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COVERING PLANE SETS WITH SETS OF THREE TIMES LESS DIAMETER

The famous Borsuk conjecture [1] that any set of diameter 1 of Euclidean space E^n can be covered by n+1 ones of smaller diameter has stimulated many similar questions. A survey of them is presented in the paper [3] of Grünbaum. Also this note remains in the sphere of problems which originate from Borsuk conjecture.

Denote by $t_n(k)$ the smallest number t such that any bounded subset of E^n can be covered by t sets of k-times less diameter. Lenz [4] showed that $t_2(2) = 7$. From the paper [2] of Borsuk it results that $t_2(3) \leqslant 20$. The following theorem shows that in fact we have $t_2(3) = 14$.

Theorem. Any bounded subset of E^2 can be covered with 14 sets of three-times less diameter. The above number is best possible: the disk of diameter 1 cannot be covered by 13 sets of diameter 1/3.

Proof. I. Obviously, to prove the first statement of our theorem it is sufficient to consider the covering of every set of diameter 1 by sets of diameter smaller or equal 1/3. What is more, it is sufficient to do this with a universal cover of sets of diameter 1, i.e. with a set which can cover any set of diameter 1. We need a relatively small universal cover, namely the octagon of Pál [5], the form of which is reminded below. In a regular hexagon S with the distance 1 between parallel sides we take two segments joining the center c of S with two vertices, the distance of which is equal to 1. Two straight lines crossing perpendicularly the segments in the distance 1/2 from c bound two half-planes containing c. The intersection of S with the half-planes is just the octagon A of Pál.

In Fig.1 there is presented the octagon A and a partition of it into 7 pairs of congruent polygons P_i, P'_i, $i = 1,\dots,7$. We describe the vertices of the polygons with the help of a perpendicular coordinate system of E^2. Namely: $a_1(0, \sqrt{3}/3)$, $a_2(1/2, \sqrt{3}/6)$, $a_3(1/2, \sqrt{3}/2-1)$, $a_4((\sqrt{3}-1)/2, (1-\sqrt{3})/2)$, $a_5(0, -\sqrt{3}/3)$, $b_1(0.16, \sqrt{3}\cdot 0.28)$, $b_2(0.4, \sqrt{3}\cdot 0.2)$, $b_3(0.5, 0.03)$, $b_4(0.415, \sqrt{3}\cdot 0.415-1)$, $b_5(0.16, -\sqrt{3}\cdot 0.28)$, $c_1(0.1, 0.3)$, $c_2(0.16, 0.16)$, $c_3(0.23, 0.16)$, $c_4(0.32, 0.03)$, $c_5(0.23, -0.16)$, $c_6(0.16, -0.16)$, $c_7(0.1, -0.3)$, $c(0, 0)$. The point symmetric to a given one respect to the line through a_1 and a_5 is denoted by the same letter with prime.

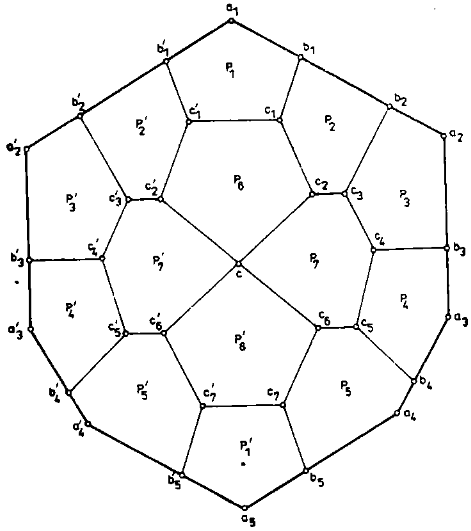

Fig.1

We omit an easy but tedious calculation which shows that the diameters of $P_1,\dots,P_7$ equal to, respectively, $d_1 = d(b_1,b'_1) = 0.32$, $d_2 = d(b_1,c_3) \approx 0.3324$, $d_3 = d(b_2,b_3) \approx 0.3318$, $d_4 = d(b_4,c_4) \approx 0.3254$, $d_5 = d(b_5,c_5) \approx 0.3324$, $d_6 = d(c_2,c'_2) = 0.32$, $d_7 = d(c_2,c_5) \approx 0.3275$. Thus the diameters of P_i and of the congruent set P'_i are even smaller than 1/3, $i=1,\dots,7$.

II. For the proof of the second part of Theorem suppose that the disk D of diameter 1 can be covered by 13 sets of diameters 1/3. Our purpose is to get a contradiction.

Since any set and its closure are of the same diameter, our supposition implies that there exists a family $\mathcal{S}$ of 13 closed sets of diameter 1/3, the union of which covers the disk D. Let $\mathcal{P}$ denote the family of sets of $\mathcal{S}$, any of which has at least two points on the circle C bounding the disk D. It is clear that the intersection of two disks with the centers in the two points and of diameter 1/3 contains this set from which the points are taken. Thus the center a of D is in a distance greater than 1/6 from any set of $\mathcal{P}$. Consequently, the sets of $\mathcal{P}$ are disjoint with a circle E with the center a and with a radius greater than 1/6. Since we need at least 3 sets of $\mathcal{S}$ to cover E, the family $\mathcal{P}$ contains at most 10 sets.

If $S \in \mathcal{P}$, then $\operatorname{diam}(S \cap C) \leq \operatorname{diam} S = 1/3$ and then the set $S \cap C$ is contained in an arc of diameter 1/3 of the circle C. This arc is based on the angle $2 \cdot \arcsin(1/3) \approx 38^\circ 56'$. Since $18 \cdot \arcsin(1/3) < 360^\circ$, we have that $\mathcal{P}$ contains at least 10 sets.

We have shown that $\mathcal{P}$ consists of exactly 10 sets.

Let an orientation of C be fixed. Any set of $\mathcal{P}$ is closed and has non-empty intersection with C and thus there exists in this set the first (in the meaning of our orientation) point of C. Since the union of any 9 sets of $\mathcal{P}$ does not cover C, the first points are different. Denote them (in order according to our orientation) by $x_1,\dots,x_{10}$ and denote the corresponding sets by $S_1,\dots,S_{10}$. Moreover, it will be convenient to denote x_{10} additionally by x_0 and x_1 by x_{11}. Observe that $x_{i+1} \in S_i$ and that $d(x_i, x_{i+1}) \leq 1/3$ for $i = 1,\dots,10$.

Let y_i be the common point of D and of the circles with the centers x_{i-1}, x_{i+1} and radius 1/3 (see Fig.2). Let $Y = \{y_1,\dots,y_{10}\}$. Denote by S_{11}, S_{12}, S_{13} the sets of $\mathcal{J} \setminus \mathcal{P}$ such that the number of points of $S_{13} \cap Y$ is not smaller than the number of points of $S_{11} \cap Y$ and than the number of points of $S_{12} \cap Y$.

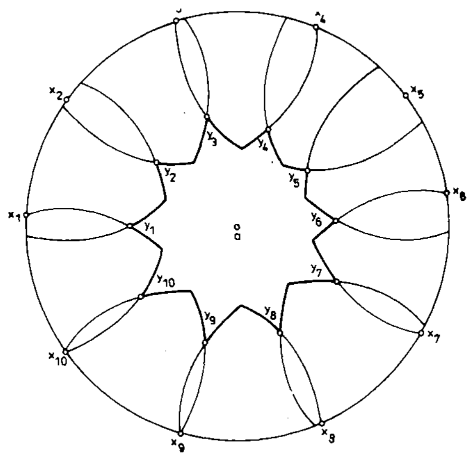

Fig.2

Let $0 < \varepsilon < 1/6$ and let y_i^{ε} be the point of D lying in the intersection of the circles with the centers x_{i-1}, x_{i+1} and radius $1/3 + \varepsilon$, $i = 1,\dots,10$. Since the intersection of C with the closed disk of radius $1/3 + \varepsilon$ and center y_i^{ε} is the arc of C between x_{i-1} and x_{i+1}, we have $d(x_j, y_i^{\varepsilon}) > 1/3$ for $j \neq i$ and therefore $y_i^{\varepsilon} \notin S_j$ (for $j \neq i$ this results from $x_j \in S_j$ and for $j = i$ this results from $x_{i+1} \in S_i$), $i, j = 1,\dots,10$. Thus $y_i^{\varepsilon} \in S_{11} \cup S_{12} \cup S_{13}$, $i = 1,\dots,10$. If ε converges to 0, then y_i^{ε} converges to y_i, $i = 1,\dots,10$. Since S_{11}, S_{12}, S_{13} are closed, we obtain that

$$Y \subset S_{11} \cup S_{12} \cup S_{13}$$

and, consequently, that

(1) S_{13} contains at least 4 points of Y.

Now, we shall prove that

(2) if $j = i+3 \pmod{10}$, then $d(y_i, y_j) > 1/3$.

We shall not make our consideration narrower fixing our mind on one such a pair of points, say on the pair y_2, y_5.

Notice that the position of y_2 is determined by the positions of x_1, x_3 and that the position of y_5 is determined by the positions of x_4, x_6. Let $\alpha = \sphericalangle y_2 a x_3$, $\beta = \sphericalangle x_4 a y_5$, $\gamma = \sphericalangle x_3 a x_4$, (see Fig.3). Obviously, $\sphericalangle x_1 a y_2 = \alpha$ and $\sphericalangle y_5 a x_6 = \beta$. Let $\delta = 2\alpha + \gamma + 2\beta$. Hence

$$\alpha + \gamma + \beta = (\gamma+\delta)/2.$$

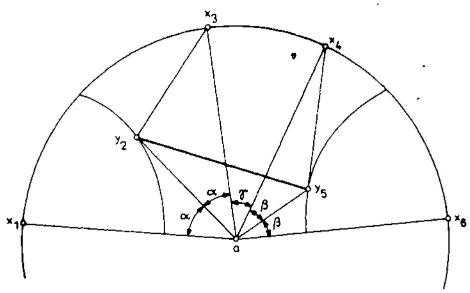

Fig.3

Observe that $d(y_2, y_5)$ is a function of angles α, β, γ. In three steps we shall show that $d(y_2, y_5)$ attains the minimum $\approx 0.3418 > 1/3$ for $\alpha_0 = \beta_0 = 90^0 - 2 \arcsin(1/3)$ and $\gamma_0 = 2 \arcsin(1/3)$.

1. Let γ and δ be fixed. We intend to show that the distance $d(y_2, y_5)$ is the smallest when $\alpha = \beta$. Since δ is fixed, we can assume that the positions of x_1, x_6 are fixed. Let for instance $\sphericalangle x_4 a x_6 \leq \sphericalangle x_1 a x_3$ (see Fig.4). When $\alpha = \beta$, the positions of x_3, x_4, y_2, y_5 are denoted by x_3', x_4', y_2', y_5', respectively. Denote by K the perpendicular bisector of the

segment x_1x_6. The straight lines containing the segments x_1x_6 and $y_2'y_5'$ are denoted by L, M, respectively. Let y_5^* be the point

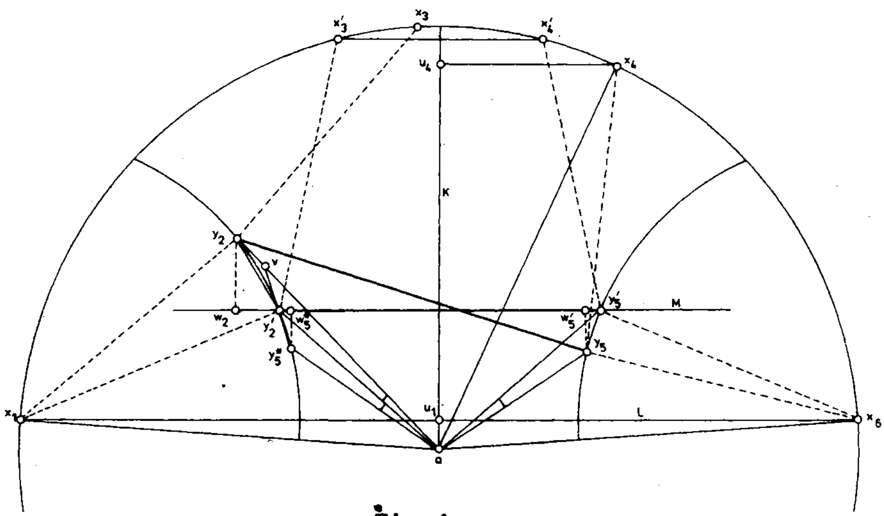

Fig.4

symmetric to y_5 with respect to K. The perpendicular projections of y_2, y_5, y_5^* on M are denoted by w_2, w_5, w_5^*, respectively. Since γ and δ are fixed, $\alpha+\gamma+\beta=(\gamma+\delta)/2$ is fixed. Hence $\sphericalangle y_2ay_5 = \sphericalangle y_2'ay_5'$. So $\sphericalangle y_2'ay_2 = \sphericalangle y_5'ay_5 = \sphericalangle y_5^*ay_2'$. Notice that the points y_5^*, y_2', y_2 lie on one side of the straight line through a and x_1 and simultaneously they lie on the circle with center x_1 and radius 1/3 in order as in Fig.4. Therefore there exists the common point v of the segment ay_2 with the straight line through y_5^*, y_2' and we obtain that $d(y_5^*,y_2') \leqslant d(y_2',v) \leqslant d(y_2',y_2)$. The orthogonal projections of x_1, x_4 on K are denoted by u_1, u_4, respectively. If $d(u_4,a) \leqslant d(u_4,u_1)$, then y_5, and so y_5^*, y_2', y_2, are on the same side of L as x_4. We shall show this also in the case when $d(u_4,a) > d(u_4,u_1)$. Since $\delta \geqslant 360^\circ - 10 \arcsin(1/3)$, we get

$\sphericalangle x_1au_1 = \delta/2 \geqslant 180^\circ - 5 \arcsin(1/3) \approx 82^\circ 40'$ and $\sphericalangle u_4ax_4 \leqslant 6 \arcsin(1/3) - \delta/2 \leqslant 11 \arcsin(1/3) - 180^\circ \approx 34^\circ 08'$. Considering the triangles x_1au_1 and x_4au_4, by an easy calculation we obtain that $d(a,u_1) < 0.0639$ and that $d(u_4,a) > 0.4139$. Hence $d(u_1,u_4) > 0.35 > 1/3 = d(x_4,y_5)$. Therefore also in this case, y_5, and so y_5^*, y_2', y_2, are on the side of L which contains x_4. This and the inequality $d(y_5^*,y_2') \leqslant d(y_2',y_2)$ and also the situation of y_5^*, y_2', y_2 on the circle with the center x_1 and radius 1/3 imply that $d(w_5^*,y_2') \leqslant d(y_2',w_2)$. Hence $d(w_5,y_5') \leqslant \leqslant d(y_2',w_2)$ and so $d(y_2',y_5') \leqslant d(w_2,w_5)$. Consequently, $d(y_2,y_5) \geqslant \geqslant d(y_2',y_5')$. Thus really, when γ and δ are fixed, the distance $d(y_2,y_5)$ is the smallest for $\alpha = \beta$.

2. If $\alpha = \beta$ and if δ is fixed, then the distance $d(y_2,y_5)$ is the smallest for γ equal to $\gamma_0 = 2 \arcsin(1/3)$. In fact: if γ decreases, then also $(\gamma+\delta)/2 = \alpha+\gamma+\beta$ decreases and consequently, $d(y_2,y_5)$ increases (note that y_2, y_5 lie on the circles of radius 1/3 with the centers x_1, x_6, respectively, and that they are on the other side of the straight line through x_1, x_6 than a). On the other hand, γ cannot be greater than $2 \arcsin(1/3)$ because $d(x_3,x_4) \leqslant 1/3$.

3. Let $\alpha = \beta$ and $\gamma = \gamma_0$. We shall show that $d(y_2,y_5)$ is minimal in the case when α, β are equal $\alpha_0 = \beta_0 = 90^\circ - 2 \arcsin(1/3)$, and that this minimal value is greater than 1/3. Obviously, $\alpha = \beta = (\delta-\gamma)/4$. Consider the trapezium $y_2y_5x_4x_3$ (the reader may imagine it as $y_2'y_5'x_4'x_3'$ in Fig.4, where $d(x_3',x_4') = 1/3$). When $\sphericalangle x_1ax_6 = \delta$ decreases, also the angles $\sphericalangle y_2x_3x_4 = \sphericalangle y_5x_4x_3$ decrease and thus, in virtue of $d(x_3,x_4) = d(x_3,y_2) = d(x_4,y_5) = 1/3$, also the distance $d(y_2,y_5)$ decreases. On the other hand, δ cannot be smaller than

$$\delta_0 = 360^\circ - 10 \arcsin(1/3) \approx 165^\circ 20'.$$

If $\delta = \delta_0$, the angle $\alpha = \sphericalangle x_1ay_2$ equals

$$\left[360^\circ - 10 \arcsin(1/3) - 2 \arcsin(1/3)\right]/4 = \alpha_0 \approx 31^\circ 36',$$

Moreover, $d(x_1,a) = 1/2$ and $d(x_1,y_2) = 1/3$. Considering the triangle x_1ay_2 we get $d(a,y_2) \approx 0.2198$. If $\delta = \delta_0$, then $\sphericalangle y_2ay_5 = \alpha_0 + \gamma_0 + \beta_0 = 2\alpha_0 + 2 \arcsin(1/3) \approx 102^\circ 08'$ and consequently, $d(y_2,y_5) \approx 0.3418 > 1/3$.

From considerations of the above steps 1-3 we conclude that $d(y_2,y_5) > 1/3$ for all possible positions of y_2 and y_5. This ends the proof of (2).

Next, we show that

(3) if $j = i+4 \pmod{10}$, then $d(y_i,y_j) > 1/3$.

To fix our mind, we shall show (3) for the points y_2, y_6, which does not make our consideration narrower.

Consider the diametral segment of C parallel to the segment x_1x_7. Since 8 arc $\sin(1/3) < 180^\circ$, points x_7, x_8, x_9, x_{10}, x_1 are on one side of this diametral segment. Its endpoints are denoted by p, t as in Fig.5. Let P, T be the circles of radius 1/3 with the centers x_1, x_7, respectively. The inter-

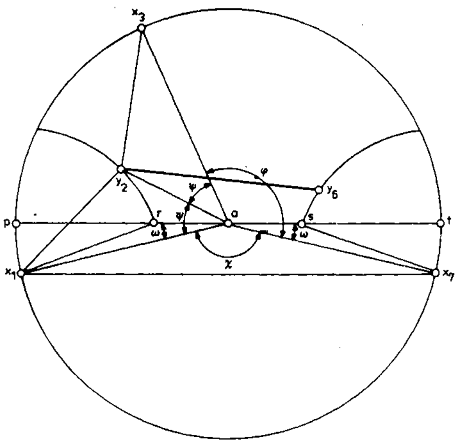

Fig.5

sections of the segment pt with P and T are denoted by r and s, respectively. Obviously, $d(p,r) < d(x_1,r) = 1/3$ and $d(s,t) < d(s,x_7) = 1/3$. Thus from $d(p,t) = 1$ we obtain that $d(r,s) > 1/3$.

Let $\varphi = \sphericalangle x_3ax_7$, $\chi = \sphericalangle x_7ax_1$, $\psi = \sphericalangle x_1ay_2$, $\omega = \sphericalangle x_1ap$. Obviously, $\sphericalangle y_2ax_3 = \psi$ and $\sphericalangle tax_7 = \omega$. Therefore $\varphi + \chi + 2\psi = 360^\circ$ and $2\omega + \chi = 180^\circ$. Hence $\psi = (360^\circ - \varphi - \chi)/2 = (360^\circ - \varphi - 180^\circ + 2\omega)/2 = (90^\circ - \varphi/2) + \omega$. Since $\varphi/2 <$

$< 4 \arcsin(1/3) \approx 77^\circ 52' < 90^\circ$, we have $\psi > \omega$. Thus y_2, and similarly also y_6, lie on this side of the diametral segment pt which does not contain points x_1, x_7. Moreover, $y_2 \in P$, $y_6 \in T$. So $d(y_2,y_6) > d(r,s)$. Consequently, from $d(r,s) > 1/3$ we get $d(y_2,y_6) > 1/3$. Thus (3) holds.

Suppose that $y_1 \in S_{13}$. Then from (2) and (3) it follows that $y_4, y_5, y_7, y_8 \notin S_{13}$. If $y_6 \in S_{13}$ then, by (2) and (3), also $y_2, y_3, y_9, y_{10} \notin S_{13}$ in contradiction with (1). So $y_6 \notin S_{13}$. Thus $S_{13} \cap Y \subset \{y_1, y_2, y_3, y_9, y_{10}\}$. If $y_3 \in S_{13}$, then from (2) and (3) we obtain that $y_9, y_{10} \notin S_{13}$ which contradicts (1). Therefore $y_3 \notin S_{13}$. Similarly, we show that $y_9 \notin S_{13}$. So $S_{13} \cap Y \subset \{y_1, y_2, y_{10}\}$. A contradiction with (1). We see that the supposition $y_1 \in S_{13}$ is false.

Analogically as we obtained that $y_1 \notin S_{13}$, we get that $y_i \notin S_{13}$, $i = 1,\dots,10$. Thus $S_{13} \cap Y = \emptyset$, which contradicts (1). This contradiction shows that our supposition about the possibility of covering D by 13 sets of diameter 1/3 is false.

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