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APPLICATION OF INTEGRAL TRANSFORMS TO SOLVING LINEAR EQUATIONS

1. Introduction

The paper includes an effective complex-analytical procedure of obtaining solutions of n-th order linear differential equations.

Let us consider the n-th order linear differential equation

$$\sum_{k=0}^{n} a_{k+1}(t)x^{(k)}(t) = z(t) \tag{1.1}$$

with given $x^{(k)}(0)$, $k=0,1,\ldots,n-1$. Functions $a_{k+1}(t)$, $t \geqslant 0$, $k=0,1,\ldots,n$, are piecewise continuously differentiable up to order n, while $z(t)$ is for $t \geqslant 0$ a function of bounded variation in every bounded interval. The additional assumptions of the functions $a_{k+1}(t)$, $t \geqslant 0$, $k=0,1,\ldots,n$, and $z(t)$ will be presented later.

2. The idea of the method presented

From equation (1.1), by taking a one-sided integral transform, we obtain

$$\int_0^{+\infty} \sum_{k=1}^{n} a_{k+1}(t)x^{(k)}(t)g(s,t)dt = \int_0^{+\infty} z(t)g(s,t)dt \tag{2.1}$$

where $g(s,t)$ is a complex function which will be calculated later and s is a complex number. After a suitable number of integrations by parts, we get

$$E(s) = P(s) + Z(s) \tag{2.2}$$

where

$$\begin{cases} E(s) = \int_0^{+\infty} x(t)\left\{\sum_{k=0}^{n}\left[\sum_{h=k}^{n}(-1)^h\binom{h}{k}a_{h+1}^{(h-k)}(t)\right]\frac{d^k}{dt^k}g(s,t)\right\}dt, \\ P(s) = \sum_{k=1}^{n}\sum_{h=0}^{k-1}(-1)^{h+1}\frac{d^h}{dt^h}\left[a_{k+1}(t)g(s,t)\right]x^{(k-1-h)}(t)\Big|_{t=0}^{t=+\infty}, \\ Z(s) = \int_0^{+\infty} z(t)g(s,t)dt. \end{cases} \tag{2.3}$$

The unknown function $g(s,t)$ is calculated from the condition that the integral transform $E(s)$ of the function $x(t)$ is the Meijer transform of $x(t)$

$$F_{k,m}(s) = \int_0^{+\infty} x(t)e^{-\frac{1}{2}ts}\, W_{k+\frac{1}{2},m}(st)(st)^{-k-\frac{1}{2}}dt, \quad k \leqslant m \leqslant -k,$$

where $W_{k,m}(z)$ is the Whittaker function of the form

$$W_{k,m}(z) = \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-k-m)}\, z^{\frac{1}{2}+m}\, e^{-\frac{1}{2}z}\, {}_1F_1\left(\tfrac{1}{2}-k+m;\ 1+2m;\ z\right) + $$
$$+ \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}-k+m)}\, e^{-\frac{1}{2}z}\, z^{\frac{1}{2}-m}\, {}_1F_1\left(\tfrac{1}{2}-k-m;\ 1-2m;\ z\right),$$

and

$${}_1F_1(a;\ b;\ z) = 1 + \frac{a}{b}\cdot\frac{z}{1!} + \frac{a(a+1)}{b(b+1)}\cdot\frac{z^2}{2!} + \dots,$$

for $b \neq 0, -1, -2, \ldots$. Then $g(s,t)$ satisfies the differential equation

$$(2.4)\qquad \sum_{k=0}^{n}\left[\sum_{h=k}^{n}(-1)^h\binom{h}{k}a_{h+1}^{(h-k)}(t)\right]\frac{d^k}{dt^k}\,g(s,t) =$$

$$= e^{-\frac{1}{2}st}(ts)^{-k-\frac{1}{2}}\,W_{k+\frac{1}{2},m}(st).$$

We reduce the solution of equation (1.1) to that of equation (2.4). The solution of equation (2.4), with respect to t, determines the unknown function $g(s,t)$.

In addition to the above hypotheses, we suppose that:

(i) The transforms $E(s)$ and $Z(s)$ converge absolutely for some $s = \sigma > 0$.

(ii) The function $P(s) + Z(s)$ has a finite number of poles $s_1, s_2, \ldots, s_l$ of order $\gamma_1, \gamma_2, \ldots, \gamma_l$, respectively, and $\operatorname{re} s_i \leqslant \sigma$, $i = 1, 2, \ldots, l$.

(iii) The integral

$$\int_{\Gamma(R)}\left[P(s)+Z(s)\right]\tilde{g}_{k,m}(st)ds,\quad (\Gamma(R) : s = \sigma - R\,e^{j\varphi},\quad \varphi \in \langle -\tfrac{\pi}{2}, \tfrac{\pi}{2}\rangle),$$

converges for sufficiently large $R > 0$ and tends to 0 as $R \to +\infty$, where

$$\tilde{g}_{k,m}(st) = (ts)^{k+m}\frac{\Gamma(1+m-k)}{\Gamma(1+2m)}\,{}_1F_1(1+m-k;\ 1+2m; st).$$

From (2.2), (2.3), by taking the inverse Meijer transform, we obtain

$$(2.5)\quad \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}E(s)\tilde{g}_{k,m}(st)ds = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}\left[Z(s)+P(s)\right]\tilde{g}_{k,m}(st)ds.$$

The left-hand side of (2.5), by the Meijer theorem ([13]), is $x(t)$, $t > 0$. Its right-hand side, by assumption (iii) and the theorem on residues, becomes

$$(2.6)\qquad \frac{1}{2\pi j}\int\limits_{\sigma-j\infty}^{\sigma+j\infty}\left[Z(s)+P(s)\right]\tilde{g}_{k,m}(st)ds =$$

$$= \lim_{R\to+\infty}\frac{1}{2\pi j}\int\limits_{\Gamma(R)+\Gamma^*(R)}\left[Z(s)+P(s)\right]\tilde{g}_{k,m}(st)ds =$$

$$= \sum_{h=1}^{l}\operatorname*{res}_{s=s_h}\left\{\left[P(s)+Z(s)\right]\tilde{g}_{k,m}(st)\right\},$$

where $\Gamma^*(R)$ is a line segment joining $\sigma - jR$ to $\sigma + jR$.

Now, we calculate

$$\operatorname*{res}_{s=s_h}\left\{\left[P(s)+Z(s)\right]\tilde{g}_{k,m}(st)\right\} =$$

$$= \operatorname*{res}_{s=s_h}\left\{\left[P(s)+Z(s)\right](ts)^{k+m}\frac{\Gamma(1+m-k)}{\Gamma(1+2m)}\,{}_1F_1(1+m-k;\,1+2m;st)\right\} =$$

$$= \frac{1}{(\gamma_h-1)!}\frac{d^{\gamma_h-1}}{ds^{\gamma_h-1}}\left\{\left[P(s)+Z(s)\right](s-s_h)^{\gamma_h}(ts)^{k+m}\times\right.$$

$$\left.\times\frac{\Gamma(1+m+k)}{\Gamma(1+2m)}\,{}_1F_1(1+m-k;1+2m;st)\right\}\Bigg|_{s=s_h} =$$

$$= \frac{1}{(\gamma_h-1)!}\sum_{i=0}^{\gamma_h-1}\binom{\gamma_h-1}{i}\frac{d^{\gamma_h-1-i}}{ds^{\gamma_h-1-i}}\left\{(s-s_h)^{\gamma_h}\left[P(s)+Z(s)\right]\right\}\times$$

$$\times\frac{d^i}{ds^i}\left[(ts)^{k+m}\frac{\Gamma(1+m-k)}{\Gamma(1+2m)}\,{}_1F_1(1+m-k;1+2m;ts)\right]\Bigg|_{s=s_h} =$$

$$= \frac{1}{(\gamma_h-1)!}\sum_{i=0}^{\gamma_h-1}\binom{\gamma_h-1}{i}\cdot\frac{d^{\gamma_h-1-i}}{ds^{\gamma_h-1-i}}\left\{(s-s_h)^{\gamma_h}\left[P(s)+Z(s)\right]\right\}\times$$

$$\times \sum_{v=0}^{i} \binom{i}{v} \frac{\Gamma(1+m-k)}{\Gamma(1+2m)} \frac{d^{i-v}}{ds^{i-v}}\, {}_1F_1(1+m-k;1+2m;st)\, \frac{d^v}{ds^v}\,(ts)^{k+m}\Big|_{s=s_h} =$$

$$= \frac{\Gamma(1+m-k)}{\Gamma(1+2m)(\gamma_h-1)!} \sum_{i=0}^{\gamma_h-1} \binom{\gamma_h-1}{i} \frac{d^{\gamma_h-1-i}}{ds^{\gamma_h-1-i}} \left\{(s-s_h)^{\gamma_h}\,[P(s)+Z(s)]\times\right.$$

$$\times \sum_{v=0}^{i} \binom{i}{v} t^{i-v} \frac{(1+m-k)(1+m-k+1)\ldots(1+m-k+i-v-1)}{(1+2m)(1+2m+1)\ldots(1+2m+i-v-1)} \times$$

$$\times\, {}_1F_1(1+m-k+i-v;1+2m;st)\, t^{k+m} \frac{(k+m)!}{(k+m-v)!}\, s^{k+m-v}\Big|_{s=s_h}.$$

Denoting, for brevity,

$$(2.7)\quad \alpha_{h,i,v} = \frac{\Gamma(1+m-k)}{\Gamma(1+2m)(\gamma_h-1)!} \binom{\gamma_h-1}{i} \frac{d^{\gamma_h-1-i}}{ds^{\gamma_h-1-i}} \times$$

$$\times \left\{(s-s_h)^{\gamma_h}\,[P(s)+Z(s)]\right\}\Big|_{s=s_h} \times$$

$$\times \binom{i}{v} \frac{(1+m-k)(1+m-k+1)\ldots(1+m-k+i-v-1)}{(1+2m)(1+2m+1)\ldots(1+2m+i-v-1)} \frac{(k+m)!}{(k+m-v)!}\, s_h^{k+m-v},$$

$h = 1,2,\ldots,l$, $i = 0,1,\ldots,\gamma_h-1$, $v = 0,1,\ldots,i$, we get

$$\operatorname*{res}_{s=s_h} \left\{[P(s)+Z(s)]\ \tilde{g}_{k,m}(st) = \sum_{i=0}^{\gamma_h-1} \sum_{v=0}^{i} \alpha_{h,i,v}\ t^{k+m+i-v} \times\right.$$

$$\times\, {}_1F_1(1+m-k+i-v;1+2m+i-v;ts_h).$$

Using the above calculations, we obtain the result as follows.

Theorem. Under the above hypotheses, the solution of equation (1.1) is given by the formula

$$(2.8) \qquad x(t) = \sum_{h=1}^{l} \sum_{i=o}^{\gamma_h - 1} \sum_{v=o}^{i} \alpha_{h,i,v} \, t^{k+m+i-v} \times$$

$$\times \, {}_1F_1(1+m-k+i-v;1+2m+i-v;ts_h),$$

where $k \leqslant m \leqslant -k$ and $\alpha_{h,i,v}$ is defined in (2.7).

3. The particular cases

The above theorem gives a solution of equation (1.1) if the solution of equation (2.4) is known. The simplest case is when equation (2.4) has constant coefficients and is of the form

$$(3.1) \qquad \sum_{i=0}^{n} C_{i+1} \frac{d^i}{dt^i} g(s,t) = \frac{(-1)^n}{a_{n+1}(t)} e^{-\frac{1}{2}ts} (ts)^{-k-\frac{1}{2}} W_{k+\frac{1}{m},m}(st),$$

where $k \leqslant m \leqslant -k$ and $a_{n+1}(t) \neq 0$ for $t \geqslant 0$, the functions $a_{i+1}(t)$, $t \geqslant 0$, $i=0,1,\ldots-n-1$, satisfying the conditions

$$(3.2) \qquad \frac{1}{a_{n+1}(t)} \sum_{h=i}^{n} (-1)^{n+h} \binom{h}{i} a_{h+1}^{(h-i)}(t) = C_{i+1},$$

where $C_{n+1} = 1$ and C_{h+1}, $h=0,1,\ldots,n-1$, are arbitrary constants. The solution of (3.2) has the following form

$$(3.3) \qquad a_{i+1}(t) = \sum_{h=0}^{n-i} (-1)^{h+n-i} \binom{i+h}{i} C_{i+1} a_{n+1}^{(h)}(t),$$

$$i=0,1,\ldots,n-1.$$

Then, assuming that the integral

$$\int_0^{+\infty} (-1)^n \cdot a_{n+1}^{-1}(t) \, e^{-\frac{1}{2}ts} (ts)^{-k-\frac{1}{2}} W_{k+\frac{1}{2},m}(st) dt$$

exists, we may obtain, by our theorem, an effective solution of (1.1) with coefficients satisfying (3.3).

E x a m p l e . Let $n = 2$, $a_3 = \frac{1}{t+1}$. From (3.3), by taking $C_1 = 4$, $C_2 = 5$, $C_3 = 1$, $k+m = 0$, we obtain

$$a_1 = \frac{2}{(t+1)^3} + \frac{5}{(t+1)} + \frac{4}{t+1} ,$$

$$a_2 = -\left[\frac{2}{(t+1)^2} + \frac{5}{t+1}\right] .$$

We solve the equation

$$(3.4) \quad \frac{\ddot{x}(t)}{t+1} - \left[\frac{2}{(t+1)^2} + \frac{5}{t+1}\right] \dot{x}(t) + \left[\frac{2}{(t+1)^3} + \frac{5}{(t+1)^2} + \frac{4}{t+1}\right] x(t) = 0.$$

The corresponding time-invariant equation has the form

$$\frac{d^2}{dt^2}\, g(s,t) + 5\,\frac{d}{dt}\, g(s,t) + 4g(s,t) = (t+1)e^{-st}.$$

Its solution has the following form

$$g(s,t) = \frac{e^{-st}}{3(s-4)^2}\left[(t+1)(s-4) + 1\right] -$$

$$- \frac{e^{-st}}{3(s-1)^2}\left[(t+1)(s-1)\right] + 1 \;+ D_1e^{-4t} + D_2e^{-t},$$

where D_1 and D_2 are arbitrary constants. For $D_1 = D_2 = 0$, we have

$$g(s,t) = \frac{e^{-st}}{3(s-4)^2}\left[(t+1)(s-4) + 1\right] - \frac{e^{-st}}{3(s-1)^2}\left[(t+1)(s-1) + 1\right],$$

$$Z(s) = 0,$$

$$P(s) = -a_3(t)g(s,t)x^{(1)}(t)\Big|_0^{+\infty} + \left[a_3(t)g(s,t)\right]^{(1)}x(t)\Big|_0^{+\infty} -$$

$$- a_2(t)g(s,t)x(t)\Big|_0^{+\infty} = \frac{(s-3)\left[\dot{x}(0) + (s-6)x(0)\right]}{3(s-4)^2} -$$

$$- \frac{s\dot{x}(0) + s(s+6)x(0)}{3(s-1)^2} + \frac{x(0)}{3(s-1)} - \frac{x(0)}{3(s-4)} .$$

The assumption (ii) is fulfilled, because the function $P(s) + Z(s) = P(s)$ has two real poles $s_1 = 4$ and $s_2 = 1$ of order two. The integrand in (iii) has a form $P(s)\cdot e^{st}$ and the assumption (iii) is fulfilled by the following remark.

R e m a r k (cf. [4], pp.45-47). A sufficient condition for (iii) to tend for all $t > 0$ to 0 as $R \to +\infty$ is, in particular, fulfilled if the integrand in (iii) is a finite sum of terms $W_k(s)\exp(\tau_k s)$ for any fixed t, where $W_k(s)$ are rational functions such that $s^i W_k(s) \to \delta_k$, (a constant) as $s \to +\infty$ if $\operatorname{re} \tau_k > 0$, $\geqslant 1$.

Now, we must calculate

$$\operatorname*{res}_{s=4} P(s)e^{ts} = ts(s-4)^2\, P(s)\Big|_{s=4} + e^{ts}\frac{d}{ds}\, P(s)(s-4)^2\Big|_{s=4} =$$

$$= \frac{1}{3}\, e^{4t}\left[\dot{x}(0) - 2x(0)\right](t+1),$$

$$\operatorname*{res}_{s=1} P(s)e^{ts} = \frac{1}{3}\,(t+1)\, e^{t}\left[-\dot{x}(0) + 5x(0)\right].$$

According to the method presented above, we obtain

$$x(t) = \operatorname*{res}_{s=4} P(s)e^{ts} + \operatorname*{res}_{s=1} P(s)e^{ts} = \frac{1}{3}\, e^{4t}(t+1)\left[\dot{x}(0) - 2x(0)\right] + \frac{1}{3}\, e^{t}(t+1)\left[-\dot{x}(0) + 5x(0)\right].$$

It is easy to calculate that the function $x(t)$ is the solution of equation (3.4). The integral transforms

$$E(s) = \int_0^{+\infty} x(t)\left\{\sum_{k=0}^{2}\left[\sum_{h=k}^{2}(-1)^h\binom{h}{k}\, a_{h+1}^{(h-k)}(t)\right]\frac{d^k}{dt^k}\, g(s,t)\right\} dt =$$

$$= \int_0^{+\infty} x(t)e^{-st}\, dt = \frac{1}{3}\left[\dot{x}(0) - 2x(0)\right]\int_0^{+\infty}(t+1)e^{4t-st}\, dt +$$

$$+ \frac{1}{3}\left[-\dot{x}(0) + 5x(0)\int_0^{+\infty}(t+1)e^{t-st}\,dt = \right.$$

$$= \frac{1}{3}\left[\dot{x}(0) - 2x(0)\right]\cdot\left[\frac{1}{(s-4)^2} + \frac{1}{s-4}\right] +$$

$$+ \frac{1}{3}\left[-\dot{x}(0) + 5x(0)\right]\cdot\left[\frac{1}{(s-1)^2} + \frac{1}{s-1}\right],$$

and $Z(s) = 0$ converge absolutely for some $s = \sigma > 4$.

REFERENCES

[1] G. Doetsch: Handbuch der Laplace Transformation, vols 1-3, Birkhauser Verlag, Basel, 1950-1956.
[2] G. Doetsch: Introduction to the theory and application of the Laplace transformation, Berlin-Heidelberg-New York, 1974.
[3] N.P. Erugin, I.J. Štokalo i dr.: Kurs obyknovennych differencialnych uravnenij (in Russian), Višca Skola, Kijev, 1974.
[4] E. Lindelöf: Le calcul des residus et ses applications à la théorie des fonctions, Paris, 1905.
[5] F. Leja: Funkcje zespolone, Warszawa, 1979.
[6] J. Ławrynowicz, W. Pełczewski: A complex analytical procedure for determining the switching instants in time-optimal control of linear plant with disturbances, Bull. Acad. Polon. Sci. Ser. Tech., 21 (1973), 59-65 (637-643).
[7] J. Ławrynowicz, W. Pełczewski: Time-optimal control of linear multivariable plant with disturbances, I. Formulation of the result, ibid., 23 (1975), 53 (621).

[8] J. Ławrynowicz, W. Pełczewski: Time-optimal control of linear multivariable plant with disturbances, II. Proof., ibid., 23 (1975), 59 (627).

[9] J. Ławrynowicz, W. Pełczewski: A complex-analytical procedure for determining the switching instants in time-optimal control of linear time-varying plant with disturbances, ibid., 25 (1977), 25 (421).

[10] J. Ławrynowicz, W. Pełczewski: A complex-analytical procedure for determining the switching instants in time-optimal control of linear time-varying plant with disturbances, II. Particular cases, ibid., 25 (1977), 33 (429).

[11] T.M. MacRobert: Functions of a complex variable, 5 th ed., London, 1962.

[12] C.S. Meijer: Integraldarstellungen für Whittakersche Functionen und ihre Produkte, I-II, Nederl. Akad. Wetensch. Proc. 44 (1941), 435-441 and 559-605.

[13] C.S. Meijer: Eine neue Erweiterung der Laplace--Transformation, I-II, Nederl. Acad. Wetensch. Proc. 44 (1941), 728-737 and 831-839.

[14] C.S. Meijer: On the G-Functions, I-VII, Nederl. Akad. Wetensch. Proc. 49 (1946), 227-237, 344-356, 457-469, 632-641, 765-772, 936-943, 1063-1072 and 1165-1175, Indag. Math. 8 (1946), 124-134, 213-225, 312-342, 391-400, 468-475, 595-602, 661-670 and 713-773.

[15] A. Szymanowska: Wyznaczanie punktów nieciągłości rozwiązań pewnych liniowych układów sterowania, Doctoral Thesis, Łódź 1980.

[16] A. Szymanowska: Application of the Meijer transform to solving linear differential equations, Differential Equations and Applications, Proceeding of the Second Conference, Rousse, Bulgaria, Rousse, 1982.

[17] S. W o ł e k : Optimal control constants determination for linear objects, Podstawy Sterowania, 1 (1971), 209-222.

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