathbb{R})$ with $\varphi(x_0) \neq \varphi(y_0)$ and $\|\varphi\| = 1$.

P r o o f . (i) $\Rightarrow$ (ii): C is normed because it is separable ([8], (12.2)). Put $z = \frac{1}{2} x_0 - \frac{1}{2} y_0$. Then $z \neq 0$ as $z = 0$ would lead to $\frac{1}{4} x_0 = \frac{1}{4} y_0$ and hence to $x_0 = y_0$ (cp. [8], (10.2)) which is a contradiction. Now, from (3.7) we get a φ with $\|\varphi\| = 1$ and $\varphi(z) = \|z\| \neq 0$ ([8], (6.9)), hence $\varphi(x_0) \neq \varphi(y_0)$.

(ii) $\Rightarrow$ (i) follows immediately from [8], (10.2). (3.8) has been proved in [8], (10.8), with the classical Hahn-Banach theorem.

R e m a r k 3.9. That the range of the Hahn-Banach theorem for totally convex spaces is actually wider than that of the classical version is shown by the following example of a normed but not separable totally convex space C. Provide $\mathbb{R}^2$ with the supremum norm $\|(x_1, x_2)\| = \sup\{|x_1|, |x_2|\}$, then $\mathbb{R}^2$ is a Banach space and $\hat{O}(\mathbb{R}^2)$ is a totally convex space. Define for $x, y \in \hat{O}(\mathbb{R}^2)$, $x = (x_1, x_2)$, $y = (y_1, y_2)$,

$$x \sim y \; :\Longleftrightarrow \; x = y \quad \text{or} \quad (|x_1| < 1 \text{ and } x_1 = y_1).$$

An easy computation shows that "$\sim$" is a compatible equivalence relation in the category of totally convex spaces (see [8], §4).

Put $C := \hat{O}(R^2)/\sim$ and let $\pi : \hat{O}(R) \to C$ be the canonical projection. C is a normed totally convex space and can be visualized as a double T in the plane R^2, namely as the set $([-1,1] \times \{0\}) \cup (\{-1\} \times [-1,1]) \cup (\{1\} \times [-1,1])$. Obviously, C is not separable and hence cannot be embedded by an injective morphism of totally convex spaces into the unit ball $\hat{O}(B)$ of any Banach space B.

4. The complex and the classical case

Like in the classical case (cp. e.g. [4], [10]) we now proceed to derive Hahn-Banach theorems for complex totally convex spaces from the Hahn-Banach theorem in the real case.

D e f i n i t i o n 4.1. Let C be a complex totally convex space. A mapping $p : C \to \hat{O}(R)$ is called a semi-norm, if the following conditions are fulfilled:

(SN 1) $p\left(\sum_i \alpha_i x_i\right) \leq \sum_i |\alpha_i|\, p(x_i)$ for every $\alpha \in \Omega$ and every $x \in C^N$.

(SN 2) $p(\alpha x) = |\alpha|\, p(x)$ for any $\alpha \in \mathbf{C}$ with $|\alpha| \leq 1$ and any $x \in C$.

As in the proof of Theorem 3.2 one shows, that for a semi-norm p on C one has

$$p\left(\tfrac{1}{2}x - \tfrac{1}{2}y\right) \geq \tfrac{1}{2}p(-y) - \tfrac{1}{2}p(-x) = \tfrac{1}{2}p(y) - \tfrac{1}{2}p(x) \text{ for } x,y \in C$$

and hence $\left|\tfrac{1}{2}p(x) - \tfrac{1}{2}p(y)\right| \leq p\left(\tfrac{1}{2}x - \tfrac{1}{2}y\right)$. In particular, one gets $p \geq 0$.

We are now in a position to prove the complex analogue of Theorem 3.2.

T h e o r e m 4.2. (The complex Hahn-Banach extension theorem). Let $C_0 < C$ be a subspace of the complex totally convex space C, and let $p : C \to \hat{O}(R)$ be a semi-norm on C. Then, for any morphism $f : C_0 \to \hat{O}(\mathbf{C})$ of totally convex spaces, such that $|f(y)| \leq p(y)$, for $y \in C_0$. There exists a morphism $\varphi : C \to \hat{O}(\mathbf{C})$ with $\varphi|C_0 = f$ and $|\varphi(x)| \leq p(x)$ for $x \in C$.

P r o o f . Decompose f into its real and imaginary part

$$f(y) = g(y) + ih(y).$$

C_0 and C are, in an obvious way, real totally convex spaces, and $g,h : C_0 \to \hat{O}(R)$ are morphism of real totally convex spaces. For the real space C, p is sublinear with $p(-x) = p(x)$ and, moreover, $|g(y)| < p(y)$, $y \in C_0$. Hence, due to (3.2), g can be extended to a $\varphi : C \to \hat{O}(R)$ with $\varphi | C_0 = g$ and $|\varphi(x)| \leq p(x)$, $x \in C$. As $h(y) = -g(iy)$ for $y \in C_0$, it is natural to define

$$\Phi(x) := \varphi(x) - i\varphi(ix) \quad \text{for} \quad x \in C.$$

Now take an operation α of the complex totally convex space C, i.e. a sequence $\alpha = (\alpha_i | i \in N)$ of complex numbers with $\sum_i |\alpha_i| \leq 1$. Decompose every α_ν into its real and imaginary part $\alpha_\nu = \gamma_\nu + i\beta_\nu$. Then, due to [8], (2.4), (2.12),

$$\frac{1}{2}\sum_\nu \alpha_\nu x_\nu = \frac{1}{2}\sum_\nu \gamma_\nu x_\nu + \frac{1}{2}\sum_\nu \beta_\nu(ix_\nu)$$

for any $x \in C^N$. An easy computation now yields

$$\Phi\left(\frac{1}{2}\sum_\nu \alpha_\nu x_\nu\right) = \frac{1}{2}\sum_\nu \alpha_\nu \Phi(x_\nu),$$

and, as $\Phi(\lambda x) = \lambda\Phi(x)$ for any $\lambda \in R$ with $|\lambda| < 1$, we get

$$\Phi\left(\sum_\nu \alpha_\nu x_\nu\right) = \sum_\nu \alpha_\nu \Phi(x_\nu).$$

Hence, Φ is a morphism of complex totally convex spaces, provided we can prove $|\Phi(x)| \leq 1$ for $x \in C$. For this, write $\Phi(x) = e^{i\delta}|\Phi(x)|$, which implies $|\Phi(x)| = e^{-i\delta}\Phi(x) = \Phi(e^{-i\delta}x) \geq 0$ and thus

$$|\Phi(x)| = \Phi(e^{-i\delta}x) = \varphi(e^{-i\delta}x) \leq p(x).$$

Consequently, our assertion is proved, as $\Phi|C_0 = f$ trivially.

Definitions 3.3 and 3.5 can be carried over verbatim to the complex case. Propositions 3.4 and 3.6 and Corollaries (3.6)-(3.8) then remain valid for complex totally convex spaces.

Finitely totally convex spaces were introduced in [8], too; they are the Eilenberg-Moore algebras for the category Vec_1 of normed vector spaces with linear contractions and the unit ball functor. They are defined quite analogously to totally convex spaces: The defining equations (TC 1) and (TC 2) are the same, only the set of operations Ω has to be restricted to the subset

$$\Omega_{fin} = \left\{(\alpha_1,\dots,\alpha_n) \,\middle|\, n \in \mathbb{N} \text{ and } \sum_{i=1}^{n} |\alpha_i| \leqslant 1\right\}.$$

Hence, they are finitary (universal) algebras. The Hahn-Banach theorem (2.2) and the extension theorems (3.2) and (4.2) remain valid for finitely totally convex spaces and are proved analogously. In the applications, however, the assumption that the space in question is normed has to be replaced by "normed and regular" (see [8], (13.7)) for finitely totally convex spaces.

We will now briefly sketch how one can easily retrieve the classical Hahn-Banach theorems from the above results. We will only consider real vector spaces, as the complex case can be treated analogously, mutatis mutandis. Let V be a real vector space and $p : V \to \mathbb{R}$ a sublinear functional in the ordinary sense. For $x \in V$ define $q(x) := \max(p(x), p(-x))$; $q : V \to \mathbb{R}$ is a seminorm. Then

$$\hat{O}_p(V) := \{x \mid x \in V \text{ and } q(x) \leqslant 1\}$$

is in an obvious way a finitely totally convex space. If we denote the restriction of p to $\hat{O}_p(V)$ again by p, we see, that, because of $-q(x) \leqslant -p(-x) \leqslant p(x) \leqslant q(x)$, $p : \hat{O}_p(V) \to \hat{O}(\mathbb{R})$ is well defined and is sublinear in the sense of (2.1) for finitely totally convex spaces. Applying now (3.1) for finitely totally convex spaces, we get a morphism $\varphi : \hat{O}_p(V) \to \hat{O}(\mathbb{R})$ of finitely totally convex spaces with $-p(-y) \leq \varphi(y) \leq p(y)$, $y \in \hat{O}_p(V)$. For every $x \in V$ there exists an $\alpha > 0$ and a $y \in \hat{O}_p(V)$ with $x = \alpha y$. It is easily seen that $\psi(x) := \alpha\varphi(y)$ is a well defined mapping and even a vector space homomorphism. Moreover, we have $-p(-x) \leq \psi(x) \leq p(x)$, $x \in V$.

If, in addition, a subspace $U < V$ is given and a homomorphism $f : U \to \mathbb{R}$ with $f(y) \leq p(y)$, $y \in U$, then it is easily verified that $\hat{O}_p(U) < \hat{O}_p(V)$ is an (isometric) subspace. We can apply (3.2) for finitely totally convex spaces, taking into account that f restricts to a morphism $\hat{O}_p(U) \to \hat{O}(\mathbb{R})$, and we get an extension φ inducing in turn an extension Ψ, as above, of the original homomorphism f. The other Hahn-Banach theorems for normed vector spaces and Banach spaces follow similarly.

Finally, let us take a look at Rode's examples in [9], pp.478, 479, and some related situations in view of the above results.

E x a m p l e 4.3. Centrally convex spaces. As an example for his theorem, Rodé in [9] considers centrally convex subsets X of real vector spaces, i.e. subsets X, s.th. for $x,y \in X$ one has $\frac{1}{2}x + \frac{1}{2}y \in X$. This is a special case of the following general situation. Define

$$D_2 := \left\{ \frac{a}{2^n} \mid n \in \mathbb{N},\ a \in \mathbb{N},\ 0 \leq a \leq 2^n \right\}$$

and for a set X set

$$CC(X) := \left\{ f \mid f \in D_2^{(X)} \text{ and } \sum_x f(x) = 1 \right\},$$

where, as usual, for a set M containing zero, $M^{(X)}$ denotes the set of mappings from X to M with finite support. For $\Omega = CC(\mathbb{N})$ call an Ω-algebra (c.p. [3]) X a centrally convex space, if the following equations hold:

$$\text{(CC 1)} \qquad \sum_i \delta_{ik} x_i = x_k,$$

(CC 2) $$\sum_{i=1}^{n} \alpha_i \Big(\sum_{k=1}^{m} \beta_{ik}x_k\Big) = \sum_{k=1}^{m} \Big(\sum_{i=1}^{n} \alpha_i\beta_{ik}\Big)x_k,$$

if $x, x_i \in X$, $\alpha, \beta_k \in \Omega$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta_k = (\beta_{1k}, \ldots, \beta_{mk})$. The centrally convex spaces together with the Ω-morphisms form a category *CCon*. CC(X) is the free centrally convex space generated by the set X, i.e. CC(-) is the left adjoint to the usual forgetful functor assigning to a centrally convex space its underlying set. *CCon* is a category of equationally defined (universal) algebras and the analogue of the Hahn-Banach theorem (2.2) holds. *CCon* is the category of Eilenberg-Moore algebras for the category of centrally convex subsets of real vector spaces.

Example 4.4. Convex spaces. As a second example, Rodé considers convex subsets of real vector spaces. Now, such convex sets are special (universal) algebras of the following type. If $\mathbf{R}_+ := \{x \mid x \in \mathbf{R} \wedge x \geqslant 0\}$, put, for a set X

$$C(X) := \Big\{f \,\Big|\, f \in \mathbf{R}_+^{(X)} \quad \text{and} \quad \sum_x f(x) = 1\Big\}.$$

For $\Omega := C(\mathbf{N})$ call an Ω-algebra X a convex space, if the following equations hold:

(C 1) $$\sum_{i=1}^{n} \delta_{ik}x_i = x_k,$$

(C 2) $$\sum_{i=1}^{n} \alpha_i \Big(\sum_{k=1}^{m} \beta_{ik}x_k\Big) = \sum_{k=1}^{m} \Big(\sum_{i=1}^{n} \alpha_i\beta_{ik}\Big)x_k,$$

for $x_i \in X$, $\alpha, \beta_k \in \Omega$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta_k = (\beta_{1k}, \ldots, \beta_{1m})$, δ_{ik} being the Kronecker delta. The convex spaces together with the Ω-morphisms between them form a category *Conv*. C(X) is the free convex space generated by the set X, i.e. C(-) is the left adjoint to the usual forgetful functor assigning to a convex space its underlying set. *Conv* is a category of equationally defined (universal) algebras and the analogue of the Hahn-Banach theorem (2.2) holds. *Conv* is the category of Eilenberg-Moore algebras for the category of convex subsets of real vector spaces V. One sees at once, that this notion of convex space can be generalized to that of an infinitely convex space, by defining $C_\infty(X) := \{f \mid f \in \mathbb{R}_+^X$ and support of f is at most countable and $\sum_x f(x) = 1\}$. Taking $\Omega_\infty := C_\infty(\mathbb{N})$, the Ω_∞-algebras fulfilling the equations analogous to (C 1) and (C 2) are then the infinitely convex spaces.

E x a m p l e 4.5. Affine spaces. Another nice example is furnished by the barycentric representation of the real affine spaces as given by Ostermann and Schmidt in [7] and by Bos and Wolff in [1], [2]. Put

$$A(X) := \left\{ f \mid f \in \mathbb{R}^{(X)} \text{ and } \sum_x f(x) = 1 \right\},$$

for any set X. Then, for $\Omega := A(\mathbb{N})$ the real affine spaces are exactly those Ω-algebras X, in which the following equations hold:

(AFF 1) $$\sum_{i=1}^{n} \delta_{ik} x_i = x_k,$$

(AFF 2) $$\sum_{i=1}^{n} \alpha_i \left(\sum_{k=1}^{m} \beta_{ik} x_k \right) = \sum_{k=1}^{m} \left(\sum_{i=1}^{n} \alpha_i \beta_{ik} \right) x_k,$$

for $x_i \in X$, $\alpha, \beta_k \in \Omega$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta_k = (\beta_{1k}, \ldots, \beta_{1m})$. Again, the affine spaces together with the Ω-morphisms form the category *AFFIN* of affine spaces, which is, of course, a category of (universal) algebras and the left adjoint to the usual forgetful functor is A(–). And again, the analogue of the Hahn-Banach theorem (2.2) holds.

The example of the category of commutative semigroups Rodé mentions in [9], p.479, is, of course, also a category of algebras. But even the last example in [9], p.479, fits into this framework. There, for two fixed real numbers $\alpha > 0$, $\beta > 0$, a function τ on a real vector space E is given with $\tau(\alpha x+\beta y) \geq \alpha\tau(x) + \beta\tau(y)$, $x,y \in E$. The (universal) algebras belonging to this example can be characterized as follows. For a set X denote by $F(X)$ the real vector space with basis X. Call a subset $M \subset F(X)$ (α,β)-closed, if for any $\xi, \eta \in M$ one has $\alpha\xi+\beta\eta \in M$. Define $H(X)$ to be the smallest (α,β)-closed subset of $F(X)$ containing X, i.e. generated by X. Take as set of operations $\Omega_{\alpha,\beta} := H(\mathbf{N})$ and as defining equations, for the algebras we want to define, all equations valid in $H(\mathbf{N})$ as subset of the vector space $F(\mathbf{N})$. We then have exactly the same situation as in the preceding examples: $H(-)$ is a left adjoint to the forgetful functor and the Hahn-Banach theorem (2.2) holds for this category of algebras, which is, by the way, the category of Eilenberg-Moore algebras of (α,β)-closed subsets of real vector spaces.

Hence it seems, as the formulation of Rodé's theorem in [9] already suggests, that appropriate categories of universal algebras over the real or complex numbers are an interesting setting for general versions of the Hahn-Banach theorem. Especially the consequences of the Hahn-Banach theorem for totally convex spaces will be investigated in a forthcoming paper.

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Received February 1, 1984.