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FOLIATIONS OF DIFFERENTIAL SPACES

Dedicated to the memory of Professor Roman Sikorski

In this paper the concept of foliation on a Sikorski differential space (see [3] and [2]) with leaves which are differential spaces is introduced. There is proved a theorem on concordance of the concept of foliation in category of differential spaces with the one in category of differentiable manifolds, when a foliated differential space is a differentiable manifold. The essential role in the proof plays the theorem (see [8]) which says that if the Cartesian product of two differential spaces is a differentiable manifold, so they are differentiable manifolds, too.

1. Introduction

For any differential space (d.s.) M of the form (S,C) the set S of all points of this d.s. will be denoted by $\underline{M}$ and its differential structure C we will denote by $F(M)$. The weakest topology on S such that all functions $\alpha \in C$ are continuous will be denoted by $\mathrm{top}M$. For any subset A of the set S the differential subspace of M with the set A as the set of all points will be denoted by M_A, i.e. $M_A = (A, C_A)$, where C_A is the set of all real functions β defined on A such that for any point p of A there exist $\alpha \in C$ and $U \in \mathrm{top}M$ fulfilling the equality $\beta|A \cap U = \alpha|A \cap U$. It is evident that $\mathrm{top}M_A$ coincides with the topology induced to A by topM. For the basic facts and definitions consult [2], [3] and [4].

The following concept of local homogenenity of a family of d.s. will play an essential part in the considerations contained in the paper. A family $\mathcal{F}$ of d.s. is said to be locally homogeneous (l.h.) iff for any K and L of $\mathcal{F}$ and any points p and q of K and L, respectively, there exist neighbourhoods U and V of points p and q such that the d.s. K_U and L_V are diffeomorphic[1]. A d.s. L is said to be locally homogeneous (l.h.) iff the family $\{L\}$ is l.h. We remark that if $\mathcal{F}$ is a l.h. family of d.s. such that there exist K and a non-empty open subset V of K with K_V being k-dimensional differentiable manifold, then every $L \in \mathcal{F}$ is a k-dimensional differentiable manifold. It is evident that if $\mathcal{F}$ is l.h. and there exist $K \in \mathcal{F}$, non-empty open set U and smooth vector fields $W_1,\dots,W_k$ on K_U such that $W_1(p),\dots,W_k(p)$ constitute a basis of the tangent space T_pK for $p \in U$, then every $L \in \mathcal{F}$ is of finite dimension k (see [4] and [5]). Similarly, if $\mathcal{F}$ is l.h. for which there exist $K \in \mathcal{F}$ and the point p in K such that T_pK is k-dimensional, then T_qL is k-dimensional for any $L \in \mathcal{F}$ and any point q in L.

Let us remark that any family of d.s. which are differentiable manifolds of the same dimension is l.h. and if all d.s. belonging to l.h. family are differentiable manifolds, then they are of the same dimension.

2. Foliations

Let M be a d.s. and l.h. family $\mathcal{F}$ of disjoined differential spaces such that $\underline{M} = \bigcup_{L \in \mathcal{F}} \underline{L}$. $\mathcal{F}$ will be called a foliation on M iff the following conditions are satisfied:

(i) L is connected and regularly lying in M for $L \in \mathcal{F}$ (see [6]);

(ii) for any point p of M there exists a neighbourhood V of p in $\mathcal{F}_p$, where p is a point of $\mathcal{F}_p \in \mathcal{F}$, a d.s. N and a diffeomorphism

1) By a diffeomorphism which sends p into q.

$$(1) \qquad \varphi : M_U \longrightarrow \mathcal{F}_{p\dot{V}} \times N$$

such that for every $L \in \mathcal{F}$ any connected in topM component of $\underline{L} \cap U$ is of the form $\varphi^{-1}[V \times \{b\}]$, where $b \in \underline{N}$, U being a neighbourhood of p in M.

The elements of $\mathcal{F}$ will be called leaves of $\mathcal{F}$.

E x a m p l e . Let L be a connected l.h. d.s., N be a d.s. and $M = L \times N$. Then the family of all differential subspaces of M with the sets of all points of the form $\underline{L} \times \{q\}$, where q is any point of N, constitute a foliation on M. In particular, L may be the set of all reals together with the structure generated by the set $\{x \mapsto \cos\frac{x}{n};\ n \in Z\} \cup \{x \mapsto \sin\frac{x}{n};\ n \in Z\}$ (see [1]). This d.s. is of dimension 1 in the sense of R. Sikorski (see [4] and [5]), but it is not a differentiable manifold.

Concordance of the concept of foliation in category of d.s. with the standard concept of foliation of a differentiable manifold when a d.s. is a differentiable manifold seems to be of importance. We recall that a family $\mathcal{F}$ of disjoined connected immersed submanifolds of a differentiable manifold M of C^∞ class is a foliation of differentiable C^∞-submanifolds of M iff there exists a natural $k \leqslant \dim M$ such that

(iii) for any point p of M there exists a chart $x: D_x \longrightarrow A_1 \times A_2$ on M such that $p \in D_x$, $x(p) = 0$, A_1 is open in R^k, A_2 is open in R^{m-k} and for any $L \in \mathcal{F}$ every connected component of $\underline{L} \cap D_x$ is of the form $x^{-1}[A_1 \times \{c\}]$, where $c \in A_2$.

T h e o r e m . If a d.s. M is a differentiable C^∞-manifold, then any family $\mathcal{F}$ of d.s. is a foliation on M if and only if $\mathcal{F}$ is a foliation of differentiable C^∞-submanifolds of M.

P r o o f . Assume that $\mathcal{F}$ is a foliation of differentiable C^∞-submanifolds of M. Let p be any point of M. Then we have exactly one $\mathcal{F}_p \in \mathcal{F}$ such that p is a point of $\mathcal{F}_p$. According to (iii) there exists a chart $x: D_x \longrightarrow A_1 \times A_2$ such that $p \in D_x$, $x(p) = 0$, A_1 and A_2 are open in R^k and R^{m-k}, respectively, and all connected components of the parts of leaves of $\mathcal{F}$ lying in D_x are of the form $x^{-1}[A_1 \times \{b\}]$, where

$b \in A_2$. We have the immersion $id: \mathcal{F}_p \longrightarrow M$. Therefore there exists a chart y around the point p on $\mathcal{F}_p$ such that $y(p) = 0$, the domain D_y of y is a connected subset of $\mathcal{F}_p$ and $y[D_y] \subset A_1$. Let N be the natural d.s. of the set A_2. We set

$$\varphi(t) = (y^{-1}(pr_1 x(t)), pr_2 x(t)) \quad \text{for} \quad t \in U,$$

where

$$(2) \qquad U = (pr_1 \circ x)^{-1}[y[V]] \cap (pr_2 \circ x)^{-1}[A_2], \; V = D_y.$$

So, we have the set U open in R^k and the bijective mapping $\varphi: U \longrightarrow V \times A_2$. It is easy to check that $U = x^{-1}[y[V] \times A_2]$ and $\varphi^{-1}(z) = x^{-1}(y(pr_1 z), pr_2 z)$ for $z \in V \times A_2$. Therefore we have the diffeomorphism (1) fulfilling the condition $\varphi(p) = (p, 0)$. Let us take any $L \in \mathcal{F}$, any connected component C of $\underline{L} \cap U$ in topM and any $q \in C$. Let S denote the component of $\underline{L} \cap D_x$ connected in topM such that $q \in S$. Then $S = x^{-1}[A_1 \times \{b\}]$, where $b \in A_2$. We have, by (2), $D_x \supset U$. So, $\underline{L} \cap D_x \supset \underline{L} \cap U \supset C \ni q$. Thus $S \supset C$, $x[C] \subset x[S] \subset A_1 \times \{b\}$, $x[C] \subset x[U] \subset y[V] \times A_2$, $x[C] \subset y[V] \times \{b\}$, and $C \subset x^{-1}[y[V] \times \{b\}]$. By the assumption, $y[V]$ is a connected subset of A_1. Therefore $x^{-1}[y[V] \times \{b\}]$ is also connected in topM and $x^{-1}[y[V] \times \{b\}] \subset S \cap U \subset \underline{L} \cap U$. This yields the inclusion $x^{-1}[y[V] \times \{b\}] \subset C$. Hence it follows that $C = x^{-1}[y[V] \times \{b\}] = \varphi^{-1}[V \times \{b\}]$, where $b \in \underline{N}$. So, $\mathcal{F}$ is a foliation of d.s. on M.

Now, assume that a family $\mathcal{F}$ of d.s. is a foliation on M. Taking a point p of M we get $\mathcal{F}_p \in \mathcal{F}$ such that there exist a neighbourhood V of p open in $\mathcal{F}_p$ and a d.s. N for which $\mathcal{F}_{pV} \times N$ is diffeomorphic to M_U, where U is a neighbourhood of p in M. Hence it follows (see [8]) that $\mathcal{F}_{pV}$ is a differentiable manifold. By l.h.-property of $\mathcal{F}$ this yields that all elements of $\mathcal{F}$ are differentiable manifolds of the same dimension, say k. From (i) it follows that they are immersed submanifolds of M. Let $p \in \underline{M}$. Hence there exists a diffeomorphism (1), where $p \in U \in \mathrm{top}M$, $\varphi(p) \in V \times \underline{N}$, $V \in \mathrm{top}\ \mathcal{F}_p$. Let us set $\varphi(p) = (p_1, p_2)$, where $p_1 \in V$, $p_2 \in \underline{N}$. Then there exist a chart y on $\mathcal{F}_p$ around p_1 and a chart z on N around p_2 such that D_y is connected and included in V, $y(p_1) = 0$ and $z(p_2) = 0$. Let us set $A_1 = y[D_y]$, $A_2 = z[D_z]$ and

$$(3) \qquad x(t) = (y(pr_1\varphi(t)),\ z(pr_2\varphi(t))) \quad \text{for} \quad t \in D_x,$$

where $D_x = \varphi^{-1}[D_y \times D_z]$. We have then the open in M set D_x and the sets A_1 and A_2 open in R^k and R^{m-k}, respectively, $m = \dim M$, such that

$$(4) \quad x^{-1}(u) = \varphi^{-1}(y^{-1}(pr_1 u),\ z^{-1}(pr_2 u)) \quad \text{for} \quad u \in A_1 \times A_2.$$

Therefore x is a chart on M around p. with $x(p) = 0$. Now, take any $L \in \mathcal{F}$ and any connected component C of $\underline{L} \cap D_x$. We have $D_x \subset U$. There exists a connected component S of the set $\underline{L} \cap U$ such that $C \subset S$. We have then $S = \varphi^{-1}[V \times \{b\}]$, where $b \in \underline{N}$. Consequently, $\varphi[C] \subset \varphi[S] \subset V \times \{b\}$, $\varphi[C] \subset (pr_1 \circ \varphi)[C] \times \{b\}$, $(pr_1 \circ \varphi)[C] \subset (pr_1 \circ \varphi)[D_x] = D_y \subset V$, $\varphi[C] \subset D_y \times \{b\}$, and $C \subset \varphi^{-1}[D_y \times \{b\}]$. The set $\varphi^{-1}[D_y \times \{b\}]$ is connected. So, $\varphi^{-1}[D_y \times \{b\}] \subset S \subset \underline{L}$. For some $p_0 \in C$ we have $\varphi(p_0) = (p_0, b)$ and $\varphi(p_0) \in \varphi[D_x] = D_y \times D_z$. This yields $b \in D_z$. So $\varphi^{-1}[D_y \times \{b\}] \subset \varphi^{-1}[D_y \times D_z] = D_x$. Thus $\varphi^{-1}[D_y \times \{b\}] \subset \underline{L} \cap D_x$. Hence it follows that $C = \varphi^{-1}[D_y \times \{b\}]$, and we get $C = \varphi^{-1}[y^{-1}[A_1] \times z^{-1}[\{c\}]]$, where $c = z(b) \in A_2$. Using (4), it is easy to check that $C = x^{-1}[A_1 \times \{c\}]$. The condition (iii) is then fulfilled. Q.E.D.

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