DEMONSTRATIO MATHEMATICA
Vol. XVIII No 2 1985

M. D. Upadhyay, V. C. Gupta

INVARIANT SUBMANIFOLDS OF AN f(3, -1)-MANIFOLD WITH COMPLEMENTED FRAMES

The f(3,+1)-manifolds have been studied by Yano [1], Ishihara and Yano [2], Goldberg and Yano ([3], [4]) have studied the f(3,1)-manifolds with complemented frames. Recently, Yano [5] has obtained certain results on invariant submanifolds of an f(3,1) manifold with complemented frames. The purpose of the present paper is to study invariant submanifolds of an f(3,-1)-manifold with complemented frames.

In section 2 we have defined and studied the normality of an f(3,-1)-structure with complemented frames. In section 3 we have proved that such a structure induces an almost product structure in the manifold, and in section 4 we have studied the relation between the integrability of this almost product structure and the normality of the f(3,-1)-structure with complemented frames.

In sections 5 and 6 we have studied invariant submanifolds of a normal f(3,-1)-manifold with complemented frames. In section 7 we have defined the f(3,-1)-structure with metric and complemented frames. We have also established some results.

1. Preliminaries

In an m-dimensional differentiable manifold M of class C^{∞}, a tensor field f ($f \neq 0, I$) of type (1,1) which satisfies

$f^3 - f = 0$ and is of constant rank r, at each point of M, is called f(3,-1)-structure of rank r and M with an f(3,-1)--structure f(3,-1)-manifold.

If we put

$$(1.1) \qquad l = f^2 \quad \text{and} \quad m = I - f^2,$$

I being the unit tensor field. Then it can be easily proved that

$$(1.2) \qquad l + m = I, \quad l^2 = l, \quad m^2 = m, \quad lm = ml = 0.$$

This shows that the tensor fields f^2 and $I - f^2$, I being the unit tensor field, are complementary projection operators which define two complementary distributions in M corresponding to the projection - operators f^2 and $I - f^2$ respectively. The distribution corresponding to f^2 is r-dimensional and that corresponding to $I - f^2$, (m-r)-dimensional.

Let there exist (m-r) vector fields U_α ($\alpha = 1,2,\dots,m-r$) spanning the distribution corresponding to $I - f^2$ and (m-r) 1-forms u^α satisfying

$$(1.3) \qquad f^2 = I - \sum_{\alpha=1}^{m-r} u^\alpha \otimes U_\alpha;$$

$$(1.4) \qquad f \cdot U_\alpha = 0, \quad u^\alpha \circ f = 0, \quad u^\alpha(U_\beta) = \delta_\beta^\alpha,$$

($\alpha, \beta = 1,2,\dots,m-r$), where δ_β^α is the Kronecker delta. Then we call the set $f(3,-1), \{U_\alpha, u^\alpha\}$ an f(3,-1)-structure with complemented frames and M an f(3,-1)-manifold with complemented frames.

Invariant submanifold

Suppose that an n-dimensional differentiable manifold $\tilde{M}$ is immersed in a f(3,-1)-manifold M by the immersion $i:\tilde{M} \to M$. If the tangent space of i(M) is invariant by the action of f, then $i(\tilde{M})$ is called an invariant submanifold of M.

In the present paper we consider an $f(3,-1)$-structure with complemented frames such that $r = m-2$.

2. $f(3,-1)$-structure with complemented frames

Let M be an m-dimensional differentiable manifold and let there be given a tensor field f of type $(1,1)$ and of rank $m-2$, two vector fields U, V and two 1-forms u, v. If the set $\{f(3,-1), U, V, u, v\}$ satisfies

$$f^2 = I - u \otimes U - v \otimes V; \tag{2.1}$$

$$fU = 0, \quad fV = 0, \quad u \circ f = 0, \quad v \circ f = 0; \tag{2.2}$$

$$u(U) = 1, \quad v(U) = 0, \quad u(V) = 0, \quad v(V) = 1, \tag{2.3}$$

then we call $\{f(3,-1), U, V, u, v\}$ an $f(3,-1)$-structure with complemented frames and M an $f(3,-1)$-manifold with complemented frames.

Let us define a tensor field S of type $(1,2)$ by

$$S(X,Y) = N(X,Y) - (du)(X,Y)U - (dv)(X,Y)V, \tag{2.4}$$

where du, dv are 2-forms and N is the Nijenhuis tensor formed with f, defined by [2]

$$N(X,Y) = [fX,fY] - f[fX,Y] - f[X,fY] + f^2[X,Y]. \tag{2.5}$$

D e f i n i t i o n 2.1. If the tensor field S vanishes identically, then the structure is said to be normal.

In consequence of (2.2) and (2.5), we have from (2.4)

$$S(X,U) = -f[fX,U] + f^2[X,U] - (du)(X,U)U - (dv)(X,U)V. \tag{2.6}$$

Let $\mathcal{L}_U$ denotes the Lie-differentiation with respect to U. Then we have (see [2])

$$-f[fX,U] + f^2[X,U] = f\{f[X,U] - [fX,U]\} = f(\mathcal{L}_U f)X,$$

$$(du)(X,U) = X(u(U)) - U(u(X)) - u([X,U]) =$$

$$= -\{u([X,U]) - [u(X),U]\} = -(\mathcal{L}_U u)(X).$$

Similarly, $(dv)(X,U) = -(\mathcal{L}_U v)(X)$.

Therefore from (2.6) we have

$$S(X,U) = f(\mathcal{L}_U f)X + (\mathcal{L}_U u)(X)U + (\mathcal{L}_U v)(X)V. \tag{2.7}$$

We can also prove that

$$S(X,V) = f(\mathcal{L}_V f)X + (\mathcal{L}_V u)(X)U + (\mathcal{L}_V v)(X)V. \tag{2.8}$$

Also from (2.4), we have in consequence of (2.2), (2.3) and (2.5)

$$u(S(X,Y)) = u([fX,fY]) - (du)(X,Y). \tag{2.9}$$

But we have

$$(du)(fX,fY) = (fX)u(fY) - (fY)u(fX) - u([fX,fY]),$$

which in view of (2.2) implies that

$$u([fX,fY]) = -(du)(fX,fY).$$

Thus from (2.9) we obtain

$$u(S(X,Y)) = -(du)(X,Y) - (du)(fX,fY). \tag{2.10}$$

Replacing X by fX in (2.10) and using (2.1), we get

$$u(S(fX,Y)) = -(du)(fX,Y) - (du)(X-u(X)U - v(X)V,fY) = \tag{2.11}$$

$$= -(du)(fX,Y) - (du)(X,fY) +$$

$$+ u(X)(du)(U,fY) + v(X)(du)(V,fY).$$

But we have

$$(du)(U,fY) = U\,u(fY) - (fY)u(U) - u([U,fY]) =$$

$$= u([fY,U]) - [u(fY),U] = (\mathcal{L}_U u)(fY).$$

Similarly,

$$(du)(V,fY) = (\mathcal{L}_V u)(fY).$$

Hence, we have from (2.11)

$$(2.12)\qquad u(S(fX,Y)) = -(du)(fX,Y) - (du)(X,fY) + $$
$$+ u(X)(\mathcal{L}_U u)(fY) + v(X)(\mathcal{L}_V u)(fY).$$

We can also prove that

$$(2.13)\qquad v(S(fX,Y)) = -(dv)(fX,Y) - (dv)(X,fY) +$$
$$+ u(X)(\mathcal{L}_U v)(fY) + v(X)(\mathcal{L}_V v)(fY).$$

T h e o r e m 2.1. If a $f(3,-1)$-structure with complemented frames $\{f(3,-1),\ U,\ V,\ u,\ v\}$ is normal, then

$$(2.14)\qquad \mathcal{L}_U f = 0,\quad \mathcal{L}_U u = 0,\quad \mathcal{L}_U v = 0;$$

$$(2.15)\qquad \mathcal{L}_V f = 0,\quad \mathcal{L}_V u = 0,\quad \mathcal{L}_V v = 0;$$

$$(2.16)\qquad du\,\pi\,f = 0,\quad dv\,\pi\,f = 0,\quad [U,V] = 0.$$

P r o o f . Let us assume that the $f(3,-1)$-structure with complemented frames $\{f(3,-1),\ U,V,\ u,v\}$ is normal. Then from (2.7) we have

$$f(\mathcal{L}_U f)X + (\mathcal{L}_U u)(X)U + (\mathcal{L}_U v)(X)V = 0,$$

which in view of (2.2) and (2.3) implies that

(2.17) $\mathcal{L}_U u = 0, \quad \mathcal{L}_U v = 0, \quad f(\mathcal{L}_U f) = 0.$

Applying f to the last equation of (2.17) and using (2.1), we obtain

$$\mathcal{L}_U f - u \circ (\mathcal{L}_U f) \otimes U - v \circ (\mathcal{L}_U f) \otimes V = 0,$$

or

$$\mathcal{L}_U f + \{(\mathcal{L}_U u) \circ f\} \otimes U + \{(\mathcal{L}_U v) \circ f\} \otimes V = 0.$$

Hence in view of (2.17) we have

(2.18) $\mathcal{L}_U f = 0.$

Similarly, from (2.8) we can prove that

(2.19) $\mathcal{L}_V u = 0, \quad \mathcal{L}_V v = 0, \quad \mathcal{L}_V f = 0.$

Let us put

(2.20) $(w\pi f)(X,Y) = w(fX,Y) + w(X,fY),$

for a 2-form w. Then in consequence of (2.17), (2.19) and (2.20), from (2.12) and (2.13), we have

$$(du)\pi f = 0 \quad \text{and} \quad (dv)\pi f = 0.$$

Now, computing $\mathcal{L}_U(fV) = 0$, we find

(2.21) $f \mathcal{L}_U V = 0.$

Applying f to (2.21) and using (2.1), we get

$$\mathcal{L}_U V - u(\mathcal{L}_U V)U - v(\mathcal{L}_U V)V = 0,$$

or

$$\mathcal{L}_U V = 0, \quad \text{i.e.} \quad [U,V] = 0.$$

3. Almost product structure ς

Let us define a tensor field ς of type (1,1) by

$$\varsigma X = fX + v(X)U + u(X)V, \tag{3.1}$$

for an arbitrary vector field X.

T h e o r e m 3.1. In order that a manifold M may admit a f(3,-1)-structure with complemented frames $\{f(3,-1), U, V, u, v\}$, it is necessary and sufficient that the manifold admits an almost product structure ς, a vector field U and a 1-form u such that

$$u(U) = 1 \quad \text{and} \quad u(\varsigma U) = 0.$$

P r o o f . In consequence of (2.1), (2.2), (2.3) and (3.1), we have

$$\begin{aligned}\varsigma^2 X = \varsigma(\varsigma X) &= f(fX + v(X)U + u(X)V) + \\ &+ v(fX + v(X)U + u(X)V)U + u(fX + v(X)U + u(X)V)V = \\ &= f^2X + u(X)U + v(X)V = X.\end{aligned}$$

Therefore, $\varsigma^2 = I$. Thus ς is an almost product structure.

Also, in view of (2.2), (2.3) and (3.1), we can easily verify that

$$\varsigma U = V, \quad \varsigma V = U; \tag{3.2}$$

$$u \circ \varsigma = v, \quad v \circ \varsigma = u. \tag{3.3}$$

Conversely, suppose that a manifold M admits an almost product structure ς, a vector field U and a 1-form u such that

$$u(U) = 1, \qquad u(\varsigma U) = 0. \tag{3.4}$$

We define a vector field V, a 1-form v and a tensor field f, respectively, by

(3.5) $$V = \zeta U,$$

(3.6) $$v = u \circ \zeta,$$

(3.7) $$f = \zeta - v \otimes U - u \otimes V.$$

Now, in consequence of (3.4), we have from (3.5) and (3.6)

(3.8) $$u(V) = 0, \quad v(U) = 0, \quad v(V) = 1.$$

Also, in view of (3.4), (3.5), (3.6) and (3.8), we have from (3.7)

(3.9) $$fU = 0, \quad fV = 0, \quad u \circ f = 0, \quad v \circ f = 0.$$

Further, by virtue of (3.6), (3.7) and (3.9), we have

$$f^2X = f(fX) = f\{\zeta X - v(X)U - u(X)V\} =$$

$$= \zeta(\zeta X) - v(\zeta X)U - u(\zeta X)V = \zeta^2 X - (u \circ \zeta)(\zeta X)\ U +$$

$$- (u \circ \zeta)(X)V = X - u(X)U - v(X)V.$$

Thus

(3.10) $$f^2 = I - u \otimes U - v \otimes V.$$

Equations (3.8), (3.9) and (3.10) show that M admits a f(3,-1)-structure with complemented frames $\{f(3,-1),\ U, V, u, v\}$.

4. <u>Integrability condition of ζ</u>

In this section, we shall obtain the relation between the integrability of an almost product structure ζ and the normality of the f(3,-1)-structure with complemented frames.

Let $N^*(X,Y)$ be the Nijenhuis tensor formed with ζ. Then we have

(4.1) $$N^*(X,Y) = [\zeta X, \zeta Y] - \zeta[\zeta X, Y] - \zeta[X, \zeta Y] + \zeta^2[X,Y],$$

or

$$(4.2)\qquad N^*(X,Y) = [\zeta X,\zeta Y] - \zeta[\zeta X,Y] - \zeta[X,\zeta Y] + [X,Y].$$

Now from (2.1), (3.1) and (4.2), we obtain

$$\begin{aligned} N^*(X,Y) = {} & [fX+v(X)U+u(X)V,\ fY+v(Y)U+u(Y)V] - \\ & - f[fX+v(X)U+u(X)V,Y] - v([fX+v(X)U+u(X)V,Y])U - \\ & - u([fX+v(X)U+u(X)V,Y])V - f[X,fY+v(Y)U+u(Y)V] - \\ & - v([X,fY+v(Y)U+u(Y)V])U - u([X,fY+v(Y)U+u(Y)V])V + \\ & + f^2[X,Y] + u([X,Y])U + v([X,Y])V. \end{aligned}$$

After some calculations, the above expression, in consequence of (2.3), (2.5) and (2.20), reduces to

$$\begin{aligned} (4.3)\qquad N^*(X,Y) = {} & N(X,Y) - (du)(X,Y)U - (dv)X,Y)V + \\ & + (dv\pi f)(X,Y)U + (du\pi f)(X,Y)V - \\ & - v(X)(\mathcal{L}_U f)Y + v(Y)(\mathcal{L}_U f)X - \\ & - u(X)(\mathcal{L}_V f)Y + u(Y)(\mathcal{L}_V f)X - \\ & - \{v(X)(\mathcal{L}_U v)Y - v(Y)(\mathcal{L}_U v)X + u(X)(\mathcal{L}_V v)Y - \\ & - u(Y)(\mathcal{L}_V v)X\}U - \{u(X)(\mathcal{L}_V u)Y - u(Y)(\mathcal{L}_V u)X + \\ & + v(X)(\mathcal{L}_U u)Y - v(Y)(\mathcal{L}_U u)X\}V - . \\ & - \{u(X)v(Y) - u(Y)v(X)\}[U,V]. \end{aligned}$$

T h e o r e m 4.1. If a $f(3,-1)$-structure with complemented frames $\{f(3,-1),\ U,V,\ u,v\}$ is normal, then the almost product structure ζ defined by (3.1) is integrable.

P r o o f . If a $f(3,-1)$-structure with complemented frames $\{f(3,-1),U,V,\ u,v\}$ is normal, then from definition (2.1), we have $S = 0$.

Thus by virtue of (2.4), (2.14), (2.15), (2.16) and (4.3), we obtain

$$N^*(X,Y) = 0.$$

Hence the almost product structure ς defined by (3.1) is integrable.

5. Invariant submanifolds

Let $\tilde{M}$ be an n-dimensional differentiable manifold $(1 < n < m)$ and suppose that $\tilde{M}$ is immersed in M by the immersion $i: \tilde{M} \to M$. Let us denote by B the differential di of the immersion i. Let us assume that the vector field U is tangent to $i(\tilde{M})$. Therefore we have

$$U = B\tilde{U}, \tag{5.1}$$

for a vector field $\tilde{U}$ of $\tilde{M}$,

$$v(B\tilde{X}) = 0, \tag{5.2}$$

for any vector field $\tilde{X}$ of $\tilde{M}$ and

$$f(B\tilde{X}) = B\tilde{f}\tilde{X}, \tag{5.3}$$

for a tensor field $\tilde{f}$ of $\tilde{M}$ and an arbitrary vector field $\tilde{X}$ of $\tilde{M}$. For convenience, we call such a submanifold an invariant submanifold with respect to U and v. Similarly, we can define an invariant submanifold with respect to V and u.

T h e o r e m 5.1. An invariant submanifold with respect to U and v of a manifold with f(3,-1)-structure and complemented frames $\{f(3,-1),U,V, u,v\}$ admits a $(\tilde{f},\tilde{U},\tilde{u})$-structure.

P r o o f . Let $\tilde{M}$ be an invariant submanifold with respect to U and v of a manifold M with f(3,-1)-structure and complemented frames $\{f(3,-1),U,V, u,v\}$.

Now, applying f to (5.1) and using (2.2) and (5.3), we obtain

$$0 = fU = f(B\tilde{U}) = B\tilde{f}\tilde{U},$$

which gives

$$\tilde{f}\tilde{U} = 0. \tag{5.4}$$

Applying f to (5.3) and using (2.1) and (5.3), we get

$$B\tilde{X} - u(B\tilde{X})U - v(B\tilde{X})V = B\tilde{f}^2\tilde{X}. \tag{5.5}$$

Let us put

$$\tilde{u}(\tilde{X}) = u(B\tilde{X}). \tag{5.6}$$

Then, in view of (5.1), (5.2) and (5.6), equation (5.5) yields

$$\tilde{f}^2\tilde{X} = \tilde{X} - \tilde{u}(\tilde{X})\tilde{U}. \tag{5.7}$$

Also from (5.3) we have

$$u(f(B\tilde{X})) = u(B\tilde{f}\tilde{X}),$$

which in view of (2.2) and (5.6) gives

$$\tilde{u}(\tilde{f}\tilde{X}) = 0. \tag{5.8}$$

Further from (5.1) we have

$$u(U) = u(B\tilde{U}),$$

which in view of (2.3) and (5.6) yields

$$\tilde{u}(\tilde{U}) = 1. \tag{5.9}$$

Combining (5.4), (5.7), (5.8) and (5.9), we have

$$\begin{cases} \tilde{f}^2 = I - \tilde{u} \otimes \tilde{U}; \\ \tilde{f}\tilde{U} = 0, \ \tilde{u} \circ \tilde{f} = 0; \\ \tilde{u}(\tilde{U}) = 1. \end{cases} \tag{5.10}$$

Structure satisfying (5.10) is called $(\tilde{f},\tilde{U},\tilde{u})$-structure.

T h e o r e m 5.2. An invariant submanifold with respect to V and u of a manifold with f(3,-1)-structure and complemented frames $\{f(3,-1), U,V, u,v\}$ admits a $(\tilde{f},\tilde{V},\tilde{v})$-structure.

P r o o f . The proof follows from the pattern of the proof of Theorem 5.1.

6. Invariant submanifolds of a normal f(3,-1)-manifold with complemented frames

Now we shall compute the expression $S(B\tilde{X},B\tilde{Y})$ for an invariant submanifold with respect to U and v.

In consequence of (2.4), (2.5) and (5.3), we have

$$\begin{aligned} S(B\tilde{X},B\tilde{Y}) &= [fB\tilde{X},fB\tilde{Y}] - f[fB\tilde{X},B\tilde{Y}] - f[B\tilde{X},fB\tilde{Y}] + \\ &+ f^2[B\tilde{X},B\tilde{Y}] - (du)(B\tilde{X},B\tilde{Y})U - (dv)(B\tilde{X},B\tilde{Y})V = \\ &= [B\tilde{f}\tilde{X},B\tilde{f}\tilde{Y}] - f[B\tilde{f}\tilde{X},B\tilde{Y}] - f[B\tilde{X},B\tilde{f}\tilde{Y}] + \\ &+ f^2[B\tilde{X},B\tilde{Y}] - (du)(B\tilde{X},B\tilde{Y})U - (dv)(B\tilde{X},B\tilde{Y})V. \end{aligned}$$

But in view of (5.1), (5.2) and (5.6), we have

$$(du)(\tilde{B}\tilde{X},\tilde{B}\tilde{Y}) = (d\tilde{u})(\tilde{X},\tilde{Y}), \quad (dv)(\tilde{B}\tilde{X},\tilde{B}\tilde{Y}) = 0.$$

Therefore

$$\begin{aligned} (6.1) \qquad S(B\tilde{X},B\tilde{Y}) &= B[\tilde{f}\tilde{X},\tilde{f}\tilde{Y}] - fB[\tilde{f}\tilde{X},\tilde{Y}] - fB[\tilde{X},\tilde{f}\tilde{Y}] + \\ &+ f^2B[\tilde{X},\tilde{Y}] - (d\tilde{u})(\tilde{X},\tilde{Y})U. \end{aligned}$$

Hence, by virtue of (5.1), equation (6.1) yields

$$\begin{aligned} (6.2) \qquad S(\tilde{B}\tilde{X},\tilde{B}\tilde{Y}) &= B\{[\tilde{f}\tilde{X},\tilde{f}\tilde{Y}] - \tilde{f}[\tilde{f}\tilde{X},\tilde{Y}] - \tilde{f}[\tilde{X},\tilde{f}\tilde{Y}] + \\ &+ \tilde{f}^2[\tilde{X},\tilde{Y}] - (d\tilde{u})(\tilde{X},\tilde{Y})\tilde{U}\}. \end{aligned}$$

T h e o r e m 6.1. An invariant submanifold with respect to U and v of a manifold with normal f(3,-1)-structure and complemented frames $\{f(3,-1),\ U,V,\ u,v\}$ admits a normal $(\tilde{f},\tilde{U},\tilde{u})$-structure.

P r o o f . If a f(3,-1)-structure with complemented frames $\{f(3,-1),\ U,V,\ u,v\}$ is normal, then $S = 0$. Therefore from (6.2) we have

$$(6.3)\qquad [\tilde{f}\tilde{X},\tilde{f}\tilde{Y}] - \tilde{f}[\tilde{f}\tilde{X},\tilde{Y}] - \tilde{f}[\tilde{X},\tilde{f}\tilde{Y}] + \\ + \tilde{f}^2[\tilde{X},\tilde{Y}] - (d\tilde{u})(\tilde{X},\tilde{Y})\tilde{U} = 0.$$

Thus by virtue of (6.3) and Theorem 5.1, the result follows.

T h e o r e m 6.2. An invariant submanifold with respect to V and u of a manifold having normal $f(3,-1)$-structure and complemented frames $\{f(3,-1), U,V, u,v\}$ admits a normal $(\tilde{f},\tilde{V},\tilde{v})$-structure.

P r o o f . The proof is similar to that of Theorem 6.1.

7. f(3,-1)-structure with metric and complemented frames

Let M be an m-dimensional differentiable manifold with $f(3,-1)$-structure and with complemented frames $\{f(3,-1),U,V, u,v\}$. Let there exist on M a Riemannian metric g satisfying

$$(7.1)\qquad g(fX,fY) = g(X,Y) - u(X)u(Y) - v(X)v(Y);$$

$$(7.2)\qquad u(X) = +g(U,X), \quad v(X) = +g(V,X),$$

for arbitrary vector fields X and Y. Then we call the structure $\{f(3,-1), U,V, u,v\}$ a $f(3,-1)$-structure with metric and complemented frames. Let us denote it by $\{f(3,-1),g,u,v\}$.

T h e o r e m 7.1. If a $f(3,-1)$-structure with metric and complemented frames $\{f(3,-1),g,u,v\}$ is normal. Then the almost product structure ζ defined by (3.1) is integrable and the manifold M defined by (ζ,g) is an almost product manifold.

P r o o f . Suppose that M admits a $\{f(3,-1),g,u,v\}$-structure. In section 3 we have proved that the tensor field ζ of type (1,1) defined by (3.1) is an almost product structure.

Also, in consequence of (2.2), (2.3), (3.1) and (7.2), we have

$$g(\zeta X,\zeta Y) = g(fX + v(X)U + u(X)V,\ fY + v(Y)U + u(Y)V) = \\ = g(fX,fY) + v(X)v(Y) + u(X)u(Y).$$

Hence, in view of (7.1), we have

$$g(\zeta X,\zeta Y) = g(X,Y). \tag{7.3}$$

Thus (ζ,g) defines an almost product manifold. The remaining part of the theorem follows from theorem (4.1).

Theorem 7.2. An invariant submanifold with respect to U and v of a manifold having f(3,-1)-structure with metric and complemented frames $\{f(3,-1),g,u,v\}$ admits a $(\tilde{f},\tilde{U},\tilde{u})$-structure with a metric.

Proof. Let $i(\tilde{M})$ be an invariant submanifold with respect to U and v of a manifold M having a f(3,-1)-structure with metric and complemented frames. The manifold M being a Riemannian manifold with metric tensor g, and $i(\tilde{M})$ is also a Riemannian manifold with metric tensor $\tilde{g}$ defined by

$$\tilde{g}(\tilde{X},\tilde{Y}) = g(B\tilde{X},B\tilde{Y}). \tag{7.4}$$

Replacing X by $B\tilde{X}$ and Y by $B\tilde{Y}$ in (7.1), we obtain

$$g(fB\tilde{X},fB\tilde{Y}) = g(B\tilde{X},B\tilde{Y}) - u(B\tilde{X})u(B\tilde{Y}) - v(B\tilde{X})v(B\tilde{Y}).$$

But $i(\tilde{M})$ being invariant, so using (5.2) and (5.3) in the above equation, we have

$$g(B\tilde{f}\tilde{X},B\tilde{f}\tilde{Y}) = g(B\tilde{X},B\tilde{Y}) - u(B\tilde{X})u(B\tilde{Y}),$$

which in view of (5.6) and (7.4) yields

$$\tilde{g}(\tilde{f}\tilde{X},\tilde{f}\tilde{Y}) = \tilde{g}(\tilde{X},\tilde{Y}) - \tilde{u}(\tilde{X})\tilde{u}(\tilde{Y}). \tag{7.5}$$

From the first equation of (7.2), we have

$$u(B\tilde{X}) = g(U,B\tilde{X}),$$

or

$$u(B\tilde{X}) = g(B\tilde{U},B\tilde{X}),$$

in consequence of (5.1). This in view of (5.6) and (7.4) yields

$$\tilde{u}(\tilde{X}) = \tilde{g}(\tilde{U},\tilde{X}). \tag{7.6}$$

Also from (7.2) we have

$$v(B\tilde{X}) = g(V,B\tilde{X}),$$

which in view of (5.2) reduces to

$$0 = g(V,B\tilde{X}). \tag{7.7}$$

This shows that V is a unit normal to the submanifold $i(\tilde{M})$. Now the theorem follows by virtue of the equations (7.5), (7.6), (7.7) and Theorem 5.1.

T h e o r e m 7.3. An invariant submanifold with respect to V and u of a manifold having f(3,-1)-structure with metric and complemented frames $\{f(3,-1),g,u,v\}$ admits a $(\tilde{f},\tilde{V},\tilde{v})$-structure with a metric.

P r o o f . The proof follows from the pattern of the proof of Theorem 7.2.

T h e o r e m 7.4. An invariant submanifold with respect to U and v of a manifold having normal f(3,-1)-structure with metric and complemented frames $\{f(3,-1),g,u,v\}$ admits a normal $(\tilde{f},\tilde{U},\tilde{u})$-structure with a metric.

P r o o f . The proof of the theorem follows from Theorems 6.1 and 7.2.

T h e o r e m 7.5. An invariant submanifold with respect to V and u of a manifold having normal f(3,-1)-structure with metric and complemented frames $\{f(3,-1),g,u,v\}$ admits a normal $(\tilde{f},\tilde{V},\tilde{v})$-structure with a metric.

P r o o f . The proof of the theorem follows from Theorems 6.2 and 7.3.

REFERENCES

[1] K. Y a n o : On a structure defined by a tensor field f of type (1,1) satisfying $f^3+f = 0$, Tensor, N.S. 14 (1963) 99-109.

[2] S. Ishihara, K. Yano: On integrability conditions of a structure f satisfying $f^3+f=0$, Quart. J. Math. Oxford ser. (2) Vol.15 (1964) 217-222.
[3] S.I. Goldberg, K. Yano: On normal globally framed f-manifolds, Tôhoku Math. J., 22 (1970) 362-370.
[4] S.I. Goldberg, K. Yano: Globally framed f-manifolds, Illinois J. Math., 15 (1971) 456-474.
[5] K. Yano: Invariant submanifolds of an f-manifold with complemented frames, Kôdai Math. Sem. Report, 25 (1973) 163-174.

DEPARTMENT OF MATHEMATICS, LUCKNOW UNIVERSITY, LUCKNOW, INDIA
Received October 28, 1983.