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ON A NONLINEAR BOUNDARY VALUE PROBLEM FOR HIGHER ORDER INTEGRODIFFERENTIAL EQUATIONS

1. Introduction

This paper is concerned with the existence, uniqueness and continuous dependence on parameter of solutions of the nonlinear boundary value problem for higher order integro-differential equations. The Banach fixed point theorem is used to establish our results.

We consider the boundary value problem (BVP, for short)

$$(1)\qquad x^{(n)} = F\Big(t,x,x',\dots,x^{(n,1)}, \int_a^t K(t,s,x,x',\dots,x^{(n-1)})ds, \mu\Big),$$

$$(2)\qquad x^{(r)}(a_j) = 0, \quad j = 1,\dots,m, \; r = 0,1,\dots,r_j-1,$$

where $a = a_1 < a_2 < \dots < a_m = b$, $1 \leqslant m \leqslant n$, $\sum_{j=1}^{m} r_j = n$, μ is a real parameter, K and F are real-valued continuous functions defined on $I^2 \times R^n$ and $I \times R^{n+2}$ respectively in which $I = [a,b]$ and R denotes the set of real numbers. Recently in [7] J. Morchalo has studied the existence and uniqueness of the solutions of equation (1) when $\mu = 0$ and the derivatives involved in K and F are absent, under different boundary conditions by using the general idea of Chaplygin. Morchalo [7] assumes that the functions $K(t,s,x)$ and $F(t,x,u)$ therein have bounded partial derivatives with respect to the variables x and u. These conditions, severely restrict the growth of the functions K and F in order to assure the existence and uniqueness of the solution.

Our purpose in this paper is to obtain the existence, uniqueness and continuous dependence of solutions of the BVP (1) - (2) without any conditions on the derivatives of the functions K and F involved in (1). Here, we use Lipschitz conditions on K and F and apply the well known Banach fixed point theorem to obtain the existence, uniqueness and continuous dependence of solutions of BVP (1) - (2). Thus our method is different from that in [7] which in turn allows us to weaken the conditions on K and F in (1). The problems of existence and uniqueness of solutions of various special forms of BVP (1) - (2) have been studied by many authors under different conditions, see [5, 10-12] and the references given therein.

2. Statement of results

In order to establish our results we need the following preliminary lemmas.

Lemma 1. There exists Green's function $G(t,s)$, $a \leq s,\ t \leq b$, such that the BVP (1) - (2) is equivalent to the integral equation

$$(3)\quad x(t) = \int_a^b G(t,s)F\Big(s, x(s), x'(s), \dots, x^{(n-1)}(s), \int_a^s K(s,\tau,x(\tau),x'(t),\dots,x^{(n-1)}(\tau))d\tau, \mu\Big)ds.$$

Lemma 2. If $G(t,s)$ is Green's function of the equation

$$(4)\qquad x^{(n)}(t) \neq 0,$$

with the boundary conditions (2), then

$$(5)\qquad \int_a^b |G(t,s)|\,ds \leqslant \frac{(n-1)^{n-1}(b-a)^n}{n!\,n^n},$$

$$(6)\qquad \int_a^b \left|\frac{\partial^{i-1}}{\partial t^{i-1}} G(t,s)\right| ds \leqslant \frac{(i-1)(b-a)^{n-i+1}}{(n-i+1)!\,n},$$

for $i = 2,\dots,n$.

Lemma 1 may be found in Coppel [4]. Lemma 2 is given in Kobelkov and Kobyakov [6].

Our main result reads as follows.

T h e o r e m 1. Suppose that the functions K and F satisfy the conditions

$$(7)\qquad |K(t,s,x,x',\dots,x^{(n-1)}) - K(t,s,y,y',\dots,y^{(n-1)})| \leqslant \leqslant \sum_{i=0}^{n-1} M_i \left|x^{(i)} - y^{(i)}\right|,$$

$$(8)\qquad |F(t,x,x',\dots,x^{(n-1)},u,\mu) - F(t,y,y,\dots,y^{(n-1)},v,\mu)| \leqslant \leqslant \sum_{i=0}^{n-1} L_i \left|x^{(i)} - y^{(i)}\right| + Q|u-v|,$$

for all $(t,s,x,x',\dots,x^{(n-1)}),(t,s,y,y',\dots,y^{(n-1)}) \in I^2\times R^n$, $(t,x,x',\dots,x^{(n-1)},u,\mu)$, $(t,y,y',\dots,y^{(n-1)},v,\mu) \in I\times R^{n+2}$, where M_i, L_i $(i=0,1,\dots,n-1)$ and Q are nonnegative constants such that

$$(9)\qquad \alpha = \max\left\{L_0\,\frac{(n-1)^{n-1}\,(b-a)^n}{n!\,n^n} + \sum_{j=1}^{n-1} L_j\left[\frac{j(b-a)^{n-j}}{(n-j)!\,n}\right],\right.$$

$$\left. M_0 = \frac{(n-1)^{n-1}\,(b-a)^n}{n!\,n^n} + \sum_{j=1}^{n-1} M_j\left[\frac{j(b-a)^{n-j}}{(n-j)!\,n}\right]\right\},$$

and

$$(10) \qquad [1+Q(b-a)]\,\alpha < 1.$$

Then, for an arbitrary $\mu \in R$, the BVP (1) - (2) has a unique solution.

Our next result deals with the problem of continuous dependence of solutions of BVP (1) - (2) on a parameter μ.

T h e o r e m 2. Assume that the conditions of Theorem 1 are satisfied. Further assume that there exists a constant $N > 0$ such that for $t \in I$, $(z, z_1, \ldots, z_{n-1}) \in R^n$, $u \in R$, μ, $\mu_0 \in R$,

$$(11) \quad \left|F(t,z,z_1,\ldots,z_{n-1},u,\mu) - F(t,z,z_1,\ldots,z_{n-1},u,\mu_0)\right| \leq \leq N\,|\mu - \mu_0|.$$

If x_μ denote the unique solution of BVP (1) - (2), then $\mu \to x_\mu$ maps continuously R into the space $C^{(n-1)}[I,R]$.

3. Proofs of Theorems 1-2

Let S be the Banach space of functions $z \in C^{(n-1)}[I,R]$ with the norm

$$\|z\| = \max\left\{\max_I \sum_{j=0}^{n-1} L_j\,|z^{(j)}(t)|,\ \max_I \sum_{j=0}^{n-1} M_j\,|z^{(j)}(t)|\right\}.$$

Define a mapping $T : S \to S$ by setting for each $x \in S$,

$$(12) \quad Tx(t) = \int_a^b G(t,s)F\Big(s,x(s),x'(s),\ldots,x^{(n-1)}(s), \int_a^s K(s,\tau,x(\tau),x'(\tau),\ldots,x^{(n-1)}(\tau))d\tau,\mu\Big)\,ds,$$

for $t \in I$. Clearly the solution of BVP (1) - (2) is a fixed point of the operator equation $Tx = x$. If $x, y \in S$, then from (12), (7), (8) and Lemma 2 we have

$$(13)\quad |Tx(t) - Ty(t)| \leq \int_a^b |G(t,s)| \left[\sum_{i=0}^{n-1} L_i |x^{(i)}(s) - y^{(i)}(s)| + \right.$$

$$\left. + Q \int_a^s \sum_{i=1}^{n-1} M_i |x^{(i)}(\tau) - y^{(i)}(\tau)| d\tau \right] ds \leq$$

$$\leq [1 + Q(b-a)] \frac{(n-1)^{n-1} (b-a)^n}{n! \, n^n} \|x-y\|,$$

and

$$(14)\quad |(Tx(t))^{(j)} - (Ty(t))^{(j)}| \leq$$

$$\leq \int_a^b \left|\frac{\partial^j}{\partial t^j} G(t,s)\right| \left[\sum_{i=0}^{n-1} L_i |x^{(i)}(s) - y^{(i)}(s)| + \right.$$

$$\left. + Q \int_a^s \sum_{i=0}^{n-1} M_i |x^{(i)}(\tau) - y^{(i)}(\tau)| d\tau \right] ds \leq$$

$$\leq [1 + Q(b-a)] \left[\frac{j(b-a)^{n-1}}{(n-1)! \, n}\right] \|x-y\|,$$

for $j = 1, \ldots, n-1$. From (13), (14) and (9) it follows that

$$(15)\quad \|Tx-Ty\| \leq [1 + Q(b-a)] \alpha \|x-y\|.$$

Since $[1+Q(b-a)]\alpha < 1$, it follows from the Banach fixed point theorem that T has a unique fixed point in S. This completes the proof of Theorem 1.

Assume $\mu_n \to \mu_0$ as $n \to \infty$. Let S be defined as in the proof of Theorem 1; denote by T_m ($m = 0,1,\ldots$) the mapping T whenever $\mu = \mu_m$. Evidently,

$$\|T_m x - T_m y\| \leq [1 + Q(b-a)]\,\alpha\,\|x-y\|,$$

for $m = 0,1,2,\ldots$ and $x,y \in S$. Arguments similar to the above and (11) imply that $\|T_n x - T_0 x\| \leq \alpha N |\mu_n - \mu_0|$, for $n=1,2,\ldots$; hence $\lim_{n\to\infty} \|T_n x - T_0 x\| = 0$ for each $x \in S$.

Consequently, Theorem 1.2 in [3, p.6] applies to the mapping T_m and the space S, and we conclude that $\|x_{\mu_n} - x_{\mu_0}\| \to 0$ as $n \to \infty$. The proof of Theorem 2 is complete.

We note that a special case of Theorem 1 for $n = 2$, $\mu = 0$ and when the integral term in (1) is absent, with different boundary conditions appears in [2, Theorem 1.1, p.5], see also [1]. The proof given there motivates the proofs of our Theorems 1 and 2, which in turn show that the recent results on existence and uniqueness of solutions of special forms of BVP (1) - (2) obtained by various authors in [5, 7, 10-12] (see, also the results in [6,8] with delay arguments) under quite dissimilar hypotheses, can in fact be derived from a common principle. The sufficient conditions obtained here are weaker than those required in [7]. We also note that the BVP (1) - (2) for $n = 2$ and $\mu = 0$ is recently studied by the present author in [9] by reducing it to the two systems of more general first order integrodifferential equations and using monotone method under different conditions.

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