DEMONSTRATIO MATHEMATICA
Vol. XVIII No 1 1985

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ON SOME EQUIVALENT CONDITIONS IN SUBCARTESIAN SPACES

Dedicated to the memory
of Professor Roman Sikorski

1. Introduction

Let C be a non-empty set of real functions defined on a set M and τ_C - the weakest topology on M such that all functions from C are continuous. By scC we denote the family of all functions of the form $\omega(\alpha_1(\cdot),\dots,\alpha_n(\cdot))$ where $\omega \in C^\infty(R^n)$, $\alpha_1,\dots,\alpha_n \in C$ and $n = 1,2,\dots$. Next, by C_A we denote the set of functions defined as follows: $f \in C_A$ iff $f:A \to R$, $A \subset M$ and, for any point $p \in A$, there exist an open set $U \ni p$ in M as well as a function $\alpha \in C$, such that $f|A \cap U = \alpha | A \cap U$.

R. Sikorski in the paper [6] introduced the concept of differential space (see also [4]). As is known [12], an ordered pair (M,C) is called a differential space if $C = (scC)_M$ and then the set C is called a differential structure on the topological space (M,τ_C).

Analogically as in the theory of C^∞-manifolds we introduce a concept of a vector tangent to (M,C) at a point p of M as an R-linear mapping $v:C \to R$ satisfying the condition $v(\alpha\beta) = \alpha(p)v(\beta) + \beta(p)v(\alpha)$ for any $\alpha,\beta \in C$. The symbol $(M,C)_p$ denotes the tangent space to (M,C) at the point p.

If (M,C) is a differential space and A a subset of M, then (A,C_A) is said to be a differential subspace of (M,C).

Let (M,C) and (N,D) be differential spaces. Then the mapping $f:M \to N$ is called a smooth mapping of the differential space (M,C) into the differential space (N,D) if $\beta \circ f \in C$ for an arbitrary $\beta \in D$. Now if f is a bijection and f and f^{-1} are smooth mappings, then f is called a diffeomorphism.

In paper [10] the author introduced the concept of a differential spaces of the class D_0 as the largest class of differential spaces, containing the class of all differentiable manifolds, for which the theorem on diffeomorphism is fulfilled. In that paper it was proved, too, that the differential space (M,C) is of class D_0 iff for any $p \in M$, there exist an open set $U \ni p$ in M and a C^∞-manifold X such that

(1) $U \subset X$,

(2) $\dim X = \dim(M,C)_p$,

(3) $C_U = (C^\infty(X))_U$.

2. Some equivalent conditions in subcartesian spaces

Let (M,τ) be a topological space. A family $\mathcal{U}$ of mappings is called an atlas on the topological space (M,τ) if the following conditions are satisfied:

(1) if $\varphi \in \mathcal{U}$ than φ is a homeomorphism of a set $D_\varphi \in \tau$ onto a set $\varphi(D_\varphi) \subset R^n$ (not necessarily open),

(2) $M = \bigcup_{\varphi \in \mathcal{U}} D_\varphi$,

(3) if $\varphi, \psi \in \mathcal{U}$, then $\varphi \circ \psi^{-1} \in C^\infty(\psi(D_\varphi \cap D_\psi), \varphi(D_\varphi))$.

It is known that if $\mathcal{U}$ is an atlas on the topological space (M,τ), then there exists exactly one maximal atlas $\mathcal{U}_M$ on (M,τ) such that $\mathcal{U} \subset \mathcal{U}_M$.

Now, a pair $(M, \mathcal{U}_M)$, where $\mathcal{U}_M$ is a maximal atlas on the topological space (M,τ), is called a subcartesian space and the maximal atlas $\mathcal{U}_M$ - a subcartesian structure on (M,τ).

The definition of a subcartesian space accepted here differs from the corresponding definition introduced in [5] in that our definition does not demand the topological space (M,τ) to be Hausdorff. Let us observe that from our definition it results, however, that any subcartesian space is a T_1-space and locally completely regular one ($T_{3\frac{1}{2}}$), which means that any finite subset of M is a closed set.

E x a m p l e 1. Let $\dot{I} = \{x \in R: 0 < x < 1\}$, $a_1, a_2 \notin \dot{I}$, and $a_1 \neq a_2$. Let us put $M = \dot{I} \cup \{a_1, a_2\}$ and let now

$$\varphi_i : \dot{I} \cup \{a_i\} \longrightarrow \dot{I} \cup \{1\}$$

for $i = 1,2$ be mappings such that $\varphi_i | \dot{I} = id_{\dot{I}}$; moreover, let $\varphi_i(a_i) = 1$ for $i = 1,2$. In the set M we introduce the weakest topology with which φ_i, $i = 1,2$, are homeomorphisms of the sets $D\varphi_i = \dot{I} \cup \{a_i\}$, $i = 1,2$, open in M onto the set $\dot{I} \cup \{1\}$.

It is clear that M is not a Hausdorff topological space because the points a_1 and a_2 have no disjoint and open neighbourhoods. Of course M is a subcartesian space determined by the atlas $\mathcal{U} = \{\varphi_1, \varphi_2\}$.

E x a m p l e 2. Let $B_1 = \dot{I} \times \dot{I} \cup \{(0,0)\}$, $B_2 = \dot{I} \times \dot{I} \cup \dot{I} \times \{0\}$ and $M = B_1 \cup B_2$. Let now $\psi_i = id_{\beta_i}$ for $i = 1,2$. Next, we introduce the weakest topology on M such that mappings ψ_1, ψ_2 will be homeomorphisms of the open sets $D_{\psi_i} = B_i$, $i = 1,2$, onto B_i.

Now, of course, M is a Hausdorff topological space, but it is not regular because the point $(0,0)$ and the closed set $\dot{I} \times \{0\}$ in M have no disjoint open neighbourhoods, M is evidently a subcartesian space determined by the atlas $\mathcal{U} = \{\psi_1, \psi_2\}$

Let M be a subcartesian space and let $\mathcal{U}$ be an atlas on M, that is $\mathcal{U} \subset \mathcal{U}_M$ where $\mathcal{U}_M$ is a subcartesian structure on M. Put

$$(1) \qquad C^\infty(M) := \left\{ f: M \longrightarrow R \;\middle|\; \forall \varphi \in \mathcal{U} \;\; f \circ \varphi^{-1} \in C^\infty(\varphi(D_\varphi)) \right\}.$$

Definition (1) is correct because it does not depend on the choice of the atlas $\mathcal{U}$ on M [11]. It is not difficult to show that $C^\infty(M)$ is a linear ring over R of continuous functions on M, closed with respect to the localization and superposition with smooth functions on R^n, $n = 1,2,\dots$, that is $C^\infty(M) = (scC^\infty(M))_M$.

Now, let us denote by $\tau_{C^\infty(M)}$ the weakest topology on M such that all functions from $C^\infty(M)$ are continuous. Of course, there is an inclusion $\tau_{C^\infty(M)} \subset \tau$, where τ is the topology of the subcartesian space M.

As is known, a subcartesian space M is said to be a $C^\infty(M)$-regular if, for any point $p \in M$ and for an arbitrary closed set $F \subset M$, $p \notin F$, there exists $f \in C^\infty(M)$ such that $f(p) = 0$ and $f(q) = 1$ for any $q \in F$.

Now we shall prove the following

P r o p o s i t i o n 1. If M is a subcartesian space, then the following conditions are equivalent:

(a) $\tau_{C^\infty(M)} = \tau$,

(b) M is a $C^\infty(M)$-regular topological space,

(c) M is a completely regular topological space,

(d) M is a regular topological space.

P r o o f . (a) $\Longrightarrow$ (b). Let p_o be an arbitrary point of M, and F an arbitrary closed set of M and $p_o \in F$. Then $M \setminus F$ is an open set in M and $p_o \in M \setminus F$. Since $\tau_{C^\infty(M)} = \tau$, there exist: a sequence of functions $f_1,\dots,f_k \in C^\infty(M)$ and $\varepsilon_o > 0$, such that

$$p_o \in V(p_o,f_1,\dots,f_k;\varepsilon_o) \subset M \setminus F,$$

where $V(p_o,f_1,\dots,f_k;\varepsilon_o) := \left\{ x \in M \,\middle|\, \max_{1\le i\le k} |f_i(k)-f_i(p_o)| < \varepsilon_o \right\}$ is an open set in M. Consider a mapping

$$\phi(x) = (f_1(x),\dots,f_k(x))$$

for x M. Let ρ be a metric on R^k defined as follows

$$\rho(\mu,\lambda) = \max_{1\le i\le k} |\lambda_i - \mu_i|,$$

where $\lambda = (\lambda_1,\dots,\lambda_k)$, $\mu = (\mu_1,\dots,\mu_k)$ and $\lambda,\mu \in R^k$. Of course, the set $A := \left\{ \lambda \in R^k : \rho(\lambda,\phi(p_o)) \ge \varepsilon_o \right\}$ is a closed set in R^k and $\phi(p_o) \notin A$.

So, there exists a function $\omega_0 \in C^\infty(R^k)$ such that $\omega_0(\Phi(p_0)) = 0$ and $\omega_0(\lambda) = 1$ for any $\lambda \in A$.

Now, putting $f_0 = \omega_0 \circ \Phi$, we have $f_0 \in C^\infty(M)$, and $f_0(p_0) = 0$ and $f_0(x) = 1$ for any $x \in F$. Hence M is a $C^\infty(M)$-regular topological space.

The implications: (b) $\Longrightarrow$ (c) and (c) $\Longrightarrow$ (d) are trivial. So, it remains to prove the implication (d) $\Longrightarrow$ (a). To this effect, it is sufficient to prove that if V is an arbitrary open set in M and $p_0 \in V$, then there exist a sequence of functions $f_1,\dots,f_k \in C^\infty(M)$ and $\varepsilon_0 > 0$, such that $p_0 \in V(p_0;f_1,\dots$ $\dots,f_k;\varepsilon_0) \subset V$.

Of course, there exists $\varphi \in \mathcal{U}_M$ such that $p_0 \in W \subset V$ where $W = D_\varphi$. Since the subcartesian space M is, by assumption, a regular topological space, there exist open neighbourhoods U_0 and U_1 such that $p_0 \in U_0$ and $M \setminus W \subset U_1$, and $U_0 \cap U_1 = \phi$. Next, W is a subspace of the subcartesian space M and, besides, it is at the same time a $C^\infty(W)$-regular subcartesian space [11]. Then there exists a function $g \in C^\infty(W)$ such that $g(p_0) = 0$ and $g(x) = 1$ for any $x \in W \setminus U_0$. Let us put

$$f(x) = \begin{cases} g(x) & \text{for } x \in W \\ 1 & \text{for } x \in M \setminus W. \end{cases}$$

Now we shall show that the function f thus defined, belongs to $C^\infty(M)$. In fact, there exists an atlas $\mathcal{U}$ on M such that if $\varphi \in \mathcal{U}$, then $D_\varphi \subset W$ or $D_\varphi \subset U_1$. So, if $D_\varphi \subset W$, then $f \circ \varphi^{-1} = g \circ \varphi^{-1} \in C^\infty(\varphi(D_\varphi))$ since $g \in C^\infty(W)$. Now, if $D_\varphi \subset U_1$, then $f \circ \varphi^{-1} = \text{const.}$ and, consequently, $f \circ \varphi^{-1} \in C^\infty(\varphi(D_\varphi))$. Hence $f \in C^\infty(M)$ and the sought-for neighbourhood is then of the form $V(p_0;f;1)$. Q.E.D.

C o r o l l a r y 1. If a subcartesian space M satisfies any of the equivalent conditions of proposition 1, then it is a differential space with the differential structure $C^\infty(M)$ defined in (1).

From Proposition 1 and from the fact that any subcartesian space is a locally completely regular there results

C o r o l l a r y 2. Any subcartesian space M is a locally differential space, i.e. for any point p M, there exists an open set $U \ni p$ in M such that $(U, C^{\infty}(U))$ is a differential space.

In paper [10] it was proved that, for any subset $A \subset R^n$, $n = 1,2,\ldots$, $(A, (C^{\infty}(R^n))_A)$ is a differential space of class D_0.

So, we obtain

C o r o l l a r y 3. Any subcartesian space M is locally a differential space of the class D_0.

C o r o l l a r y 4. If a subcartesian space M satisfies any of the equivalent conditions of Proposition 1, then it is a differential space of class D_0.

Moreover, the following proposition is true.

P r o p o s i t i o n 2. Let M and N be subcartesian spaces and let $f : M \to N$ be a smooth mapping, $p \in M$. If the mapping $T_p f : M_p \to N_{f(p)}$ is an isomorphism, then there exists an open neighbourhood of the point p in M such that the mapping $f|_U : U \to f(U)$ is a diffeomorphism.

P r o o f . It is sufficient to observe that the theorem on diffeomorphism wears a local complexion and that any subcartesian space is locally a differential space of class D_0 [9].

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Received October 20, 1984.