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ON THE CONJUGATES OF SOME OPERATOR SPACES, I

0. Preliminaries

In this paper we prove that the dual space of $\mathcal{L}^1_{m\psi}(\mathcal{A})$ (non-commutative Lorentz space) is the non-commutative Marcinkiewicz space $\mathcal{M}_\psi(\mathcal{A})$ and in general, $\mathcal{M}_\psi(\mathcal{A})^* \neq \mathcal{L}^1_{m\psi}(\mathcal{A})$. In the last section we determine the predual of $\mathcal{L}^1_{m\psi}(\mathcal{A})$ is some cases.

Throughout, the paper $\mathcal{A}$ is a semifinite von Neumann algebra on a Hilbert space $\mathcal{H}$. The lattice of orthogonal projections from $\mathcal{A}$ is denoted by $\mathrm{Proj}(\mathcal{A})$ (see [3]).

D e f i n i t i o n 0.1. A projection p in $\mathcal{A}$ is minimal (atom) if each non-zero projection q in $\mathcal{A}$ such that $q \leqslant p$ is equal to p.

$(\mathcal{H}, \mathcal{A}, m)$ is the regular gage space in the sence of [8], $\mathcal{L}_m(\mathcal{A}) = \mathcal{L}_m = \mathcal{L}$ is the $*$-algebra of m-measurable operators (see [4], [1]) and $\mathcal{J}_m(\mathcal{A}) = \mathcal{J}_m = \mathcal{J} = \{a \in \mathcal{L}_m(\mathcal{A}) : m(e^{\perp}_{\varepsilon}) < \infty,\ \varepsilon > 0\}$, where $\sqrt{a^*a} = |a| = \int_0^\infty \lambda\, de_\lambda$ is the spectral resolution of $|a|$. $\mathcal{J}_m$ is the $*$-subalgebra of $\mathcal{L}_m$. The functional: $\tilde{\rho}_m: \mathcal{L}\times\mathcal{L} \to [0,\infty)$ (see [1]) $\tilde{\rho}_m(a,b) = \rho_m(a-b) = \inf\{\varepsilon>0 : m(e^{\perp}_{\varepsilon}) \leqslant \varepsilon\} = \inf\{\varepsilon>0: (a-b)(\varepsilon) \leqslant \varepsilon\}$ where $|a-b| = \int_0^\infty \lambda\, de_\lambda$, is a metric in $\mathcal{L}$. $\mathcal{L}_m(\mathcal{A})$ in this metric is a Frechet space.

D e f i n i t i o n 0.2 (cf. [9], [4], [1]). A sequence of m-measurable operators $\{a_n\}$ is said to be m-convergent (convergent in measure) to a measurable operator a ($a_n \xrightarrow{m} a$) if $\rho_m(a-a_n) \to 0$ as $n \to \infty$.

For any $a \in \mathcal{L}_m(\mathcal{A})$ and $\alpha > 0$ we define (see [5], [10], [2]):

(i) $a(\alpha) = \inf\{0 \leq \lambda < \infty : m(e_\lambda^\perp) \leq \alpha,\ \alpha > 0\}$, $|a| = \int_0^\infty \lambda de_\lambda$,

(ii) $\underset{\sim}{a}(\alpha) = 1/\alpha \int_0^\alpha a(\beta)d\beta$,

(iii) $\tilde{a}(\alpha) = \begin{cases} \sup\{1/m(p)\cdot m(|pa|) : \alpha \leq m(p) < \infty\}, & 0 < \alpha \leq m(1) \\ 0 & \alpha > m(1). \end{cases}$

It is not difficult to observe that (see [5], [2])

$\underset{\sim}{(a + b)}(\alpha) \leq \underset{\sim}{a}(\alpha) + \underset{\sim}{b}(\alpha)$, $\widetilde{(a + b)}(\alpha) \leq \tilde{a}(\alpha) + \tilde{b}(\alpha)$, $a, b \in \mathcal{L}_m(\mathcal{A})$, $\alpha > 0$.

P r o p o s i t i o n 0.1 ([1], proposition 2.7). If $a_n \xrightarrow{m} a$ $(a, a_n \in \mathcal{L})$ then $a_n(\alpha) \to a(\alpha)$ at each point of continuity of the function $a(\alpha)$.

P r o p o s i t i o n 0.2 ([2], proposition 2.4). For any $a \in \mathcal{J}_m(\mathcal{A})$ and $\alpha > 0$,

$$a(\alpha) \leq \tilde{a}(\alpha) \leq \underset{\sim}{a}(\alpha).$$

Throughout, $\psi(\alpha)$, $0 \leq \alpha < \infty$ is a non-negative, concave continuous function such that $\psi(0) = 0$, $\psi(\infty) = \lim_{\alpha \to \infty} \psi(\alpha) = \infty$.
It is easy to see that $\psi'(\alpha)$ is a non-negative, non-increasing function and $\psi(\alpha) = \int_0^\alpha \psi'(\beta)d\beta$. We say that (*) or (**) is satisfied if:

(*) $\lim_{\alpha \to \infty} \alpha/\psi(\alpha) = \sup\{\alpha/\psi(\alpha) : \alpha > 0\} = \infty$, $\psi(\alpha)/\alpha \leq \Gamma \psi'(\alpha)$ for $\alpha > 0$ and some constant $\Gamma > 0$.

(**) $0 < \Gamma_1 \leq \alpha/\psi(\alpha) \leq \Gamma_2 < \infty$ for $\alpha > 0$ and some constants Γ_1, Γ_2.

1. Non-commutative Lorentz spaces

D e f i n i t i o n 1.1 (see [5], [1], [2]). For any $a \in \mathcal{L}_m(\mathcal{A})$ and concave function $\psi(\alpha)$, we define

$$\|a\|_{1,m\psi} = \int_0^\infty \psi'(\alpha)\ a(\alpha)\ d\alpha$$

and

$$\mathcal{L}^1_{m\psi}(\mathcal{A}) = \mathcal{L}^1_{m\psi} = \left\{a \in \mathcal{L}_m(\mathcal{A}) : \|a\|_{1,m\psi} < \infty\right\}$$

$\mathcal{L}^1_{m\psi}$ are called non-commutative Lorentz spaces. Non-commutative Lorentz space is a Banach space with the norm $\|\cdot\|_{1,m\psi}$ (see [7], [1], [2]). By $\mathcal{L}^1_m$ $(\mathcal{L}^{\delta 1}_m)$ we denote the space $\mathcal{L}^1_{m\psi}$ in the case $\psi(\alpha) = \alpha$ $(\psi(\alpha) = \alpha^{1/\delta}$, $1 < \delta < \infty)$ with the norm $\|\cdot\|_{1,m}$ $(\|\cdot\|_{\delta,1,m})$. Note that $\mathcal{L}^1_{m\psi}(\mathcal{A}) = \mathcal{L}^1_m(\mathcal{A})$ and $1/\Gamma_2\cdot\|a\|_{1,m} \leqslant$ $\leqslant \|a\|_{1,m\psi} \leqslant 1/\Gamma_1\cdot\|a\|_{1,m}$ if the condition (**) is satisfied.

Theorem 1.1 (cf. [1], th.6.3). A linear functional h is continuous on $\mathcal{L}^1_{m\psi}(\mathcal{A})$ if and only if

$$h(b) = h_a(b) = m(ba),\ b \in \mathcal{L}^1_{m\psi}(\mathcal{A}),$$

where a is an operator locally measurable in the sense of [10] and $\sup\{1/\psi(m(p))\cdot m(|pa|) : p \in \mathrm{Proj}(\mathcal{A}), m(p) < \infty\} < \infty$. Then

$$\|h_a\| = \sup\{1/\psi(m(p))\cdot m(|pa|) : p \in \mathrm{Proj}(\mathcal{A}),\ m(p) < \infty\}.$$

Theorem 1.2. Assume that $h_a \in \mathcal{L}^1_{m\psi}(\mathcal{A})^*$. Then:

(i) $a \in \mathcal{L}^1_m(\mathcal{A}) + \mathcal{A}$,

(ii) $a \in \mathcal{J}_m$ if $\lim\limits_{\alpha\to\infty} \alpha/\psi(\alpha) = \sup\{\alpha/\psi(\alpha) : \alpha > 0\} = \infty$,

(iii) $a \in \mathcal{A}$ if $\lim\limits_{\alpha\to 0} \alpha/\psi(\alpha) = \inf\{\alpha/\psi(\alpha) : \alpha > 0\} = \gamma > 0$,

(iv) $1/\psi'(\alpha)\cdot\tilde{a}(\alpha) \leqslant \Gamma\|h_a\|$ if (*) is satisfied.

Proof. (i). Suppose that $|a^*| = \int_0^\infty \lambda\, de_\lambda$ and $m(e^\perp_{\lambda_0}) = \infty$. For any $p \leqslant e^\perp_{\lambda_0}$, $m(p) \geqslant 1$, we have

$$p|a^*|p = pe^\perp_{\lambda_0}\ |a^*|\, e^\perp_{\lambda_0} p \geqslant \lambda_0 p$$

and

$$\|h_a\| \geqslant 1/\psi(m(p))\cdot m(|pa|) \geqslant 1/\psi(m(p))\cdot m(|a^*|\,p) \geqslant$$
$$\geqslant \lambda_0 m(p)/\psi(m(p)) \geqslant \lambda_0/\psi(1)$$

that is $\lambda_0 \leqslant \psi(1)\|h_a\| < \infty$. In consequence, $a^* \in \mathcal{L}_m(\mathcal{A})$, which is equivalent to $a \in \mathcal{L}_m(\mathcal{A})$. Moreover, for any $\lambda > \|h_a\|\,\psi(1)$ we have

$$a^* = a^* e_\lambda + a^* e_\lambda^\perp \in \mathcal{A} + \mathcal{L}_m^1, \text{ where } |a^*| = \int_0^\infty \lambda de_\lambda.$$

(ii). With the notations of the proof of (i) we have the following $\|h_a\| \geq m(p)/\psi(m(p))\cdot\lambda_0 \to \infty$ as $m(p) \to \infty$. In consequence, $m(e_{\lambda_0}^\perp) < \infty$ for any $\lambda_0 > 0$ that is $|a^*| \in \mathcal{J}_m(\mathcal{A})$ which is equivalent to $a \in \mathcal{J}_m(\mathcal{A})$.

(iii). Let $p \leq e_{\lambda_0}^\perp$, $m(e_{\lambda_0}^\perp) \neq 0$. Then $\|h_a\| \geq m(p)/\psi(m(p))\cdot$ $\cdot\lambda_0 \geq \gamma\lambda_0$ which implies $\lambda_0 \leq \|h_a\|/\gamma$ that is $|a^*| \in \mathcal{A}$ and $\||a^*|\| \leq \|h_a\|/\gamma$. Finally, $a \in \mathcal{A}$, $\|a\| \leq \|h_a\|/\gamma$.

(iv). For any $p \in \mathrm{Proj}(\mathcal{A})$, $m(p) < \infty$, we have

$$1/m(p)\cdot m(|pa|) = \psi(m(p))/m(p)\cdot 1/\psi(m(p))\cdot m(|pa|) \leq$$

$$\leq \psi(m(p))/m(p)\cdot\|h_a\| \leq \psi(\alpha)/\alpha\cdot\|h_a\| \quad \text{if} \quad m(p) \geq \alpha.$$

In consequence, $\tilde{a}(\alpha) \leq \psi(\alpha)/\alpha\cdot\|h_a\|$ and $1/\psi'(\alpha)\cdot\tilde{a}(\alpha) \leq \Gamma\|h_a\|$.

2. <u>Non-commutative Marcinkiewicz spaces</u>

D e f i n i t i o n 2.1 (see [7], [1]). For any $a \in \mathcal{L}_m^1 + \mathcal{A}$ and $\psi(\alpha)$, we define

$$M_\psi(a) = \sup\left\{1/\psi(\alpha) \cdot \int_0^\alpha a(\beta)\, d\beta : \alpha > 0\right\}$$

and

$$\mathcal{M}_\psi(\mathcal{A}) = \mathcal{M}_\psi = \left\{a \in \mathcal{L}_m^1 + \mathcal{A} : M_\psi(a) < \infty\right\}.$$

$\mathcal{M}_\psi(\mathcal{A})$ are called non-commutative Marcinkiewicz spaces. $\mathcal{L}_m^{\delta\infty}(\mathcal{A})$ denotes the space $\mathcal{M}_\psi(\mathcal{A})$ in the case $\psi(\alpha) = \alpha^{1/\delta'}$, $1/\delta + 1/\delta' = 1$, δ, $\delta' \geq 1$. It is easy to see that $\mathcal{M}_\psi(\mathcal{A})$ is a normed linear space with norm $M_\psi(\cdot)$. We shall prove that Marcinkiewicz space $\mathcal{M}_\psi(\mathcal{A})$ is a Banach space. First we prove the following

P r o p o s i t i o n 2.1. The topology of m-convergence on $\mathcal{M}_\psi(\mathcal{A})$ is weaker than the topology of $M_\psi(\cdot)$ on $\mathcal{M}_\psi(\mathcal{A})$.

P r o o f . It is to be shown that, for any $\varepsilon > 0$: $\{a \in \mathcal{M}_\psi(\mathcal{A}) : \rho(a) \leq \varepsilon\}$ is a neighbourhood of zero in the topology of $M_\psi(\cdot)$.

Case I. $\lim_{\alpha \downarrow 0} \alpha/\psi(\alpha) = \gamma > 0$.

Let $O_{\gamma,\varepsilon} = \{a \in \mathcal{M}_\psi(\mathcal{A}) : M_\psi(a) < \gamma\varepsilon\}$. Assume that $a \in O_{\gamma,\varepsilon}$ $|a| = \int_0^\infty \lambda de_\lambda$, is the spectral resolution of $|a| = \sqrt{a^*a}$; then, $e_\varepsilon^\perp \leq 1/\varepsilon\, |a|$. In consequence, $M_\psi(e_\varepsilon^\perp) \leq 1/\varepsilon \cdot M_\psi(a) < \gamma$. It is clear that for any $0 \neq p \in \mathrm{Proj}(\mathcal{A})$, $M_\psi(p) \geq \gamma$. Hence $e_\varepsilon^\perp = 0$, that is $a \in \mathcal{A}$ and $\|a\| \leq \varepsilon$. Finally

$$O_{\gamma,\varepsilon} \subset \{a \in \mathcal{M}_\psi(\mathcal{A}) : \|a\| \leq \varepsilon\} \subset \{a \in \mathcal{M}_\psi(\mathcal{A}) : \rho(a) \leq \varepsilon\}.$$

Case II. $\lim_{\alpha \downarrow 0} \alpha/\psi(\alpha) = 0$.

Suppose that $\alpha/\psi(\alpha) \leq \gamma \Rightarrow \alpha \leq \varepsilon$. Then for any $a \in O_{\gamma,\varepsilon}$, $|a| = \int_0^\infty \lambda de_\lambda$ $M_\psi(e_\varepsilon^\perp) = m(e_\varepsilon^\perp)/\psi(m(e_\varepsilon^\perp)) < \gamma$. Hence $m(e_\varepsilon^\perp) \leq \varepsilon$ and, in consequence, $\rho(a) \leq \varepsilon$.

P r o p o s i t i o n 2.2. $\mathcal{M}_\psi(\mathcal{A})$ is a Banach space.

P r o o f . Assume that $\{a_n\}$ is a Cauchy sequence in $\mathcal{M}_\psi(\mathcal{A})$. By Proposition 2.1, $a_n \xrightarrow{m} a$ for some $a \in \mathcal{L}_m(\mathcal{A})$. Hence and from Proposition 0.1 $(a_k - a_n)(\alpha) \xrightarrow[k]{} (a - a_n)(\alpha)$ almost everywhere on $(0,\infty)$ with respect to the Lebesgue measure. Thus

$$1/\psi(\alpha) \cdot \int_0^\alpha (a-a_n)(\beta)d\beta \leq \liminf_k 1/\psi(\alpha) \cdot \int_0^\alpha (a_k-a_n)(\beta)d\beta \leq$$

$$\leq \liminf_k M_\psi(a_k-a_n)$$

and, in consequence,

$$M_\psi(a-a_n) \leq \liminf_k M_\psi(a_k-a_n).$$

Finally, $a = (a-a_n) + a_n \in \mathcal{M}_\psi(\mathcal{A})$ and $M_\psi(a-a_n) \xrightarrow[n]{} 0$, which ends the proof of the proposition.

D e f i n i t i o n 2.2. For any: $a \in \mathcal{L}_m(\mathcal{A})$ and $\psi(\alpha)$, we define

(i) $\|a\|_\psi = \sup\{1/\psi'(\alpha)\cdot a(\alpha) : \alpha > 0\}$,

(ii) $^{\sim}\|a\|_\psi = \sup\{1/\psi'(\alpha)\cdot \tilde{a}(\alpha) : \alpha > 0\}$,

(iii) $_{\sim}\|a\|_\psi = \sup\{1/\psi'(\alpha)\cdot \underset{\sim}{a}(\alpha) : \alpha > 0\}$.

It is clear that

$$\|a + b\|_\psi \leqslant \mathrm{const}(\|a\|_\psi + \|b\|_\psi),$$

$$^{\sim}\|a + b\|_\psi \leqslant {}^{\sim}\|a\|_\psi + {}^{\sim}\|b\|_\psi,$$

$$_{\sim}\|a + b\|_\psi \leqslant {}_{\sim}\|a\|_\psi + {}_{\sim}\|b\|_\psi,$$

so that $^{\sim}\|\cdot\|_\psi$ ($_{\sim}\|\cdot\|_\psi$) is a norm and $\|\cdot\|_\psi$ is a quasi-norm.

P r o p o s i t i o n 2.3. (i). Suppose that (*) is satisfied. Then the same topology is defined on $\mathcal{M}_\psi(\mathcal{A})$ by all four functionals $M_\psi(\cdot)$, $\|\cdot\|_\psi$, $^{\sim}\|\cdot\|_\psi$, $_{\sim}\|\cdot\|_\psi$.

(ii). $\mathcal{M}_\psi(\mathcal{A}) = \mathcal{A}$ and $\Gamma_1\|a\| \leqslant M_\psi(a) \leqslant \Gamma_2\|a\|$

if (**) is satisfied.

P r o o f . (i). For an arbitrary $a \in \mathcal{M}_\psi(\mathcal{A})$ we have

$$M_\psi(a) = \sup\left\{1/\psi(\alpha)\cdot\int_0^\alpha a(\beta)\,d\beta : \alpha > 0\right\} \leqslant$$

$$\leqslant \|a\|_\psi\left\{\sup 1/\psi(\alpha)\cdot\int_0^\alpha \psi'(\beta)\,d\beta : \alpha > 0\right\} =$$

$$= \|a\|_\psi \leqslant {}^{\sim}\|a\|_\psi \leqslant {}_{\sim}\|a\|_\psi =$$

$$= \sup\left\{1/\psi'(\alpha)\cdot 1/\alpha\int_0^\alpha a(\beta)\,d\beta : \alpha > 0\right\} =$$

$$= \sup\left\{\psi(\alpha)/\alpha\,\psi'(\alpha)\cdot 1/\psi(\alpha)\int_0^\alpha a(\beta)\,d\beta : \alpha > 0\right\} \leqslant \Gamma M_\psi(a).$$

(ii). Obvious.

The main result concerning the dual space of $\mathcal{L}^1_{m\psi}(\mathcal{A})$ is the following

Theorem 2.1 (cf. [1], th.6.5). The mapping

$$\mathcal{M}_\psi(\mathcal{A}) \ni a \longrightarrow h_a \in \mathcal{L}^1_{m_\psi}(\mathcal{A})^*,$$

where $h_a(b) = m(ba)$, $b \in \mathcal{L}^1_{m_\psi}$ is a linear isometry of $\mathcal{M}_\psi(\mathcal{A})$ onto $\mathcal{L}^1_{m_\psi}(\mathcal{A})^*$.

Proof. It is clear that for any $p \in \mathrm{Proj}(\mathcal{A})$, $m(p) < \infty$ and $a \in \mathcal{M}_\psi$

$$1/\psi(m(p)) \cdot m(|pa|) \leqslant 1/\psi(m(p)) \cdot \int_0^{m(p)} a(\alpha)\, d\alpha \leqslant M_\psi(a).$$

In other words (see th.1.1) $h_a \in \mathcal{L}^{1*}_{m_\psi}$ and $\|h_a\| \leqslant M_\psi(a)$.

Conversely, suppose that $h_a \in \mathcal{L}^1_{m_\psi}(\mathcal{A})^*$, where $a \in \mathcal{L}^1_m + \mathcal{A}$ (see th.1.2). Let

$$g(\lambda) = \begin{cases} \lambda/\psi(\alpha), & 0 \leqslant \lambda \leqslant \alpha \\ \alpha/\psi(\alpha), & \lambda \geqslant \alpha . \end{cases}$$

By Lemma 2.3, [12] we have

$$1/\psi(\alpha) \cdot \int_0^\alpha a(\beta)\, d\beta =$$

$$= \sup\left\{|m(ba)| \ : \ b \in \mathcal{L}^1_m + \mathcal{A},\ \int_0^\lambda b(\beta)\, d\beta \leqslant g(\lambda)\right\} \leqslant \|h_a\|,$$

since

$$1 = 1/\psi(\alpha) \cdot \int_0^\alpha \psi'(\lambda)d\lambda = \int_0^\infty \psi'(\lambda)d_\lambda g(\lambda) =$$

$$= \alpha/\psi(\alpha)\cdot \psi'(\infty) - \int_0^\infty g(\lambda)d_\lambda \psi'(\lambda) \geqslant$$

$$\geqslant \alpha/\psi(\alpha)\cdot\psi'(\infty) - \int_0^\infty \left[\int_0^\lambda b(\beta)d\beta\right] d_\lambda \psi'(\lambda) =$$

$$= \psi'(\infty)\left[\alpha/\psi(\alpha) - \int_0^\infty b(\beta)d\beta\right] + \int_0^\infty \psi'(\lambda)b(\lambda)d\lambda =$$

$$= \psi'(\infty)\left[\alpha/\psi(\alpha) - \int_0^\infty b(\beta)d\beta\right] + \|b\|_{1,m_\psi}.$$

Consequently $a \in \mathcal{M}_\psi(\mathcal{A})$ and $M_\psi(a) \leqslant \|h_a\|$, which ends the proof of the theorem.

3. $\mathcal{M}_\psi(\mathcal{A})^* \neq \mathcal{L}^1_{m_\psi}(\mathcal{A})$

It is evident that $\mathcal{M}_\psi(\mathcal{A})^* \supset \mathcal{L}^1_{m_\psi}(\mathcal{A})$. We shall prove that in general $\mathcal{M}_\psi(\mathcal{A})^* \neq \mathcal{L}^1_{m_\psi}(\mathcal{A})$. In this section $\mathcal{F}_m(\mathcal{A}) = \mathcal{F}_m = \left\{a \in \mathcal{A} : m(\operatorname{supp}|a|) < \infty\right\}$ denotes the ideal of m-elementary operators.

L e m m a 3.1. Suppose that N is a non-trivial continuous semi-norm on $\mathcal{M}_\psi(\mathcal{A})$ such that $N(a) = 0$ for any $a \in \mathcal{F}_m(\mathcal{A})$. Then $|h_b(a)| = |m(ba)| \leqslant N(a)$ for $a \in \mathcal{M}_\psi(\mathcal{A})$ and some $b \in \mathcal{L}^1_{m_\psi}(\mathcal{A})$ implies $b = 0$.

P r o o f . Let $b = u|b|$ be the polar decomposition of b and $|b^*| = u|b|u^* = \int_0^\infty \lambda de_\lambda$ the spectral resolution of $|b^*|$. For any arbitrary $\varepsilon > 0$, we have $m(e_\varepsilon^\perp) < \infty$, since $\mathcal{L}^1_{m_\psi} \subset \mathcal{I}_m$. Therefore $a = u^* e_\varepsilon^\perp \in \mathcal{F}_m(\mathcal{A}) \subset \mathcal{M}_\psi(\mathcal{A})$ and

$$|h_b(a)| = |m(ba)| = |m(bu^*e_\varepsilon^\perp)| = |m(|b^*|e_\varepsilon^\perp)| \leqslant N(u^*e_\varepsilon^\perp) = 0.$$

In other words $|b^*|e_\varepsilon^\perp = 0$ for any $\varepsilon > 0$ and, in consequence, $|b^*| = 0$ which is equivalent to $b = 0$.

C o r o l l a r y 3.1. N defines a continuous linear functional h on $\mathcal{M}_\psi(\mathcal{A})$ which is not of the form $h_b(a) = m(ba)$ with $b \in \mathcal{L}^1_{m_\psi}(\mathcal{A})$.

P r o o f . By Hahn-Banach theorem and Lemma 3.1.

D e f i n i t i o n 3.1. We define a continuous semi-norm N_0 (N_∞) on $\mathcal{M}_\psi(\mathcal{A})$ by

(i) $N_0(a) = \limsup_{\alpha \to 0} 1/\psi(\alpha) \cdot \int_0^\alpha a(\beta)\, d\beta \leq M_\psi(a)$,

(ii) $N_\infty(a) = \limsup_{\alpha \to \infty} 1/\psi(\alpha) \cdot \int_0^\alpha a(\beta)\, d\beta \leq M_\psi(a)$.

It is easy to check that $N_\infty(a) = 0$, $a \in \mathcal{F}_m$. Let $\lim_{\alpha \to 0} \alpha/\psi(\alpha) = 0$. Then $N_0(a) = 0$, $a \in \mathcal{F}_m$.

P r o p o s i t i o n 3.1. (i). Assume that there exists a projection $0 \neq p \in \mathrm{Proj}(\mathcal{A})$, such that p contains no atoms. If $m(p) < \infty$ $(m(p) = \infty)$ then N_0 (N_∞) is non-trivial.

(ii). Let $\lim_{\alpha \to \infty} \alpha/\psi(\alpha) = \Gamma < \infty$, $m(1) = \infty$. Then N_∞ is non-trivial.

P r o o f . (i). Let $m(p) < \infty$ and let $p = \sum_{n=1}^{\infty} p_n$, $p_i \perp p_j$, $m(p_n) = m(p)/2^n$. Let $a = \sum_{n=1}^{\infty} \psi' \left(m\left(\sum_{k=n}^{\infty} p_k\right)\right) p_n$. It is easy to see that $a(\alpha) \leq \psi'(\alpha)$ and $a(\alpha) \geq \psi'(2\alpha)$, $0 < \alpha < m(p)$. Therefore

$$M_\psi(a) \leq 1 \text{ and } N_0(a) \geq \limsup_{\alpha \to 0} 1/\psi(\alpha) \cdot \int_0^\alpha \psi'(2\beta)\, d\beta =$$

$$= 1/2 \limsup_{\alpha \to 0} \psi(2\alpha)/\psi(\alpha) \geq 1/2.$$

Similarly if $m(p) = \infty$, then N_∞ is non-trivial. Indeed, let $\varphi(\alpha)$ be a decreasing, continuous on the right step function inscribed between the two curves $\psi'(\alpha)$ and $2\psi'(\alpha)$. Suppose that $\varphi(\alpha)$ has the jumps in the points $\alpha = \alpha_n$, where $\alpha_n \downarrow 0$ as $n \to -\infty$, $\alpha_n \uparrow \infty$ as $n \to \infty$, if $\lim_{\alpha \to 0} \psi'(\alpha) = \infty$. If $\lim_{\alpha \to 0} \psi'(\alpha) = \gamma < \infty$ let the jumps of $\varphi(\alpha)$ occur at the points $\alpha = \alpha_n$, $n = 1,2,\ldots$, $\alpha_n \uparrow \infty$, $\varphi(0) = \gamma$.

We get in the first case

$$a = \sum_{n=-\infty}^{\infty} \psi'(\alpha_n) p_n,$$

where $p = \sum_{n=-\infty}^{\infty} p_n$, $p_i \perp p_j$, $i \neq j$, $m(p_n) = \alpha_n - \alpha_{n-1}$

and in the second case

$$a = \gamma p_1 + \sum_{n=2}^{\infty} \psi'(\alpha_n) p_n,$$

where $p = \sum_{n=2}^{\infty} p_n$, $p_i \perp p_j$, $i \neq j$, $m(p_1) = \alpha_1$, $m(p_n) = \alpha_n - \alpha_{n-1}$, $n = 2,3,4,\ldots$.

It is clear that $a(\alpha) = \varphi(\alpha)$ and, in consequence, $M_\psi(a) \leq 2$, $N_\infty(a) \geq 1$.

(ii) In this case $1 \in \mathcal{M}_\psi(\mathcal{A})$ and $M_\psi(1) = N_\infty(1) = \Gamma$.•

C o r o l l a r y 3.2. $\mathcal{M}_\psi(\mathcal{A})^* \neq \mathcal{L}^1_{m_\psi}(\mathcal{A})$ if one of the conditions (i) - (iii) is satisfied:

(i) There exists $p \in \mathrm{Proj}(\mathcal{A})$, such that $m(p) = \infty$ and p contains no atoms.

(ii) There exists $p \in \mathrm{Proj}(\mathcal{A})$ such that $m(p) < \infty$, p contains no atoms and $\lim_{\alpha \to 0} \alpha/\psi(\alpha) = 0$.

(iii) $m(1) = \infty$ and $\lim_{\alpha \to \infty} \alpha/\psi(\alpha) = \Gamma < \infty$.

Consider the case when an arbitrary projection $p \in \mathrm{Proj}(\mathcal{A})$ has atoms.

P r o p o s i t i o n 3.2. Assume that there exists a projection $p \in \mathrm{Proj}(\mathcal{A})$, such that $p = \sum_{n=1}^{\infty} p_n$, $0 \neq p_n \in \mathrm{Proj}(\mathcal{A})$, $p_i \perp p_j$, $i \neq j$. Then $\mathcal{M}_\psi(\mathcal{A})^* \neq \mathcal{L}^1_{m_\psi}(\mathcal{A})$ if one of the three conditions:

(i) $m(p) < \infty$, $2m(p_n) \leq \sum_{k=n}^{\infty} m(p_k)$, $\lim_{\alpha \to 0} \alpha/\psi(\alpha) = 0$,

(ii) $m(p) < \infty$, $\lim\limits_{\alpha \to 0} \alpha/\psi(\alpha) = 0$ and (*) is satisfied,

(iii) $m(p) = \infty$ and (*) is satisfied holds.

P r o o f . Without loss of generality we may assume that p_n is an atom, $n = 1,2,\dots$.

(i) In that case N_0 is non-trivial. The proof of the above fact is the same as that of Proposition 3.1(i).

(ii) With no loss of generality we may assume that $m(p_n) > 1/2 \sum\limits_{k=n}^{\infty} m(p_k)$, $n = 1,2,\dots$. For an arbitrary $a \in \mathcal{M}_\psi(\mathcal{A})$: $p_n a p_n = \lambda_n p_n$. We define a continuous semi-norm N on $\mathcal{M}_\psi(\mathcal{A})$ by

$$N(a) = \limsup_{n\to\infty} |\lambda_n| / \psi'(m(p_n)) = \limsup_{n\to\infty} |\lambda_n| \, \|p_n\|_\psi =$$

$$= \limsup_{n\to\infty} \|p_n a p_n\|_\psi \leqslant \|a\|_\psi \leqslant \Gamma M_\psi(a).$$

It is not difficult to observe that $N(a) = 0$, $a \in \mathcal{F}_m$. If $a = \sum\limits_{n=1}^{\infty} \psi'\Big(\sum\limits_{k=n}^{\infty} m(p_k)\Big) p_n$ then $a \in \mathcal{M}_\psi(\mathcal{A})$ and

$$N(a) = \limsup_{n\to\infty} \psi'\Big(\sum_{k=n}^{\infty} m(p_k)\Big) / \psi'(m(p_n)) \geqslant$$

$$\geqslant \limsup_{n\to\infty} \psi'(2m(p_n)) / \psi'(m(p_n)) \geqslant 1/2$$

that is N is non-trivial.

Proposition (ii) results immediately from Lemma 3.1 and from the above.

(iii) Putting $a = \sum\limits_{n=1}^{\infty} \psi'\Big(\sum\limits_{k=1}^{n} m(p_k)\Big) p_n$ we have $a \in \mathcal{M}_\psi(\mathcal{A})$ and

$$N_\infty(a) = \limsup_{\alpha\to\infty} 1/\psi(\alpha) \cdot \int_0^\alpha a(\beta)\, d\beta \geqslant$$

$$\limsup_{n\to\infty} 1/\psi\Big(m\Big(\sum_{i=1}^{n} p_i\Big)\Big) \int_0^{m(\sum_{i=1}^{n}) p_i} a(\beta)\, d\beta =$$

$$= \limsup_{n\to\infty} 1/\psi\Big(m\Big(\sum_{i=1}^{n} p_i\Big)\Big) \cdot \sum_{i=1}^{n} \psi'\Big(\sum_{k=1}^{i} m(p_k)\Big) \cdot m(p_i) \geq$$

$$\geq \limsup_{n\to\infty} m\Big(\sum_{i=1}^{n} p_i\Big)\,\psi\Big(m\Big(\sum_{i=1}^{n} p_i\Big)\Big) \cdot \psi'\Big(m\Big(\sum_{k=1}^{n} p_k\Big)\Big) \geq 1/\Gamma > 0.$$

In other words N_∞ is non-trivial.

C o r o l l a r y 3.3. $\mathcal{L}_m^{\delta\infty}(\mathcal{A})^* \neq \mathcal{L}_m^{\delta' 1}(\mathcal{A})$ for an arbitrary infinite dimensional algebra $\mathcal{A}$.

P r o p o s i t i o n 3.3. $\mathcal{M}_\psi(\mathcal{A})^* \neq \mathcal{L}^1_{m_\psi}(\mathcal{A})$ if one of the two conditions:

(i) $m(p) = \infty$, $\psi'\Big(m\Big(\sum_{i=1}^{k} p_i\Big)\Big) \leq \mathrm{const}\cdot \psi'\Big(m\Big(\sum_{i=1}^{k+1} p_i\Big)\Big)$,

(ii) $m(p) < \infty$,

$$\psi'\Big(m\Big(\sum_{i=n}^{\infty} p_i\Big)\Big) \leq \mathrm{const}\cdot \psi'\Big(m\Big(\sum_{i=n-1}^{\infty} p_i\Big)\Big), \quad \lim_{\alpha\to 0} \alpha/\psi(\alpha) = 0$$

holds, where $p = \sum_{n=1}^{\infty} p_n$, $0 < m(p_n) < \infty$, $p_n \in \mathrm{Proj}(\mathcal{A})$, $p_i \perp p_j$, $i \neq j$.

P r o o f . As in the proof of proposition 3.1 (ii) (for the semi-norm N_0 in the case (ii)).

As an example suppose that $\mathcal{A} = \mathfrak{B}(\mathcal{H})$ is the von Neumann algebra of all bounded operators acting in an infinite dimensional Hilbert space $\mathcal{H}$, while $m(p) = \dim p(\mathcal{H})$ is the ordinary trace on $\mathcal{A}$ and $\psi(\alpha) = \ln(1+\alpha)$.

R e m a r k 3.1. Let $m(1) < \infty$ and $\lim_{\alpha\to 0} \alpha/\psi(\alpha) = \gamma > 0$. In this case $\mathcal{M}_\psi(\mathcal{A}) = \mathcal{A}$, $\gamma\,\|a\| \leq M_\psi(a) \leq m(1)/\psi(m(1))\cdot\|a\|$ and $\mathcal{L}^1_{m_\psi}(\mathcal{A}) = \mathcal{L}^1_m(\mathcal{A})$, $\psi(m(1))\,\|a\|_{1,m} \leq \|a\|_{1,m_\psi} \leq \gamma\,\|a\|_{1,m}$. Consequently, $\mathcal{M}_\psi(\mathcal{A})^* = \mathcal{L}^1_{m_\psi}(\mathcal{A})$ implies the reflexivity of $\mathcal{A}$, but one can easily see that a reflexive von Neumann algebra is finite dimensional.

4. The predual of $\mathcal{L}^1_{m_\psi}(\mathcal{A})$

In this section we denote by $\overline{\mathcal{F}}_m$ the closure of $\mathcal{F}_m$ in $\mathcal{M}_\psi(\mathcal{A})$.

P r o p o s i t i o n 4.1. Let $p \in \mathrm{Proj}(\mathcal{A})$, $m(p) < \infty$ and $|b_n| \leqslant p$, $m(|b_n|) = \|b_n\|_{1,m} \to 0$. Then $M_\psi(b_n) \to 0$ if one of the conditions (i)-(ii) is satisfied:

(i) $\lim\limits_{\alpha \to 0} \alpha/\psi(\alpha) = \inf\{\alpha/\psi(\alpha) : \alpha > 0\} = 0$,

(ii) $\inf\{m(p) : p \in \mathrm{Proj}(\mathcal{A})\} = \gamma > 0$.

P r o o f . (i) Fix $\varepsilon > 0$. Let $\alpha/\psi(\alpha) \leqslant \varepsilon/2$ for $\alpha \leqslant \delta$ and $m(p)/\psi(\alpha) \leqslant \varepsilon/2$ for $\alpha \geqslant 1/\delta$. Then

$$1/\psi(\alpha) \int_0^\alpha b_n(\beta)\, d\beta \leqslant \begin{cases} \varepsilon/2 & \text{for } \alpha \leqslant \delta , \\ \varepsilon/2 & \text{for } \alpha \geqslant 1/\delta \end{cases}$$

and for $\delta < \alpha < 1/\delta$ we have

$$1/\psi(\alpha) \int_0^\alpha b_n(\beta)\, d\beta \leqslant \delta/\psi(\delta) + \|b_n\|_{1,m}/\psi(\delta) \leqslant \varepsilon$$

for sufficiently large n. In other words, $\limsup\limits_n M_\psi(b_n) \leqslant \varepsilon$. From the arbitrariness of $\varepsilon > 0$ the proposition (i) follows.

(ii) In this case $\mathcal{L}_m(\mathcal{A}) = \mathcal{A}$ (topologically and algebraically) and $M_\psi(b_n) \leqslant m(p)/\psi(m(p)) \cdot \|b_n\|$. To finish the proof, it suffices to use the following simple fact: $\|b_n\|_{1,m} \to 0$ implies $b_n \xrightarrow{m} 0$.

C o r o l l a r y 4.1. Assume that (i) or (ii) is satisfied and h is a continuous, linear functional on $\overline{\mathcal{F}}_m$, where the closure is taken in $\mathcal{M}_\psi(\mathcal{A})$. Then $h(a) = h_b(a) = m(ba)$, $a \in \overline{\mathcal{F}}_m$ while b is an operator locally measurable in the sense of [10].

Proof follows from Theorem 4.3 [11], Proposition 2.1 and the proof of Theorem 4.4 [11].

T h e o r e m . 4.1. The mapping $\mathcal{L}^1_{m_\psi} \ni b \to h_b \in \overline{\mathcal{F}}^*_m$, where $h_b(a) = m(ba)$, $a \in \overline{\mathcal{F}}^*_m$ is a linear isometry of $\mathcal{L}^1_{m_\psi}$ onto $\overline{\mathcal{F}}^*_m$ if one of the conditions (i)-(ii) is satisfied:

(i) $\lim \alpha/\psi(\alpha) = 0$ as $\alpha \to 0$,

(ii) $\inf\{m(p) : p \in \mathrm{Proj}(\mathcal{A})\} = \gamma > 0$.

P r o o f . Suppose that $b \in \mathcal{L}^1_{m\psi}$. By Theorem 2.1 $h_b = m(b\cdot) \in \mathcal{M}_\psi(\mathcal{A})^*$ and $\|h_b\| \leqslant \|b\|_{1,m}$. Hence $h_b|\bar{\mathcal{F}}_m \in \bar{\mathcal{F}}^*_m$ and $\|h_b|\bar{\mathcal{F}}_m\| \leqslant \|b\|_{1,m\psi}$. It is easy to see that $h_b|\bar{\mathcal{F}}_m \equiv 0$ implies $b = 0$.

Conversely, let $h \in \bar{\mathcal{F}}^*_m$. By Corollary 4.1 $h = h_b$, where b is locally measurable in the sense of [10]. We shall prove that $b \in \mathcal{L}^1_{m\psi}(\mathcal{A})$. Let $|b^*| = \int_0^\infty \lambda de_\lambda$ be the spectral resolution of $|b^*|$ and $b = u|b|$ the polar decomposition of b. For any $p \in \mathrm{Proj}(\mathcal{A})$, $m(p) < \infty$, $a = \psi(m(p))/m(p)\cdot u^* p \in \mathcal{F}_m$ and $h_b(a) = \psi(m(p))/m(p)\cdot m(|b^*|\ p) \leqslant \|h_b\|$. Assume that $m\left(e^\perp_{\lambda_0}\right) = \infty$, $\lambda_0 > 0$ and $0 \neq p_n \leqslant e^\perp_{\lambda_0}$, $m(p_n) \to \infty$. Then

$$\infty > \|h_b\| \geqslant \psi(m(p_n))/m(p_n)\cdot m(|b^*|\ p_n) \geqslant$$

$$\geqslant \lambda_0 \psi(m(p_n)) \to \infty \quad \text{as} \quad n \to \infty ,$$

a contradiction. In consequence, $|b^*| \in \mathcal{J}_m(\mathcal{A})$ which is equivalent to $b \in \mathcal{J}_m(\mathcal{A})$. Let

$$\varphi_n(\alpha) = \begin{cases} \psi'(1/n), & 0 < \alpha < 1/n, \\ \psi'(\alpha)\chi_{[1/n,n]}, & 1/n \leqslant \alpha \leqslant n, \\ 0, & \alpha > n. \end{cases}$$

By Lemma 2.3 [12] we have

$$\int_0^\infty \varphi_n(\alpha)\ b(\alpha)\ d\alpha =$$

$$= \sup\left\{|m(ba)| : a \in \mathcal{L}^1_m + \mathcal{A},\ \int_0^\alpha a(\beta)\ d\beta \leqslant \int_0^\alpha \varphi_n(\beta)\ d\beta\right\}.$$

Moreover, for $\alpha \leqslant 1/n$, $\alpha a(\alpha) \leqslant \int_0^\alpha a(\beta)d\beta \leqslant \psi'(1/n)\alpha$. Hence $a(0+) = \|a\| \leqslant \psi'(1/n)$ and $ae_\varepsilon^\perp \in \mathcal{F}_m$ for any $\varepsilon > 0$, where $|a| = \int_0^\infty \lambda de_\lambda$. Consequently,

$$\int_0^\infty \varphi_n(\alpha)b(\alpha)d\alpha = \sup\left\{|m(ba)| \ : \ a \in \mathcal{F}_m, \ \int_0^\alpha a(\beta)d\beta \leqslant \int_0^\alpha \varphi_n(\beta)d\beta\right\} \leqslant \|h_b\|.$$

Finally

$$\|b\|_{1,m_\psi} = \int_0^\infty \psi'(\alpha)b(\alpha)d\alpha = \sup_n \int_0^\infty \varphi_n(\alpha)b(\alpha)d\alpha \leqslant \|h_b\|,$$

which ends the proof of the theorem.

P r o p o s i t i o n 4.2.

(i) Assume that $\lim \alpha/\psi(\alpha) = 0$ as $\alpha \to 0$. Then

$$\overline{\mathcal{F}}_m = \left\{a \in \mathcal{M}_\psi(\mathcal{A}) \ : \ \lim_{\alpha \to 0,\infty} 1/\psi(\alpha) \int_0^\alpha a(\beta)d\beta = 0\right\} \subset \mathcal{J}_m(\mathcal{A}).$$

(ii) Suppose that $\inf\{m(p) \ : \ p \in \mathrm{Proj}(\mathcal{A})\} = \gamma > 0$. Then $\overline{\mathcal{F}}_m = \mathcal{J}_m(\mathcal{A})$ if (**) is satisfied.

P r o o f . (i) Let $\lim_{\alpha \to 0,\infty} 1/\psi(\alpha) \cdot \int_0^\alpha a(\beta)d\beta = 0$. It is clear that $a \in \mathcal{J}_m$. Fix $\varepsilon > 0$. Suppose that $1/\psi(\alpha) \cdot \int_0^\alpha a(\beta)d\beta \leqslant \varepsilon$ for $\alpha \leqslant \delta$ and $\alpha \geqslant 1/\delta$. For any $n = 1,2,\ldots$ $a_n = a(e_n - e_{1/n}) \in \mathcal{F}_m$ and $a - a_n = a(e_n^\perp + e_{1/n})$, where $|a| = \int_0^\infty \lambda de_\lambda$. Hence

$$(a - a_n)(\alpha) \leqslant (ae_n^\perp)(\alpha/2) + (ae_{1/n})(\alpha/2) \leqslant a(\alpha/2)\chi_{(0,2m(e_n^\perp)]} + 1/n,$$

and consequently,

$$\int_0^{1/\delta} (a - a_n)(\alpha)d\alpha \leqslant 1/(n\delta) + \int_0^{2m(e_n^\perp)} a(\alpha/2)d\alpha =$$

$$= 1/(n\delta) + 2\int_0^{m(e_n^\perp)} a(\alpha)d\alpha \leqslant 1/(n\delta) + 2\varepsilon\psi(m(e_n^\perp))$$

$3\varepsilon\psi(\delta)$ for $n \geqslant N_\varepsilon$.

Since $(a - a_n)(\alpha) = (a(e_n^{\perp} + e_{1/n}))(\alpha) \leq a(\alpha)$, therefore $M_\psi(a - a_n) \leq 3\varepsilon$ for $n \geq N_\varepsilon$. In other words $a \in \overline{\mathcal{F}}_m$.

Conversely, let $a \in \overline{\mathcal{F}}_m$ and $M_\psi(a - a_n) \to 0$ as $n \to \infty$, where $a_n \in \mathcal{F}_m$. Fix $\varepsilon > 0$ and let $2/\psi(\alpha) \cdot \int_0^\alpha (a - a_n)(\beta)d\beta \leq \varepsilon$ for $n \geq N_\varepsilon$, $\alpha > 0$. We have

$$1/\psi(\alpha) \cdot \int_0^\alpha a(\beta)d\beta \leq$$

$$\leq 1/\psi(\alpha) \cdot \int_0^\alpha (a - a_n)(\beta/2)d\beta + 1/\psi(\alpha) \cdot \int_0^\alpha a_n(\beta/2)d\beta$$

$$\leq 2/\psi(\alpha) \cdot \int_0^\alpha (a - a_n)(\beta)d\beta + 2/\psi(\alpha) \cdot \int_0^\alpha a_n(\beta)d\beta, \qquad n \geq N_\varepsilon.$$

Hence $\limsup_{\alpha \to 0,\infty} 1/\psi(\alpha) \cdot \int_0^\alpha a(\beta)d\beta \leq \varepsilon$. From the arbitrariness of $\varepsilon > 0$ the proposition (i) follows.

(ii) It is easy to see that the closure of $\mathcal{F}_m$ in $\mathcal{L}_m(\mathcal{A})$ is $\mathcal{J}_m(\mathcal{A})$. Since, in this case, $\mathcal{L}_m(\mathcal{A}) = \mathcal{A} = \mathcal{M}_\psi(\mathcal{A})$ and $a_n \xrightarrow{m} a \Leftrightarrow \|a - a_n\| \to 0$, therefore $\overline{\mathcal{F}}_m = \mathcal{J}_m(\mathcal{A})$.

C o r o l l a r y 4.2. Assume that $\mathcal{A} = \mathcal{B}(\mathcal{H})$ is the von Neumann algebra of all bounded operators acting in a Hilbert space $\mathcal{H}$, $C(\mathcal{H})$ the space of all compact operators on $\mathcal{H}$, while $m(p) = \dim p(\mathcal{H})$ is the ordinary (von Neumann) trace on $\mathcal{A}$. Then

$$\mathcal{L}_n^1(\mathcal{A}) \ni b \to h_b = m(b\cdot) \in C(\mathcal{H})^*$$

is a linear isometry of $\mathcal{L}_m^1(\mathcal{A})$ onto $C(\mathcal{H})^*$.

R e m a r k . By Theorem 1.1 and the proof of Theorem 4.1 we easily obtain

$$h_b(\cdot) = m(b\cdot) \in \mathcal{M}_\psi^* \quad \text{iff} \quad b \in \mathcal{L}_{m_\psi}^1.$$

In the particular case

$$\sup\left\{|m(ba)| \;:\; \|a\|_{\delta\infty} \leqslant 1\right\} < \infty \quad \text{iff} \quad b \in \mathcal{L}_m^{\delta'1},\ 1/\delta + 1/\delta' = 1.$$

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