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GENERAL APPROACH TO LINE GRAPHS OF GRAPHS

1. Introduction

A unified approach to the notion of a line graph of general graphs is adopted and proofs of theorems announced in [6] are presented. Those theorems characterize five different types of line graphs. Both Krausz-type and forbidden induced subgraph characterizations are provided.

So far other authors introduced and dealt with single special notions of a line graph of graphs possibly belonging to a special subclass of graphs. In particular, the notion of a simple line graph of a simple graph is implied by a paper of Whitney (1932). Since then it has been repeatedly introduced, rediscovered and generalized by many authors, among them are Krausz (1943), Izbicki (1960; a special line graph of a general graph), Sabidussi (1961; a simple line graph of a loop-free graph), Menon (1967; adjoint graph of a general graph) and Schwartz (1969; interchange graph which coincides with our line graph defined below).

In this paper we follow another way, originated in our previous work [6]. Namely, we distinguish special subclasses of general graphs and consider five different types of line graphs each of which is defined in a natural way.

Note that a similar approach to the notion of a line graph of hypergraphs can be adopted.

We consider here the following line graphs: line graphs, loop-free line graphs, simple line graphs, as well as augmented line graphs and augmented loop-free line graphs.

The natural reason for introducing the concept of augmented line graphs is that they can be underlying general graphs of line digraphs of general digraphs.

The corresponding five operations from a graph to its line graph are considered and some relations between them are observed and proved. In particular, restrictions of those operations to the class of simple graphs all coincide.

We deal mainly with the problem of the characterization of line graphs under consideration. Two different characterizations are given. The first one is Krausz-type characterization in terms of a decomposition of the graph into subgraphs, and the second one in terms of forbidden induced subgraphs. The first characterization of simple line graphs of simple graphs on one hand and of general graphs on the other hand is given by Krausz [5] and by Hemminger [4] as well as Bermond an Meyer [2], respectively. Because classes of loop-free line graphs and augmented loop-free line graphs coincide we give Krausz-type characterizations of loop-free line graphs, line graphs and augmented line graphs only.

Simple line graphs of simple graphs are characterized in the second way by Beineke [1], and those of general graphs by Bermond and Meyer [2].

Beineke's result will play an imprtant role in proofs of our second characterizations of remaining line graphs.

Some comments and remarks about above-mentioned related results can be found in [6].

2. Definitions

Terms and symbols not defined here will be used in their common meaning unless it is otherwise stated below. In particular we shall make use of conventions introduced in our paper [6].

By a graph (general graph, called sometimes pseudomultigraph) G we mean an ordered triple $G = (V,E,\eta_G)$ where $V = V(G)$ is the vertex set, $E = E(G)$ the edge set, and η_H is the function $E \to \mathcal{P}_{1,2}(V)$ which associates with each edge of G the (one- or two-element) set of its end-vertices. $\mathcal{L}(G)$ denotes the set of loops in G. Letters $\mathcal{M}$, $\mathcal{F}$ and $\mathcal{O}$ stand for the class of graphs, loop-free graphs, and simple (ordinary) graphs, respectively.

$\mathcal{M}_{1,2}$ denotes the class of graphs having at most one loop at any vertex and no more than two edges connecting any pair of different vertices. $\mathcal{F}_2$ is the subclass of loop-free graphs with at most two edges connecting any two different vertices. Attaching a single loop to each vertex of a complete graph gives a complete 1-graph. In general, an s-graph with s being a non-negative integer contains at most s different edges connecting any pair of (possibly the same) vertices. If $\mathcal{N}$ stands for $\mathcal{M}$, $\mathcal{M}_{1,2}$, $\mathcal{F}$, $\mathcal{F}_2$, or $\mathcal{O}$ then the elements of $\mathcal{N}$ are called $\mathcal{N}$-graphs. So a loop-free 2-graph is called simply $\mathcal{F}_2$-graph.

A loop-graph contains one vertex and one loop.

The degree of a vertex x of a graph G, denoted by $d_G(x)$, is the number of edges incident to x, each loop being counted twice. A vertex x is called naked, resp. hanging, in G if $d_G(x) = 0$ and $d_G(x) = 1$, respectively. An edge is hanging if it is incident to a hanging vertex. Two vertices x and y are adjacent (neighbours) in G if there exists an edge e connecting them, i.e. with $\eta_G(e) = \{x,y\}$ (and $x \neq y$). A vertex without any neighbour is said to be isolated. A loop-free vertex has no attached loops. A vertex incident to a loop is self-adjacent.

The multiple of adjacency of two (possibly the same) vertices is the number of edges connecting them. If this number is one then the vertices are simply adjacent. Two edges are multiple edges if they are different and have the same set of end-vertices; if in addition they are not loops, they are called doubly adjacent.

Two graphs G and H are isomorphic, in symbols $G \cong H$, if there is a bijection $\varphi : V(G) \rightarrow V(H)$ which preserves the multiple of adjacency of vertices.

3. Line graphs

Given two graphs H and M consider the following conditions:

(i) $V(H) = E(M)$;

(ii) The number of edges in H connecting any two different vertices $e_1, e_2 (e_1, e_2 \in E(M),\ e_1 \neq e_2)$ is equal to 2 if $\eta_M(e_1) = \eta_M(e_2)$ otherwise it is equal to $|\eta_M(e_1) \cap \eta_M(e_2)|$;

(iii) H has exactly one loop incident to a vertex e if e is a loop of M;

(iv) The number of edges in H joining any two different vertices $e_1, e_2, (e_1, e_2 \in E(M),\ e_1 \neq e_2)$ is equal to $|\eta_M(e_1) \cap \eta_M(e_2)|$;

(v) Two vertices of H are simply adjacent if they are different and adjacent edges in M.

(3.1) Definition. We define operations L, AL, $L_{\mathfrak{F}}$, $AL_{\mathfrak{F}}$, and $L_{\mathcal{O}}$ by the following implications under the assumptions that, for a given M, the graph H is the smallest possible (cf. Fig.1):

(i), (iii), (iv) $\Rightarrow H = L(M)$, the line graph of M;

(i), (ii), (iii) $\Rightarrow H = AL(M)$, the augmented line graph of M;

(i), (iv) $\Rightarrow H = L_{\mathfrak{F}}(M)$, the line $\mathfrak{F}$-graph (or loop-free line graph of M);

(i), (ii) $\Rightarrow H = AL_{\mathfrak{F}}(M)$, the augmented line $\mathfrak{F}$-graph of M;

(i), (v) $\Rightarrow H = L_{\mathcal{O}}(M)$, the line $\mathcal{O}$-graph (or simple line graph) of M.

A graph H is called a line-graph of a fixed type if there is a graph M such that H is isomorphic to the line graph of M of the type in question.

Given a graph M, let ρ be a 1-1 mapping from the set $\mathcal{L}(M) = \mathcal{L}$ of loops of M onto a set V_1 of new vertices which can be added to M.

Let $\overline{V}_1$ denote the set of new vertices obtained from V_1 by substituting a single vertex for each set of images under ρ of all mutually adjacent loops and let $\bar{\rho} : \mathcal{L} \to \overline{V}_1$ be a resulting mapping. Now we define loop-free transform $\psi(M)$ [(resp. $\overline{\psi}(M)$)] of M by adding V_1 (resp. $\overline{V}_1$) and replacing each loop

$\ell \in \mathcal{L}(M)$ by a new edge connecting $\rho(\ell)$ [resp. $\bar{\rho}(\ell)$] with the vertex of ℓ.

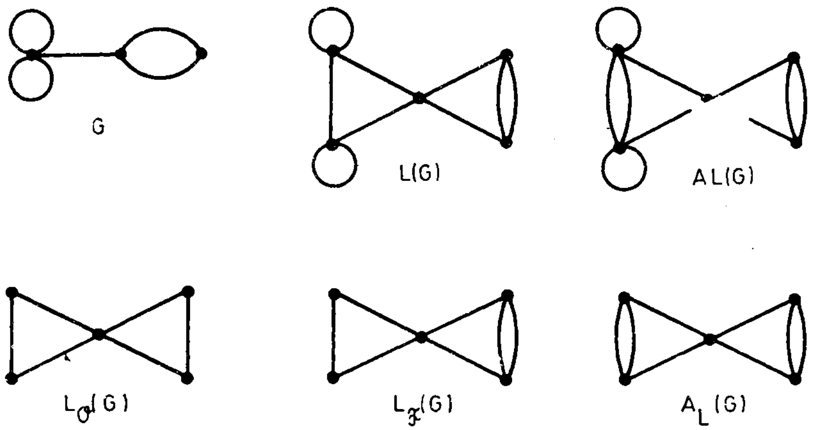


Fig.1

Thus nonadjacent loops in M are replaced by nonadjacent edges both in $\psi(M)$ and $\bar{\psi}(M)$. Moreover, adjacent loops in M correspond to adjacent new edges which are doubly adjacent in $\bar{\psi}(M)$ only. For a loop-free graph M, $\psi(M) = M = \bar{\psi}(M)$.

Now observe that, for each M,

$$L_\mathcal{F}(M) = L_\mathcal{F}(\psi(M)) \quad \text{and} \quad AL_\mathcal{F}(M) = AL_\mathcal{F}(\bar{\psi}(M)).$$

Moreover, restrictions of operations $L_\mathcal{F}$ and $AL_\mathcal{F}$ to the class $\mathcal{F}$ coincide. Therefore the images of $\mathcal{M}$ and $\mathcal{F}$ under the operations $L_\mathcal{F}$ and $AL_\mathcal{F}$ all coincide, i.e.,

(3.2) $$L_\mathcal{F}[\mathcal{M}] = L_\mathcal{F}[\mathcal{F}] = AL_\mathcal{F}[\mathcal{F}] = AL_\mathcal{F}[\mathcal{M}].$$

Note also that Bermond and Meyer observed [2] that

$$L_\sigma(M) \cong L_\sigma(\psi(M)) \quad \text{whence}$$

(3.3) $$L_\sigma[\mathcal{M}] = L_\sigma[\mathcal{F}].$$

4. Krausz-type characterizations

(4.1) D e f i n i t i o n s . Given a graph H, let $\mathcal{C}$ be a set of edge-disjoint subgraphs whose edge sets, those non-empty, form a partition of the edge set E(H) of H and such that:

(4.2) each naked vertex is in exactly one and each non-naked vertex in exactly two members of $\mathcal{C}$.

Let sets $\tilde{\mathcal{C}}$, $\tilde{\mathcal{C}}_o$ and $\tilde{\mathcal{C}}_{oo}$ consist of complete subgraphs, mutually disjoint loop-graphs, and mutually disjoint complete 1-graphs, respectively. Then $\mathcal{C}$ is called:

1. D-decomposition of H if $\mathcal{C} = \tilde{\mathcal{C}}$;
2. D_o-decomposition of H if $\mathcal{C} = \tilde{\mathcal{C}}_o \cup \tilde{\mathcal{C}}$;
3. D_{oo}-decomposition of H if $\mathcal{C} = \tilde{\mathcal{C}}_{oo} \cup \tilde{\mathcal{C}}$, every two loops incident to adjacent vertices of H belong to one 1-graph in $\tilde{\mathcal{C}}_{oo}$ and, for each member of $\tilde{\mathcal{C}}_{oo}$, its vertex set is contained in the vertex set of a member of $\tilde{\mathcal{C}}$.

In what follows α stands for the empty symbol, o, or oo.

(4.3) Definition. The definition of $\tilde{D}_\alpha$-decomposition is obtained from the above definition of D_α-decomposition by replacing the condition (4.2) with the less restrictive one:

(4.4) each vertex is in one but not in more than two of members of $\mathcal{C}$.

(4.5) Proposition. The existence of a D_α-decomposition is equivalent to the existence of a $\tilde{D}_\alpha$-decomposition (α = o, oo or α is the empty symbol).

In fact, a D_α-decomposition is clearly a $\tilde{D}_\alpha$-decomposition. Conversely, adding an appropriate set of trivial subgraphs to a given $\tilde{D}_\alpha$-decomposition of H gives a D_α-decomposition (in which each non-naked vertex is contained in two members). Q.E.D.

In what follows we shall make use of D_α-decompositions only. Note, however, that Harary [3] presents Krausz' theorem in terms of a $\tilde{D}$-decomposition.

Now note that only $\mathcal{F}_2$-graph can have a D-decomposition, only $\mathcal{M}_{1,2}$-graph in which no vertex incident to a loop is doubly adjacent to another vertex can have a D_o-decomposition, and only $\mathcal{M}_{1,2}$-graph can have a D_{oo}-decomposition.

(4.6) Theorem. A graph H is a line $\mathcal{F}$-graph iff there exists a D-decomposition of H.

P r o o f . Necessity. Let H be the line $\mathcal{F}$-graph of a graph M. Assume (without loss of generality) that M has no isolated vertices. Then to every vertex x of M with ℓ_x loops attached to x there corresponds a complete subgraph $K^{(x)}$ of H (= $L_{\mathcal{F}}(M)$) of order $d_M(x) - \ell_x$ whose vertices are edges of M incident to x. Let $\tilde{\mathcal{C}}_2 = \{K^{(x)} : x \in V(M)\}$.

Since, by the definition of $L_{\mathcal{F}}(M)$, doubly adjacent edges of M are doubly adjacent vertices of H, we may assume that each edge of H belongs to exactly one member of $\tilde{\mathcal{C}}_2$. Let V_1 be the set of non-isolated vertices of H each of which belongs to exactly one member of $\tilde{\mathcal{C}}_2$. So if $b \in V_1$ then the only member of $\tilde{\mathcal{C}}_2$ containing b is clearly a non-trivial subgraph. Therefore the union of $\tilde{\mathcal{C}}_2$ and the set $\{K_1 : V(K_1) \subseteq V_1\}$ is a D-decomposition of H.

Sufficiency. Let $\mathcal{C} = \tilde{\mathcal{C}}$ be a D-decomposition of a graph H. Let $\tilde{\mathcal{C}}_1$ be the set of duplicates of those (trivial) members K_1 of $\tilde{\mathcal{C}}$ each of which contains an isolated vertex of H. Let M be an $\mathcal{F}$-graph with $V(M) = \tilde{\mathcal{C}} \cup \tilde{\mathcal{C}}_1$ and with $|V(C_i) \cap V(C_j)|$ edges connecting different vertices $C_i, C_j \in \tilde{\mathcal{C}} \cup \tilde{\mathcal{C}}_1$. Now it is easily seen that $L_{\mathcal{F}}(M)$ is isomorphic to H. ■

(4.7) C o r o l l a r y . (Krausz [5]). A simple graph H is a line $\mathcal{G}$-graph iff there exists a D-decomposition of H.

Arguments similar to that used above (with modifications in dealing with loops) can be used to prove two following theorems.

(4.8) T h e o r e m . A graph H is a line graph iff H has a D_o-decomposition.

(4.9) T h e o r e m . A graph H is an augmented line graph iff there exists a D_{oo}-decomposition of H.

5. Forbidden subgraph characterizations

Our aim is to characterize different types of line graphs under consideration in large enough classes of graphs by giving the minimal (with respect to the inclusion) lists of forbidden induced subgraphs.

Such characterization of simple line graphs for simple graphs is found by Beineke [1] and that for loop-free as well as (cf. (3.3)) for general graphs is found by Bermond and Meyer [2] and Hemminger [4].

The following theorem of Beineke will play a crucial role in proving the sufficiency of the conditions given in the next theorems.

(5.1) Theorem. (Beineke [1]). An $\mathcal{O}$-graph H is the line $\mathcal{O}$-graph of an $\mathcal{O}$-graph iff none of the nine graphs shown in Fig.2 is isomorphic to an induced subgraph of H.

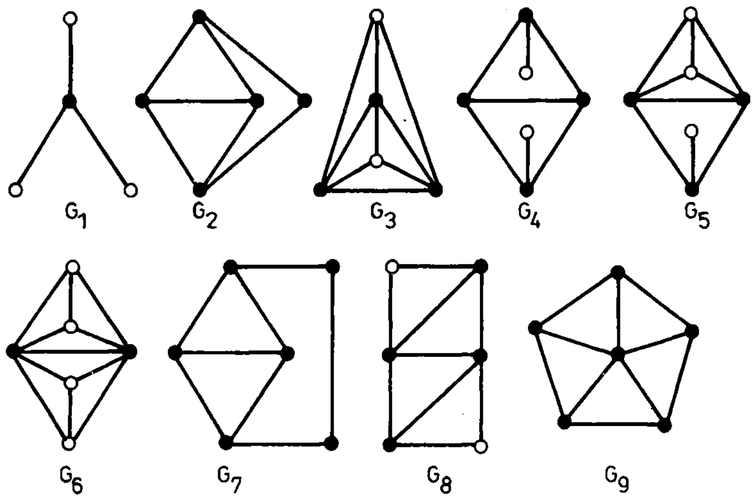


Fig.2

In what follows similar lists of forbidden induced subgraphs will be displayed. It can be easily seen that Beineke's list as well as those given below are minimal in the sense that no graph on a list is an induced subgraph of another one on the list.

The necessity of conditions given in any of forbidden subgraph characterizations can be easily proved by using the corresponding Krausz-type theorem.

In the proofs the following easy observation is essentially needed.

(5.2) P r o p o s i t i o n. Any induced subgraph of a particular type of a line graph is a line graph of the same type.

In what follows proofs of necessity are left to the reader. In general, the proofs of sufficiency are more complicated.

It can be observed that the lists given in Beineke's theorem as well as in our below-stated theorems are finite because those theorems provide characterizations in restricted classes of graphs. The corresponding characterizations in the class $\mathcal{M}$ of all graphs can be obtained by adding suitable infinite lists of flowers, that is, graphs either with one vertex and attached loops or with two multiply adjacent vertices possibly with single loops at them.

5.1. Loop-free line graphs

Note that, by (3.2), the set of loop-free line graphs and that of augmented loop-free line graphs both coincide. Thus a single theorem will characterize the two types of loop-free line graphs.

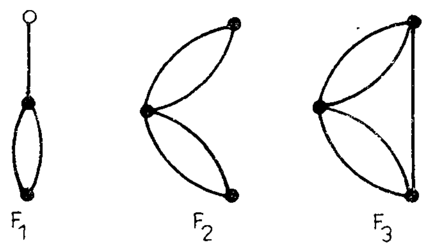


Fig.3

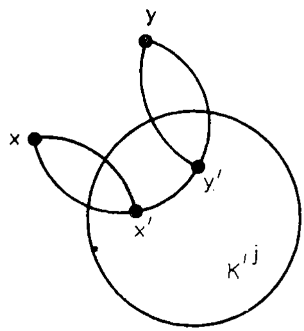


Fig.4

Given an $\mathcal{F}_2$-graph H, let A_H and D_H be respectively adjacency relation and double adjacency relation both in V(H). So if $H \in \mathcal{F}_2$ then

$$xA_Hy \Leftrightarrow x,y \in V(H),\ x \neq y, \text{ and } \exists e \in E(H):\ \eta_H(e) = \{x,y\},$$

$$xD_Hy \Leftrightarrow x,y \in V(H),\ x \neq y, \text{ and } \exists e_1,e_2 \in E(H):\ e_1 \neq e_2 \text{ and } \eta_H(e_i) = \{x,y\} \text{ for } i = 1,2.$$

Observe that $D_H \subseteq A_H$ and both A_H and D_H are symmetric. Define

$$R_H = D_H \cup I_H$$

where I_H is the identity relation over V(H). So R_H is symmetric and reflexive. The transitivity of R_H and a certain useful property of A_H and D_H are equivalent to that the 2-graphs F_1, F_2 and F_2 of Fig.3 do not occur in H as induced subgraphs. This is precised in the following obvious lemma.

(5.3) Lemma. For any $\mathcal{F}_2$-graph H, the following equivalences hold true:

(i) R_H is an equivalence relation iff neither F_2 nor F_3 is isomorphic to an induced subgraph of H.

(ii) For any two doubly adjacent vertices x,y of H, if a vertex z is adjacent to x then either z is adjacent to y or z = y, that is (by the symmetry of A_H and D_H),

$$A_H \circ D_H \subset A_H \cup I_H \quad \text{and} \quad D_H \circ A_H \subset A_H \cup I_H, \tag{5.4}$$

iff neither F_1 nor F_2 is isomorphic to an induced subgraph of H. ∎

The statement (ii) is equivalent to the following

(5.5) Corollary. Sets of neighbours of any two doubly adjacent vertices of H coincide iff neither F_1 nor F_2 is isomorphic to an induced subgraph of H.

(5.6) Theorem. An $\mathcal{F}_2$-graph H is a line $\mathcal{F}$-graph [an augmented line $\mathcal{F}$-graph] iff none of the 12 graphs of Figs.2 and 3 is isomorphic to an induced subgraph of H.

Sufficiency. Because complete $\mathcal{F}_2$-graphs are line $\mathcal{F}$-graphs, without any loss of generality, assume that H is a non-trivial graph whose no component is a complete $\mathcal{F}_2$-graph.

Now, by Lemma (5.3), $R = R_H$ is an equivalence relation over V(H) and (5.4) holds true.

Let f be a choice function for the quotient set V/R with $V = V(H)$ and $R = R_H$. So $f: V/R \to V$ and the following condition is satisfied

$$A \in V/R \Rightarrow f(A) \in A.$$

Moreover, let H' be a subgraph of H induced by $f[V/R]$, the image of V/R under f. Because R is no total relation therefore H' contains at least two vertices. Note that, by definitions of R and H', H' is a simple graph. Now suppose that x is an isolated vertex of H'. Then, by Corollary (5.5), the set $[x]_R$ induces a component of H being a complete $\mathcal{F}_2$-graph, a contradiction. Hence H' is an $\mathcal{O}$-graph without trivial components. Since, moreover, none of the graphs of Fig.2 occurs in H' as an induced subgraph, H' is the line graph of an $\mathcal{O}$-graph.

Therefore, by Krausz theorem, there is a D-decomposition $\{K'^j : j \in J\}$ of H' into complete subgraphs K'^j. We shall construct a D-decomposition $\{K^j : j \in J\}$ of H such that $K'^j \subseteq K^j$ for all $j \in J$. Put

$$V_j = \bigcup_{x \in V(K'^j)} [x]_R \quad \text{for} \quad j \in J. \tag{5.7}$$

In connection with this definition of V_j note that $[x]_R = f^{-1}(x)$ for any $x \in V(H')$ and, moreover, $V_i \neq V_j$ for $i \neq j$ because then $K'^i \neq K'^j$. By the definition of f, and properties of the D-decomposition of H', each vertex from V is in the relation R with an element of $V(K'^j)$ for some $j \in J$. Hence, the family $\{V_j : j \in J\}$ of sets V_j is a covering of the vertex set V of the graph H. Moreover,

(5.8) For each $j \in J$, every two different elements x, y of V_j are adjacent (or doubly adjacent) vertices in H.

In fact, this is clear if either $x, y \in V(K'^j)$ or xRy. So consider the remaining case. Then either x or y, say x, does not belong to $V(K'^j)$. Therefore there exists $x' \in V(K'^j)$ such that xRx' and $x \neq x'$, that is, x is doubly adjacent to x'. Now, if $y \in V(K'^j)$ then y is adjacent to x. Indeed, this is obvious if $y = x'$. But if $y \neq x'$ then y is adjacent to x' and $x' D_H x$. Therefore, by (5.4), y is adjacent also to x. Otherwise, if $y \notin V(K'^j)$ then there is a vertex $y' \in V(K'^j)$, which is doubly adjacent to y. Moreover, either $y' = x'$ or y' is adjacent to x' (cf. Fig.4).

Therefore, by (5.4), y is adjacent to x' and consequently also to x, which completes the proof of (5.8).

Now since H' has no trivial component, each vertex of H' lies in exactly two graphs from the D-decomposition. Hence, by (5.7), we have

(5.9) each vertex of H belongs to exactly two sets from the family $\{V_j : j \in J\}$.

Moreover, any two different vertices x, y doubly adjacent in H, i.e., belonging to the same element $[x]$ of V/R, belong simultaneously to exactly two different sets defined by (5.7). In fact, $f([x])$ is a vertex of exactly two different graphs from the D-decomposition of H', say $f([x]) \in V(K'^i \cap K'^j)$, $i, j \in J$, $i \neq j$. Therefore, by (5.7), $x, y \in V_i \cap V_j$ as desired.

Conversely, if two different vertices x, y belong simultaneously to two different sets V_i and V_j, $i, j \in J$, then they are doubly adjacent in H, i.e., $[x] = [y]$. Really, for otherwise $f([x]) \neq f([y])$ and (since $\{x,y\} \subseteq V_i \cap V_j$) $\{f([x]), f([y])\} \subseteq V(K'^i \cap K'^j)$. Thus there is in H a single edge with end-vertices $f([x])$ and $f([y])$, which belongs to H' and lies in two different graphs K'^i, K'^j, a contradiction to the definition of the D-decomposition of H'.

So we have proved that any two different vertices belong simultaneously to two different sets V_i, V_j iff they are doubly adjacent in H.

Furthermore, we can prove that for any two vertices x and y which are adjacent but not doubly adjacent in H, there is a single set V_j containing them. Indeed, then, by (5.4), $f([x])$ and $f([y])$ are adjacent vertices of H' which belong therefore to a single graph K'^i. Hence, by (5.7), $x, y \in V_i$. Since x and y are not doubly adjacent, there is no other set V_j containing both x and y. Hence and from (5.8) it follows the following proposition.

(5.10) The number of different sets V_j containing any two different vertices x, y of H is equal to the number of edges joining x and y in H.

Let $g : E(H) \to J$ be a function satisfying the conditions:

(1) $g(e) = j \Rightarrow \eta_H(e) \subseteq V_j$,

(2) g associates different values with any multiple edges e_1, e_2, that is, $(e_1 \neq e_2, \eta_H(e_1) = \eta_H(e_2)) \Rightarrow g(e_1) \neq g(e_2)$. The existence of the function g is guaranted by (5.10).

Define

$$K^j = \langle V_j, g^{-1}(j) \rangle \quad \text{for} \quad j \in J.$$

Observe that each K^j is a well-defined subgraph of H, which in fact is a complete subgraph of H. All the graphs K^j are mutually edge-disjoint and cover the vertices and edges of H. Furthermore, by (5.9), any vertex of H belongs to precisely two of graphs K^j. So $\{K^j : j \in J\}$ is a D-decomposition of H. Hence, by Theorem (4.6), H is a line $\mathcal{F}$-graph.

5.2. General line graphs

(5.11) L e m m a . A graph G is a line graph iff G has no adjacent loops and its loop-free transform $\psi(G)$ is a line $\mathcal{F}$-graph.

P r o o f . It is easily seen that ψ transforms a $\tilde{D}_0$-decomposition of G onto a D-decomposition of $\psi(G)$. Conversely, from a D-decomposition of $\psi(G)$ it is possible to get a $\tilde{D}_0$-decomposition of G by replacing some K_2's or pairs K_2, K_1 with loop-graphs. Thus Lemma follows from Proposition (4.5) and Krausz-type Theorems (4.6) and (4.8). ■

Let M be a connected $\mathcal{F}$-graph non-isomorphic to K_2. Denote by $V_h = V_h(M)$ and $E_h = E_h(M)$ the set of hanging vertices and the set of hanging edges of M, respectively.

Suppose that J is a subset of E_h that is a matching of M. Given $M \neq K_2$, let $S_J(M)$ denote the graph obtained from M by replacing each edge e of J by a single loop joined to the non-hanging vertex of e.

Let W be a subset of vertices of M that are not incident to any loop. Denote by $T_W(M)$ the graph, obtained from M by attaching a single loop to every vertex in W.

(5.12) **Lemma.** Let a graph H have no adjacent loops, let $M \neq K_2$ and let $\psi(H)$ be the loop-free transform of H. Then M is isomorphic to an induced subgraph of $\psi(H)$ iff there exists a subset J of $E_h(M)$ that is a matching of M and a subset W of loop-free vertices of $S_J(M)$ such that $T_W(S_J(M))$ is isomorphic to an induced subgraph of H.

Proof. If M is an induced subgraph of $G = \psi(H)$ and H has no adjacent loops then the set $J = J' \cap E(M)$ where $J' = \{e \in E(G) : \eta_G(e) \cap (V(G) - V(H)) \neq \emptyset\}$ is a matching of M and no element of the set

$$W = \{x \in V(M) \cap V(H) \mid \exists e \in J' - E(M) : x \in \eta_G(e)\}$$

is incident to a loop of $S_J(M)$. Then obviously $T_W(S_J(M))$ is an induced subgraph of H.

Conversely, if there are appropriate sets W and J such that $N := T_W(S_J(M))$ is an induced subgraph of H then $\psi(S_J(M))$ is equal to M, and $\psi(N)$, an induced subgraph of G, is equal to M together with hanging edges which replace those loops of N which are incident to the vertices belonging to W. Thus M is an induced subgraph of G.

(5.13) **Theorem.** A 2-graph H is a line graph iff H has no adjacent loops and none of 29 graphs of Figs.2,3,5 and 6 is isomorphic to an induced subgraph of H.

Proof. Denote by A and B the set of 12 graphs of Figs.2 and 3 and the set of 29 graphs of Figs.2,3,5 and 6, respectively.

Sufficiency. Assume that H is a graph without adjacent loops and no element of B is isomorphic to an induced subgraph of H. Let $M \in A$. Put

$P_M = \{N : N = T_W(S_J(M))$, where $J \subseteq E_h(M)$, J is a matching of M, and W is a set of loop-free vertices of $S_J(M)\}$.

Let $C = \bigcup_{M \in A} P_M$.

It follows from the Lemma (5.12) that the following statement is true:

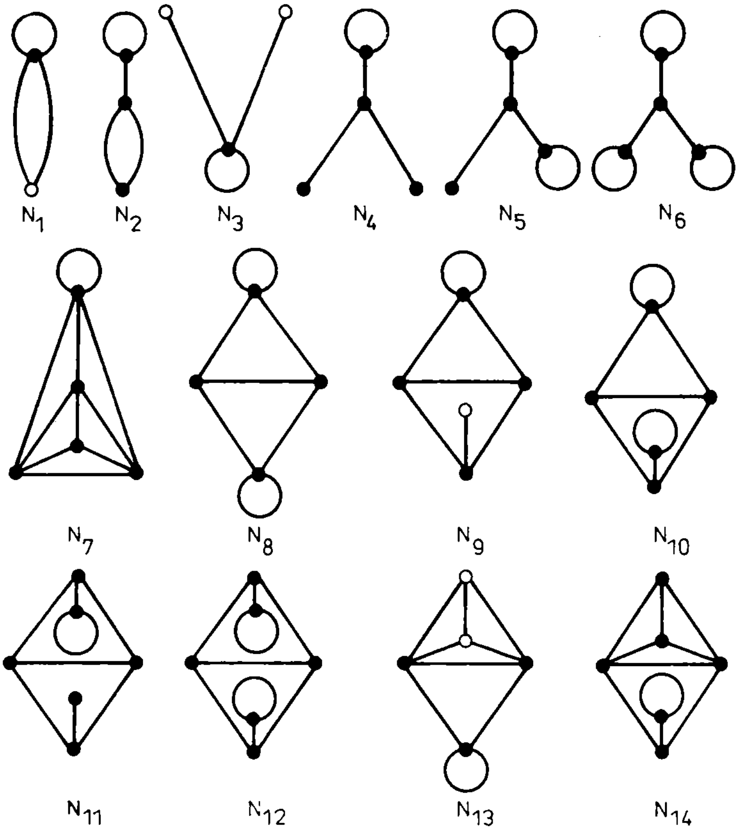


Fig.5

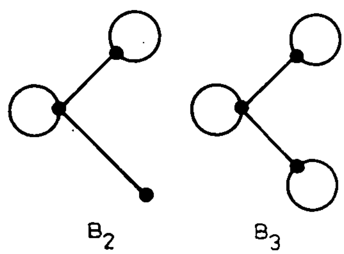


Fig.6

None of the graphs belonging to C is an induced subgraph of H iff no graph from A occurs as induced subgraph of $\psi(H)$.

So if H has no induced subgraph isomorphic to a graph from C, then, by the Lemma (5.11) and Theorem (5.6), H is a line graph.

Observe that the family B is minimal, that is, no element of B is isomorphic to an induced subgraph of another member of B. Moreover, by Proposition (5.2), any induced subgraph of a line graph is also a line graph and, as it will be clear later, B is contained in C.

Therefore, in order to end the proof it suffices to show that each member of C has an induced subgraph isomorphic to a certain graph belonging to B.

Observe first that the only graphs in A which have hanging edges are G_1, G_4, G_5, and F_1. Let us list the images of those last 4 graphs under S_J, for all J permissible:

$$S_{\{e\}}(G_1) = N_3,\quad S_{\{e\}}(G_4) = N_9,\quad S_{\{e,f\}}(G_4) = N_8,$$

$$S_{\{e\}}(G_5) = N_{13},\quad S_{\{e\}}(F_1) = N_1$$

(see the Figs.2, 3 and 5; e,f stand for non-adjacent hanging edges in corresponding graphs).

Let us apply the transformations T_W to all graphs from the family $\tilde{A} = A \cup \{N_1, N_3, N_8, N_9, N_{13}\}$. First W is taken as a subset of "white" vertices depicted in Figs.2, 3 and 5 as small circles. We get the following list:

$$N_2 = T_{\{x\}}(F_1),\quad N_4 = T_{\{x\}}(G_1),\quad N_5 = T_{\{x,y\}}(G_1),$$

$$N_6 = T_{\{x,y,z\}}(G_1),\quad N_7 = T_{\{x\}}(G_3),\quad N_{10} = T_{\{x\}}(N_9),$$

$$N_{11} = T_{\{x\}}(G_4),\quad N_{12} = T_{\{x,y\}}(G_4),\quad N_{14} = T_{\{x\}}(G_5)\ (x \in V_h(G_5)),$$

$$B_1 = T_{\{x\}}(N_1),\quad B_2 = T_{\{x\}}(N_3),\quad B_3 = T_{\{x,y\}}(N_3).$$

For other admissible non-empty subsets W of "white" vertices we obtain graphs which have N_8 (for $G = G_3$ or $G = N_{13}$), N_9 or N_{10} (for $G = G_5$ or $G = G_8$) and N_8 or N_{13} (for $G = G_6$) as induced subgraphs.

Let $G \in \tilde{A}$. It is easy to verify that by adding new loops to some of loop-free vertices of G that include one which is drawn as a black point in Figs.2, 3 and 5, we get a graph which has an induced subgraph isomorphic to N_1, N_3, B_1, B_2 or B_3.

5.3. Augmented line graphs

Let M be a 2-graph without adjacent loops. Denote by $\varphi(M)$ a subgraph M' obtained from M by deleting exactly one edge from each pair of parallel edges whose both end-vertices are incident to loops.

Let $\mathcal{C}$ be a D_0-decomposition of a 2-graph M' and let $\mathcal{R}$ be a complete graph belonging to $\mathcal{C}$. Let $V_0(\mathcal{R})$, $\mathcal{L}_{\mathcal{R}}$, and $\tilde{\mathcal{L}}_{\mathcal{R}}$ be the set of all vertices of $\mathcal{R}$ incident to loops in M', the corresponding set of loops of M', and the corresponding set of loop-graphs, respectively.

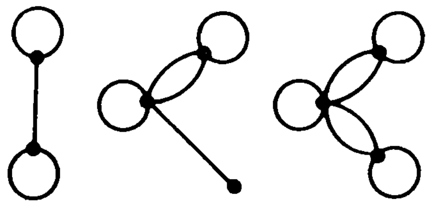
Let us add a duplicate e' of every edge of $\mathcal{R}$ whose both end-vertices belong to $V_0(\mathcal{R})$ (this operation will be denoted φ^{-1}). If $V_0(\mathcal{R}) \neq \emptyset$ then $V_0(\mathcal{R})$ together with the union of the set of new edges e' and the subset $\mathcal{L}_{\mathcal{R}}$ of loops of M' form a complete 1-graph, $Q_{\mathcal{R}}$, which is disjoint from the members of $\mathcal{C} - (\{\mathcal{R}\} \cup \tilde{\mathcal{L}}_{\mathcal{R}})$. It is clear that the family of all complete graphs $\mathcal{R}$ in $\mathcal{C}$ together with the graphs $Q_{\mathcal{R}}$ is a D_{00}-decomposition of $\varphi^{-1}(M')$.

Conversely, one can easily construct a D_0-decomposition of $\varphi(M)$ from a given D_{00}-decomposition of M. Therefore, due to Theorems (4.8) and (4.9), the following Lemma holds true.

(5.14) L e m m a . A 2-graph M without adjacent loops is an augmented line graph iff $\varphi(M)$ is a line graph and $M = \varphi^{-1}(\varphi(M))$. ■

(5.15) T h e o r e m . A 2-graph H is an augmented line graph iff H has no adjacent loops and none graph of Figs.2, 3, 5 and 7 is isomorphic to an induced subgraph of H.

P r o o f . Sufficiency. Suppose that H has no adjacent loops and none of graphs in Figs.2, 3, 5 and 7 is isomorphic to an induced subgraph of H.



A_1 A_2 A_3

Fig.7

Since the graph A_1 in Fig.7 is not isomorphic to any induced subgraph of H then any two adjacent vertices of H both incident to loops are doubly adjacent. Moreover, if M is an induced subgraph of H then $\varphi(M)$ is an induced subgraph of $\varphi(H)$. The converse statement is also true, i.e., if N is an induced subgraph of $\varphi(H)$ then $M = \varphi^{-1}(N)$ is an induced subgraph of H.

Now it is easily seen that if H has no induced subgraph isomorphic to any graph of Figs.2, 3, 5 or to the graphs A_2 or A_3 then none of the graphs of Figs.2, 3, 5, 6 is isomorphic to an induced subgraph of $\varphi(H)$, and, by Lemma (5.14) and Theorem (5.13), H is an augmented line graph. ■

The natural reason for introducing the concept of augmented line graphs is that they can be underlying graphs (i.e., multigraphs) of line digraphs of general digraphs.

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