DEMONSTRATIO MATHEMATICA
Vol. XVIII No 1 1985

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ON THE UNIQUENESS OF SOLUTIONS TO A PARABOLIC SYSTEM OF DIFFERENTIAL-FUNCTIONAL EQUATIONS

Dedicated to the memory of Professor Roman Sikorski

The paper is concerned with the uniqueness of solutions (belonging to an appropriate Sobolev space) to the first Fourier problem for a divergence structure quasi-linear parabolic system of differential-functional equations in a cylindric domain possibly unbounded.

The theory of differential-functional equations and inequalities of parabolic type was originated and developed by J. Szarski [5-9]. These topics were also treated by other authors (e.g. [1, 11-13]).

In all the papers quoted above the functionals appearing in the equations act only on the unknown functions and the solutions are meant in the classical sense. In our paper the uniqueness is proved in a more general function class - a Sobolev space. Moreover, our functionals act both on the unknown functions and their first order x-derivatives. Also the systems treated here are more general than those of the mentioned papers, since they are not assumed to be weakly coupled. Even in the particular case of the differential systems (not containing the functionals) our result is more general because of the same reasons as above (cf. [10]). The results concerning such general differential and differential-functional systems are however obtained mainly due to an increasing condition which, apart from the solutions, is imposed on the first order x-derivatives of the solutions as well.

The method used here is an adaptation of those applied in [1], [10].

Let $t \in \langle 0,T\rangle$, $x = (x_1,\dots,x_n) \in R^n$ and $|x| = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$. Denote by Ω an arbitrary open set (bounded or not) of the space R^n. We define $Q_{t_1,t_2} = \Omega\times(t_1,t_2)$, $Q_t = Q_{0,t}$, $Q = \Omega\times(0,T)$, $\Omega^r = \Omega\cap(|x|<r)$, $Q^r = \Omega^r\times(0,T)$, $S = \partial\Omega\times(0,T)$. Further, we define $u(x,t) = (u^1(x,t),\dots,u^N(x,t)) = (u^k(x,t))_{k=1}^{N}$, $u_x^k(x,t) = (u_{x_1}^k(x,t),\dots,u_{x_n}^k(x,t))$, $u_x(x,t) = (u_x^1(x,t),\dots,u_x^N(x,t))$. Let a vector function u be defined on Q. Then for any fixed $t\in(0,T)$ we denote by $u(\cdot,t) = (u^1(\cdot,t),\dots,u^N(\cdot,t))$ the function $u(\cdot,t):\Omega\ni x \to u(x,t)\in R^n$. The symbols $u_x^k(\cdot,t)$, $(k = 1,\dots,N)$ and $u_x(\cdot,t)$ are defined similarly.

We shall treat of the uniqueness of solutions for the following Fourier problem

$$(1)\quad u_t^k(x,t) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} A_i^k(x,t,u(x,t),u_x(x,t),u(\cdot,t),u_x(\cdot,t)) + \\ + B^k(x,t,u(x,t),u_x(x,t),u(\cdot,t),u_x(\cdot,t)),\ (x,t))\ Q,$$

$$(2)\qquad u^k(x,t)\big|_S = 0,\quad u^k(x,0) = \psi_0^k(x),\quad x\in\Omega,$$

for $k = 1,\dots,N$, where A_i^k, B^k, ψ_0^k are given functions.

The boundary of Q is assumed to be smooth in the sense of the following definition

D e f i n i t i o n 1 ([2], p.224). Let F be an open set of R^{n+1}. The boundary ∂F of F will be said to be smooth, if there exist: numbers $\varepsilon > 0$, $M > 0$, an integer N and a sequence $(V_i)_{i=1}^{\infty}$ of open sets such that

1) if $x \in \partial F$, then the ball $B(x, \varepsilon)$ with the center at x and the radius ε belongs to V_i for a certain i,

2) no point of R^{n+1} belongs to more than N sets V_i,

3) for any i there is a set

$$F_i = \left\{ (\xi, \eta) \in R^{n+1} : \eta > \varphi_i(\xi), \ \xi \in R^n \right\}$$

such that the function φ_i is Lipschitz continuous with the Lipschitz constant $M_i \leq M$ and $V_i \cap F = V_i \cap F_i$.

We shall denote by $W_2^{1,1}(Q^r)$ the Sobolev space of (scalar) functions belonging together with their first order weak derivatives to the space $L^2(Q^r)$. Further $V_2^{1,0}(Q^r)$ will stand for the Sobolev space of all functions measurable on Q^r, continuous with respect to t in the norm of $L^2(\Omega^r)$ and having the finite norm

$$\|u\|_{V_2^{1,0}(Q^r)} = \max_{0 \leq t \leq T} \|u(\cdot, t)\|_{2,\Omega^r} + \|u_x\|_{2,Q^r} .$$

The continuity of u with respect to t in the norm $L^2(\Omega^r)$ means that

$$\|u(\cdot, t+h) - u(\cdot, t)\|_{2,\Omega^r} \longrightarrow 0 \quad \text{as} \quad h \longrightarrow 0.$$

The space $V_2^{1,0}(Q^r)$ is the closure of $W_2^{1,1}(Q^r)$ in the norm of $V_2^{1,0}(Q^r)$ ([3], p. 15).

D e f i n i t i o n 2. Let $E_2(K, \lambda; Q)$ denote the class of functions $u = (u^1, \ldots, u^N)$ measurable on Q and such that

$$\int_Q \sum_{k=1}^{N} \left[(u^k)^2 + |u_x^k|^2 \right] \exp(-K\rho^\lambda) dx dt < \infty ,$$

where $\rho = (|x|^2 + 1)^{\frac{1}{2}}$.

D e f i n i t i o n 3. A function φ will be said to belong to the class $\phi(Q)$, if

1^o $\varphi \in W_2^{1,1}(Q)$,

2^o there exists $r(\varphi) > 0$ such that $\varphi = 0$ on $Q \setminus Q^r$,

3^o the trace of φ on the plane $t = T$ and on the surface S is equal to zero.

D e f i n i t i o n 4. A function $u = (u^1, \ldots, u^N)$ defined on the set Q will be called a weak solution in this set to the Fourier problem (1)-(2), if $u^k \in V_2^{1,0}(Q^r)$ $(k=1,\ldots,N)$, for any r, and, for any system of functions $(\varphi^k)_1^N$ belonging to $\phi(Q)$, the following identities are satisfied

$$(3)\quad \int_Q \Big[-u^k\varphi^k_t + \sum_{i=1}^n \varphi^k_{x_i} A^k_i(x,t,u,u_x,u(\cdot,t),u_x(\cdot,t))\Big] dx\, dt +$$

$$-\int_\Omega \psi_0^k \varphi^k(x,0)dx = \int_Q \varphi^k B^k(x,t,u,u_x,u(\cdot,t),u_x(\cdot,t))dx\, dt,$$

$$k=1,\ldots,N.$$

We make the following assumptions.

A s s u m p t i o n H_1. Let $A_i^k(x,t,u,p,v,q)$ $(i=1,\ldots,n;\ k=1,\ldots,N)$ be measurable in $(x,t) \in Q$ and continuous in the remaining variables: $u \in R^N$, $p \in R^{n+N}$, $v \in \left(L^2_{loc}(\Omega)\right)^N$, $q \in \left(L^2_{loc}(\Omega)\right)^{n+N}$. There exist positive constants $\varkappa$, B_0, B, C, K_1 and $0 < \lambda \leqslant 2$ such that

$$(i)\quad \sum_{i=1}^n \sum_{k=1}^N \left(p_i^k - \tilde{p}_i^k\right)\left[A_1^k(x,t,u,p,v,q) - A_i^k(x,t,\tilde{u},\tilde{p},\tilde{v},\tilde{q})\right] \geqslant$$

$$\geqslant \varkappa \sum_{i=1}^n \sum_{k=1}^N \left(p_i^k\right)^2 - B\,\varrho^\lambda\Big(\sum_{k=1}^N (u^k - \tilde{u}^k)^2 +$$

$$+ \sum_{k=1}^N \|v^k - \tilde{v}^k\|^2_{K_1}\Big) - B_0\,\varrho^\lambda \sum_{i=1}^n \sum_{k=1}^N \|q_i^k - \tilde{q}_i^k\|^2_{K_1},$$

where $\|h\|_{K_1} = \left(\int_\Omega (h(x))^2 \exp(-K_1 \varrho^\lambda) dx \right)^{\frac{1}{2}}$,

(ii) $$\left| A_i^k(x,t,u,p,v,q) - A_i^k(x,t,\tilde u,\tilde p,\tilde v,\tilde q) \right| \le$$

$$\le C \Big(\varrho \sum_{l=1}^{N} |u^l - \tilde u^l| + \varrho^{\frac{2-\lambda}{2}} \sum_{j=1}^{n} \sum_{l=1}^{N} |p_j^l - \tilde p_j^l| +$$

$$+ \varrho^{\frac{2-\lambda}{2}} \sum_{l=1}^{N} \|v^l - \tilde v^l\|_{K_1} + \varrho^{\frac{2-\lambda}{2}} \sum_{j=1}^{n} \sum_{l=1}^{N} \|q_j^l - \tilde q_j^l\|_{K_1} \Big),$$

$i=1,\dots,n;\ k=1,\dots,N.$

A s s u m p t i o n H_2. Let $B^k(x,t,u,p,v,q)$ $(k=1,\dots,N)$ be measurable in (x,t) and continuous in the remaining variables (as in Assumption H_1). There exists positive constants D, K_1 and $0<\lambda\le 2$ such that

$$\left| B^k(x,t,u,p,v,q) - B^k(x,t,\tilde u,\tilde p,\tilde v,\tilde q) \right| \le$$

$$\le D \Big(\varrho^\lambda \sum_{l=1}^{N} |u^l - \tilde u^l| + \varrho^{\frac{\lambda}{2}} \sum_{i=1}^{n} \sum_{l=1}^{N} |p_i^l - \tilde p_i^l| +$$

$$+ \varrho^\lambda \sum_{l=1}^{N} \|v^l - \tilde v^l\|_{K_1} + \varrho^{\frac{\lambda}{2}} \sum_{i=1}^{n} \sum_{l=1}^{N} \|q_i^l - \tilde q_i^l\|_{K_1} \Big), \quad k=1,\dots,N.$$

A s s u m p t i o n H_3. $\psi_0^k \in L^2_{loc}(\Omega)$.

Now we shall show that Assumptions H_1, H_2 ensure the measurability of the integrands in identities (3).

We denote by $M(X,F)$ the linear space of functions measurable on X with the range in the space F.

L e m m a 1. Let F be a separable metrizable space. Suppose that a functional $f: Q \times R^N \times F \longrightarrow R^1$ satisfies two conditions:

(i) for any $u \in R^N$, $v \in F$ the function $f(x,t,u,v)$ is measurable in (x,t),

(ii) for almost all $(x,t) \in Q$ the function f is continuous in (u,v).

Then for arbitrary fixed measurable functions $u \in M(Q,R^N)$, $v \in M(\langle 0,T\rangle, F)$ the function $\tilde{f} : Q \longrightarrow R^1$, where $\tilde{f}(x,t) = f(x,t,u(x,t), v(t))$, is measurable.

Proof. Let $u(x,t)$ be measurable on Q and let $v \in F$. Then $f(x,t,u(x,t),v(t))$ is a measurable function. Indeed, let (u_n) be a sequence of step functions such that $u_n(x,t) \longrightarrow u(x,t)$ as $n \longrightarrow \infty$ almost everywhere on Q. By (i), the functions $f_n(x,t) = f(x,t,u_n(x,t),v(t))$ are measurable on Q. Further, (ii) implies that $\lim_{n\to\infty} f_n(x,t) = f(x,t,u(x,t),v(t))$ almost everywhere on Q. Let $y_0 \in M(\langle 0,T\rangle, F)$. Then, by virtue of the theorem 23_2 of [4], there is a sequence (y_n) of step functions belonging to $M(\langle 0,T\rangle, F)$ such that $y_n(t) \longrightarrow y_0(t)$, as $n \longrightarrow \infty$, in the metric of the space F, for almost all $t \in \langle 0,T\rangle$. By the previous facts the functions $g_n(x,t) = f(x,t,u(x,t), y_n(t))$ are measurable on Q. This and (ii) imply that $\lim_{n\to\infty} g_n(x,t) = f(x,t,u(x,t), y_0(t))$ for almost all $(x,t) \in Q$, and whence, in view of the theorem 23 of [4] the measurability of the function $f(x,t,u,(x,t), y_0(t))$ follows. The proof of the lemma is thus completed.

Corollary 1. By Assumptions $H_1 - H_2$, Lemma 1 and Corollary 1 of [4] the functions

$$(*) \quad \tilde{A}_i^k(x,t) = A_i^k(x,t,u(x,t),\ u_x(x,t),\ u(\cdot,t),\ u_x(\cdot,t)),$$

$$(**) \quad \tilde{B}^k(x,t) = B^k(x,t,u(x,t),\ u_x(x,t),\ u(\cdot,t),\ u_x(\cdot,t))$$

$(i=1,\dots,n;\ k=1,\dots,N)$ are measurable on Q.

Theorem 1. If Assumptions $H_1 - H_2$ are satisfied and $L < \varkappa$, where $L = B_0 \int_\Omega \varrho^\lambda \exp(-K\varrho^\lambda)dx$, then the Fourier problem (1)-(2) has at most one solution in the class $E_2(K,\lambda;Q)$ with $0 < K < K_1$.

R e m a r k 1. If in equations (1) there appear the functionals of the form

$$A_i^k = A_i^k(x,t,u,u_x,u(\cdot,t)),$$

$$B^k = B^k(x,t,u,u_x,u(\cdot,t),\ u_x(\cdot,t)),$$

then the condition $L < \varkappa$ in Theorem 1 is superfluous. If, moreover, $B^k = B^k(x,t,u,u_x,u(\cdot,t))$ $(k=1,\dots,N)$, then the problem (1)-(2) possesses at most one solution in the wider class

$$\tilde{E}_2(K,\lambda;Q) = \left\{ u : \int_Q \sum_{k=1}^{N} (u^k)^2 \exp(-K\varrho^\lambda)\ dx\ dt < \infty \right\}.$$

P r o o f of Theorem 1. Let u, $\tilde{u}$ be any solutions of problem (1)-(2), of class $E_2(K,\lambda;Q)$ and let $w = u - \tilde{u}$. We shall write $A_i^k(u)$, $B^k(u)$ short for the right-hand sides of $(*)$, $(**)$, respectively. According to the definition of a solution, for any system of functions $(\varphi^k)_1^N$ which belong to the class $\phi(Q)$ we have the equalities

$$(4)\quad -\int_Q \sum_{k=1}^{N} w^k \varphi_t^k\ dx\ dt + \int_Q \sum_{i=1}^{n} \sum_{k=1}^{N} \varphi_{x_i}^k \left(A_i^k(u) - A_i^k(\tilde{u})\right) dx\ dt =$$

$$= \int_Q \sum_{k=1}^{N} \varphi^k \left(B^k(u) - B^k(\tilde{u})\right) dx\ dt.$$

A function g defined on Q being given, we introduce its averages:

$$g_h(x,t) = \frac{1}{h} \int_t^{t+h} g(x,\tau)d\tau, \quad 0 < h \leq \delta, \quad (x,t) \in Q_{T-\delta},$$

$$g^h(x,t) = \frac{1}{h} \int_{t-h}^{t} g(x,\tau)d\tau, \quad 0 < h \leq \delta, \quad (x,t) \in Q_{\delta,T}.$$

We shall make use of the following property of the averages ([3], p.167).

If functions f, g together with their squares are summable in the interval $\langle -\delta, T\rangle$ and one of them vanishes in the intervals $\langle -\delta, 0\rangle$ and $\langle T-\delta, T\rangle$, then

$$\int_0^T f(t)g^h(t)dt = \int_0^{T-\delta} f_h(t)g(t)dt.$$

As the test functions in (4) we take $(\varphi^k)^h$ $(k=1,\dots,N)$, where $\varphi^k \in \phi(Q_{-\delta,T})$, $\varphi^k(x,t) = 0$ for $t \leqslant 0$ and $t \geqslant T-\delta$. The first integral in (4) is transformed in accordance with the mentioned property of the averages as follows

$$-\int_0^T\int_\Omega w^k[(\varphi^k)^h]_t\, dxdt =$$

$$= -\int_0^{T-\delta}\int_\Omega (w^k)_h\varphi_t^k dxdt = \int_0^{T-\delta}\int_\Omega (w^k)_{ht}\,\varphi^k dxdt.$$

Here we took the advantage of the fact that, if $u \in V_2^{1,0}(Q^r)$, then $u_h \in W_2^{1,1}(Q^r_{T-\delta})$ (see Lemma 4.7 of [3], p.101) and of the properties of φ^k. Consequently, (4) can be rewritten as

$$\int_0^{T-\delta}\int_\Omega \sum_{k=1}^N (w^k)_{ht}\varphi^k dxdt + \tag{5}$$

$$+\int_0^{T-\delta}\int_\Omega \sum_{k=1}^N \sum_{i=1}^n \varphi^k_{x_i}[A_i^k(u) - A_i^k(\tilde{u})]_h\, dxdt =$$

$$= \int_0^{T-\delta}\int_\Omega \sum_{k=1}^N \varphi^k[B^k(u) - B^k(\tilde{u})]_h\, dxdt.$$

Equality (5) holds for arbitrary functions $\varphi^k \in \phi(Q_{-\delta,T})$ such that $\varphi^k(x,t) = 0$ for $t \leqslant 0$ and $t \geqslant T-\delta$.

We shall construct special test functions for (5). To this end we extend functions u^k and $\tilde{u}^k$ in the following way

$$\hat{u}^k(x,t) = \begin{cases} u^k(x,2T-t), & t\in\langle T,T+\delta\rangle, \\ u^k(x,t), & t\in\langle 0,T\rangle, \\ u^k(x,-t), & t\in\langle -\delta,0\rangle, \end{cases}$$

$$\check{u}^k(x,t) = \begin{cases} \tilde{u}^k(x,2T-t), & t\in\langle T,T+\delta\rangle, \\ \tilde{u}^k(x,t), & t\in\langle 0,T\rangle, \\ \tilde{u}^k(x,-t), & t\in\langle -\delta,0\rangle. \end{cases}$$

Note that since $\hat{u}^k$, $\check{u}^k\in V_2^{1,0}(Q^r_{-\delta,T+\delta})$ for all r therefore $(\tilde{w}^k)_h \equiv (\hat{u}^k - \check{u}^k)_h \in W_2^{1,1}(Q^r_{-\delta,T})$ for all r. Further, suppose $\xi(x,t)$ is a non-negative smooth function defined on the zone $R^n\times(0,T)$ with bounded support while

$$J_s(t) = \begin{cases} 0, & t\leqslant 0, \\ st, & 0\leqslant t\leqslant \frac{1}{s}, \\ 1, & \frac{1}{s}\leqslant t\leqslant t_1 - \frac{1}{s}, \\ s(t_1-t), & t_1 - \frac{1}{s}\leqslant t\leqslant t_1, \\ 0, & t\geqslant t_1, \end{cases}$$

with $t_1\leqslant T-\delta$. Notice also that $(\tilde{w}^k)_h\big|_S = 0$.

Now it is easy to see that functions

(6) $$\varphi^k_s(x,t) = (\tilde{w}^k)_h\, \xi\, J_s(t), \quad (x,t)\in Q_{-\delta,T} \quad (k=1,\ldots,N)$$

have all the properties required from the test functions for (5). Let us substitute functions (6) into (5) and let $s\to\infty$. Hence, by Assumptions H_1, H_2, we obtain for any $t_1\leqslant T-\delta$

$$(7)\qquad \int_0^{t_1}\int_\Omega \sum_{k=1}^{N} (w^k)_{ht}(w^k)_h\,\xi\,dxdt\;+$$

$$+\int_0^{t_1}\int_\Omega \sum_{i=1}^{n}\sum_{k=1}^{N} \left[(w^k)_h\xi\right]_{x_i}\left[A_i^k(u) - A_i^k(\tilde u)\right]_h dxdt =$$

$$= \int_0^{t_1}\int_\Omega \sum_{k=1}^{N} (w^k)_h\cdot\xi\cdot\left[B^k(u) - B^k(\tilde u)\right]_h dxdt.$$

Now we transform the first integral in (7) in the following way

$$\int_0^{t_1}\int_\Omega \sum_{k=1}^{N} (w^k)_{ht}(w^k)_h\,\xi\,dxdt = \frac{1}{2}\int_0^{t_1}\int_\Omega \sum_{k=1}^{N} \left\{\left[(w^k)_h\right]^2\right\}_t \xi\,dxdt =$$

$$= \frac{1}{2}\int_\Omega \sum_{k=1}^{N}\left[(w^k)_h\right]^2\xi\,dx\Big|_0^{t_1} - \frac{1}{2}\int_0^{t_1}\int_\Omega \sum_{k=1}^{N}\left[(w^k)_h\right]^2\xi_t\,dxdt.$$

From Assumptions H_1, H_2 and Lemma 4.7 of [3], p.101, it follows that $(w^k)_h \longrightarrow w^k$ in the norm of the space $V_2^{1,0}(Q^r)$, $\left[A_i^k(u) - A_i^k(\tilde u)\right]_h \longrightarrow A_i^k(u) - A_i^k(\tilde u)$ in the norm of $L^2(Q^r)$, $\left[B^k(u) - B^k(\tilde u)\right]_h \longrightarrow B^k(u) - B^k(\tilde u)$ in the norm of $L^2(Q^r)$ with any r. Therefore in (7) one can take the limit as $h \longrightarrow 0$. Thus, for $t_1 \in \langle 0, T-\delta\rangle$ we get

$$(8)\qquad \frac{1}{2}\int_\Omega \sum_{k=1}^{N}\left[w^k(x,t_1)\right]^2\xi(x,t_1)dx - \frac{1}{2}\int_0^{t_1}\int_\Omega \sum_{k=1}^{N}(w^k)^2\xi_t\,dxdt\;+$$

$$+\int_0^{t_1}\int_\Omega \sum_{i=1}^{n}\sum_{k=1}^{N}(w^k\xi)_{x_i}\left[A_i^k(u) - A_i^k(\tilde u)\right]dxdt =$$

$$= \int_0^{t_1} \int_\Omega \sum_{k=1}^{N} w^k \xi \left[B^k(u) - B^k(\tilde{u}) \right] dxdt.$$

Equality (8) also holds for $t_1 \in (0,T\rangle$. To this purpose it is sufficient to extend, in (4), the functions u and $\tilde{u}$ onto the interval $\langle 0,T+\delta\rangle$ in the same way as it was done in the construction of the test functions φ_s^k. Having extended functions $\hat{u}$ and $\hat{\tilde{u}}$ we then extend functions A_i^k and B^k onto $\langle 0, T+\delta\rangle$. Reasoning as before leads to (8) with $t_1 \in (0,T\rangle$.

Let us define $\xi(x,t) = \gamma(x)\, G(x,t)$, $G(x,t) = \exp(-\bar{K}\,\varrho^\lambda e^{\mu t})$, where $K < \bar{K} < K_1$ and $\gamma(x)$ is of class $C^2(R^n)$ such that $0 \leqslant \gamma(x) \leqslant 1$, $\gamma(x) = 1$ for $|x| < r$, $\gamma(x) = 0$ for $|x| > r+1$, $|\gamma_x(x)| \leqslant \gamma_0$, γ_0 being a constant independent of r.

Denote

$$T_1 = \min\left(\frac{1}{\mu},\ \frac{1}{\mu}\ln\frac{K_1}{\bar{K}},\ T\right).$$

By the inequalities

(a) $ab \leqslant \varepsilon a^2 + \dfrac{b^2}{4\varepsilon}$, $\varepsilon > 0$,

(b) $\exp(-K_1\varrho^\lambda) \leqslant G(x,t) \leqslant \exp(-K\varrho^\lambda)$, $t \in \langle 0,\dot{T}_1\rangle$,

and Assumptions H_1, H_2, the third and fourth integrals in (8) can be estimated as follows

$$I_3 \geqslant \varkappa \int_0^{t_1}\int_\Omega \xi \sum_{i,k} \left(w^k_{x_i}\right)^2 dx\, dt -$$

$$- B_0 \int_0^{t_1}\int_\Omega \xi\,\varrho^\lambda \sum_{i,k} \left\| w^k_{x_i}(\cdot,t)\right\|^2_{K_1} dxdt +$$

$$- B\int_0^{t_1}\int_\Omega \xi\,\varrho^\lambda \left[\sum_k (w^k)^2 + \sum_k \left\|w^k(\cdot,t)\right\|^2_{K_1}\right] dxdt +$$

$$- C\int_0^{t_1}\int_\Omega \sum_{i,k} |w^k|\,|\xi_{x_i}| \left[\sum_{j,l} \varrho^{\frac{2-\lambda}{2}} \left|w^l_{x_j}\right| + \right.$$

$$+\varrho\sum_{l}|w^l| + \varrho^{\frac{2-\lambda}{2}}\sum_{l}\|w^l(\cdot,t)\|_{K_1} +$$

$$+\varrho^{\frac{2-\lambda}{2}}\sum_{j,l}\left\|w^l_{x_j}(\cdot,t)\right\|_{K_1}\Bigg] dxdt \geqslant$$

$$\geqslant \frac{1}{5}(4\varkappa+L)\int_0^{t_1}\int_\Omega \gamma\, G\sum_{i,k}(w^k_{x_i})^2\, dxdt +$$

$$-\frac{1}{5}(2\varkappa+3L)\int_0^{t_1}\int_\Omega G\sum_{i,k}\left(w^k_{x_i}\right)^2 dxdt -$$

$$-\int_0^{t_1}\int_\Omega\sum_{i,j,k}|\gamma_{x_i}|\, G\left(w^k_{x_j}\right)^2 dxdt -$$

$$-NC\int_0^{t_1}\int_\Omega \varrho\, G\sum_{i,k}|\gamma_{x_i}|\,(w^k)^2\, dxdt -$$

$$-M_1\Bigg[\int_0^{t_1}\int_\Omega \gamma\, G\,\varrho^\lambda\sum_k (w^k)^2\, dxdt + \int_0^{t_1}\int_\Omega G\sum_k (w^k)^2\, dxdt +$$

$$+\int_0^{t_1}\int_\Omega\sum_{i,k}|\gamma_{x_i}|\,\varrho^{2-\lambda}\, G(w^k)^2\, dxdt\Bigg],$$

where the positive constant M_1 depends on $\varkappa$, B_0, B, C, n, N, λ and K. Then we have

$$I_4 \leqslant D\int_0^{t_1}\int_\Omega \xi\sum_k |w^k|\Bigg[\varrho^\lambda\sum_l |w^l| + \varrho^{\frac{\lambda}{2}}\sum_{i,l}|w^l_{x_i}| +$$

$$+\varrho^\lambda\sum_l \|w^l(\cdot,t)\|_{K_1} + \varrho^{\frac{\lambda}{2}}\sum_{i,l}\|w^l_{x_i}(\cdot,t)\|_{K_1}\Bigg] dxdt \leqslant$$

$$\leqslant M_2\left[\int_0^{t_1}\int_\Omega \gamma G \varrho^\lambda \sum_k (w^k)^2\, dxdt + \int_0^{t_1}\int_\Omega G \sum_k (w^k)^2\, dxdt\right] +$$

$$+ \frac{1}{5}(\varkappa - L)\left[\int_0^{t_1}\int_\Omega \gamma G \sum_{i,k}\left(w^k_{x_i}\right)^2 dxdt + \right.$$

$$\left. + \int_0^{t_1}\int_\Omega G \sum_{i,k}\left(w^k_{x_i}\right)^2 dxdt\right],$$

M_2 being a positive constant depending on $\varkappa$, B_0, D, n, N, λ and K. Taking into consideration the estimates of I_3 and I_4 we obtain from (8), for $t_1 \in (0,T_1\rangle$,

$$(9)\quad \frac{1}{2}\int_\Omega \sum_k \left[w^k(x,t_1)\right]^2 \gamma(x) G(x,t_1)dx +$$

$$+ \frac{1}{2}\int_0^{t_1}\int_\Omega \sum_k (w^k)^2 \gamma G \bar{K} \varrho^\lambda e^{\mu t}\cdot \mu\, dxdt +$$

$$+ \frac{1}{5}(3\varkappa + 2L)\int_0^{t_1}\int_\Omega \gamma G \sum_{i,k}\left(w^k_{x_i}\right)^2 dxdt \leqslant$$

$$\leqslant \int_0^{t_1}\int_\Omega \sum_{i,j,k} |\gamma_{x_i}|\; G\left(w^k_{x_j}\right)^2 dxdt +$$

$$+ \frac{1}{5}(3\varkappa + 2L)\int_0^{t_1}\int_\Omega G \sum_{i,k}\left(w^k_{x_i}\right)^2 dxdt +$$

$$+ NC\int_0^{t_1}\int_\Omega \varrho\, G \sum_{i,k} |\gamma_{x_i}|\; (w^k)^2\, dxdt +$$

$$+ (M_1+M_2)\left[\int_0^{t_1}\int_\Omega \gamma\, G\, \varrho^\lambda \sum_k (w^k)^2\, dxdt + \right.$$

$$\left. + \int_0^{t_1}\int_\Omega G \sum_k (w^k)^2\, dxdt + \int_0^{t_1}\int_\Omega \sum_{i,k} |\gamma_{x_i}|\, \varrho^{2-\lambda} G (w^k)^2\, dxdt\right].$$

Notice that for a sufficiently large μ_0 we have

$$2(M_1+M_2) - \frac{1}{2} K\mu_0 \leq 0.$$

Moreover, since $w \in E_2(K,\lambda;Q)$, for any $\varepsilon > 0$ there exists r so large that

$$(M_1+M_2)\left[\int_0^{T_0}\int_{\Omega^{r+1}\setminus\Omega^r} G\, \varrho^\lambda \sum_{k=1}^N (w^k)^2\, dxdt + \right.$$

$$\left. + \int_0^{T_0}\int_{\Omega\setminus\Omega^r} G \sum_{k=1}^N (w^k)^2\, dxdt\right] < \frac{\varepsilon}{4},$$

$$NC \int_0^{T_0}\int_{\Omega^{r+1}\setminus\Omega^r} G\, \varrho \sum_{k=1}^N \sum_{i=1}^n |\gamma_{x_i}| (w^k)^2\, dxdt +$$

$$+ (M_1+M_2)\int_0^{T_0}\int_{\Omega^{r+1}\setminus\Omega^r} G\, \varrho^{2-\lambda} \sum_{i,k} |\gamma_{x_i}|\, (w^k)^2\, dxdt < \frac{\varepsilon}{4},$$

$$\frac{1}{5}(3\varkappa+2L)\int_0^{T_0}\int_{\Omega\setminus\Omega^r} G \sum_{i=1}^n \sum_{k=1}^N \left(w^k_{x_i}\right)^2 dxdt < \frac{\varepsilon}{4},$$

$$\int_0^{T_0}\int_{\Omega^{r+1}\setminus\Omega^r} \sum_{i,j=1}^n \sum_{k=1}^N G\, |\gamma_{x_i}| \left(w^k_{x_j}\right)^2 dxdt < \frac{\varepsilon}{4},$$

where $T_0 = \min\left(\frac{1}{\mu_0}, \frac{1}{\mu_0} \ln\frac{K_1}{\bar{K}}, T\right)$. Consequently, from (9) we obtain

$$\int_{\Omega^r} \sum_{k=1}^{N} (w^k)^2 G \, dx \Big|_{t=\bar{t}} < \varepsilon \quad \text{for} \quad \bar{t} \in (0,T_0) \text{ and any } r.$$

This means that $\int_{\Omega} \sum_{k=1}^{N} (w^k)^2 G \, dx = 0$ in $(0,T_0)$. We have thus shown that $u = \tilde{u}$ on $\Omega \times (0,T_0)$. If $T_0 = T$ the proof is complete. In case $T_0 < T$ the proof is obtained after a finite number of repetitions.

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Received September 11, 1984.