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INDUCING AND COINDUCING OF ANALYTICAL PREMANIFOLDS BY MAPPINGS

Dedicated to the memory of Professor Roman Sikorski

In the present paper the concept of analytical premanifold (a.p.) is examined. This concept being related to Sikorski's concept [3] (cf. [1]) of differential space (see [4]) and to Postnikov's concept of premanifold [2] was introduced by W. Waliszewski [6]. The present paper contains constructions related to the ones in Waliszewski's paper [5] of the a.p. induced by a set of mappings, the coinduced one, and the universal characterisations of them.

0. Generating of a.p. and restricted mappings

We adopt all concepts and denotations as in [6]. There is proved there that for any set G of real functions the set $(\text{an } G)_{\underline{G}}$ is the smallest among all a.p. containing G. This a.p. is called the a.p. generated by G. We have Top G = = $\text{Top}((\text{an } G)_{\underline{G}})$.

P r o p o s i t i o n 1. If M and N are a.p. and N is generated by a set G of real functions then for any function f which maps the set $\underline{M}$ into $\underline{N}$ we have smooth mapping

(1) $$f: M \longrightarrow N$$

if and only if $\beta \circ f \in M$ for $\beta \in G$.

(For any functions g and f without restriction for domain D_g of g we have well defined the composition $g \circ f$ on the set $f^{-1}[D_g]$ of all points $p \in D_f$ such that $f(p) \in D_g$ by the usual formula $(g \circ f)(p) = g(f(p))$).

P r o o f . Assume that $\beta \circ f \in M$ for $\beta \in G$. First, we remark that we have a continuous mapping f: Top M $\longrightarrow$ Top G. Let $B \in$ Top G and $p \in f^{-1}[B]$. Then there exist sets $B_1, \ldots, B_s$ open in R and $\beta_1, \ldots, \beta_s \in G$ such that $f(p) \in \beta_1^{-1}[B_1] \ldots \cap \cap \beta_s^{-1}[B_s] \subset B$. Hence, $p \in (\beta_1 \circ f)^{-1}[B_1] \cap \ldots \cap (\beta_s \circ f)^{-1}[B_s] \subset \subset f^{-1}[B]$, where $\beta_1 \circ f, \ldots, \beta_s \circ f \in M$. So, $f^{-1}[B] \in$ Top M. Let φ be a real analytic function on an open subset of R^n and $\beta_1, \ldots, \beta_n \in G$. We have then $\varphi(\beta_1, \ldots, \beta_n) \circ f = \varphi(\beta_1 \circ f, \ldots, \beta_n \circ f) \in$ an M = M. Now, let $\gamma \in N$. Set $\alpha = \gamma \circ f$. Let $p \in D_\alpha$. Thus, we have $f(p) \in \gamma$. Therefore, there exist $\beta \in$ an G and a set $V \in$ Top G such that $V \subset D_\beta$, $f(p) \in V \subset D_\gamma$ and $\gamma|V = \beta|V$. Setting $U = f^{-1}[V]$ we have $U \in$ Top M, $U \subset f^{-1}[D_\beta] = D_{\beta \circ f}$, $p \in U \subset D_\alpha$ and $\alpha|U = \beta \circ f|U$. So, $\alpha \in M_{\underline{M}} = M$. Q.E.D.

P r o p o s i t i o n 2. For any smooth mapping (1) and any sets A and B such that $f[A] \subset B \subset \underline{N}$ we have the smooth mapping $f|A$: $M_A \longrightarrow N_B$.

P r o o f . First, we assume that $A = \underline{M}$. Let $\beta \in N_B$ and $p \in \underline{M}$. We have $f(p) \in B$. Thus, there exist $\alpha \in N$ and a set $V \in$ Top N such that $V \subset D_\alpha$, $p \in V \cap B \subset D_\beta$ and $\beta|V \cap B = \alpha|V \cap B$. So, $\beta \circ f|f^{-1}[V \cap B] = \alpha \circ f|f^{-1}[V \cap B]$. From the inclusion $f[A] \subset B$ it follows that $f^{-1}[V \cap B] = f^{-1}[V]$. Therefore, $\beta \circ f|f^{-1}[V] = \alpha \circ f|f^{-1}[V]$, $\alpha \circ f \in M$, $f^{-1}[V] \in$ Top M and $p \in f^{-1}[V] \subset D_{\beta \circ f}$. Thus, $\beta \circ f \in M_{\underline{M}} = M$. Now, remark that for $A \subset \underline{M}$ we have a smooth mapping id_A: $M_A \longrightarrow M$. So, $f|A = f \circ id_A$: $M_A \longrightarrow N$. Hence it follows that $f|A$: $M_A \longrightarrow N_B$. Q.E.D.

P r o p o s i t i o n 3. If $f: \underline{M} \longrightarrow \underline{N}$, N is any a.p., $Q \subset$ Top M covers the set $\underline{M}$, and for each $A \in Q$ we have a smooth mapping $f|A$: $M_A \longrightarrow N$, so we have a smooth mapping (1).

P r o o f . Take any $\rho \in N$ and set $\gamma = \rho \circ f$. For any $A \in Q$ we have $\gamma \circ id_A = \rho \circ f | A \in M_A$. Let $p \in D_\gamma$. Then, $p \in A$ for some $A \in Q$. Setting $\beta = \gamma \circ id_A$ we have $\beta \in M_A$. So, there exist $V \in \operatorname{Top} M$ and $\alpha \in M$ such that $V \subset D_\alpha$, $p \in V \cap A \subset D_\beta$ and $\beta | V \cap A = \alpha | V \cap A$. Taking $U = V \cap A$ we have then $U \in \operatorname{Top} M$, $U \subset D_\alpha$, $p \in U \subset D_\gamma$ and $\gamma | U = \alpha | U$. Therefore, $\gamma \in M$. Q.E.D.

By an easy verification we get

P r o p o s i t i o n 4. If M is a.p. and $A \subset \underline{M}$, then for any $\alpha \in M$ such that $D_\alpha \subset A$ we have $\alpha \in M_A$. If $A \in \operatorname{Top} M$, then $M_A \subset M$.

1. The a.p. induced by a set of mappings

Let N be a.p. and F be a set of functions with values in $\underline{N}$. For any functions β and f we set $f^*(\beta) = \beta \circ f$. Then for any $f \in F$ we have the set $f^*[N]$ of real functions. Denote by F^*N the a.p. generated by the union of all sets $f^*[N]$, where $f \in F$. F^*N will be called the a.p. induced from N by the set F of mappings. The following theorem gives a characterization of F^*N.

T h e o r e m 1. The a.p. induced from N by the set F of functions with values in $\underline{N}$ is the exactly one a.p. M satisfying the following conditions:

(i) $f: M_{D_f} \longrightarrow N$ for $f \in F$;

(ii) $\underline{M} = \bigcup_{f \in F} D_f$ and Top M is the induced topology to the set $\underline{M}$ from Top N by all the functions $f \in F$, i.e. Top M coincides with the smallest topology on $\underline{M}$ containing all sets $f^{-1}[A]$, where $A \in \operatorname{Top} N$ and $f \in F$;

(iii) for any a.p. L and any continuous function c from Top L into Top M we have the smooth mapping $c: L \longrightarrow M$ if and only if we have smooth mappings

$$f \circ c: L_{c^{-1}[D_f]} \longrightarrow N \tag{2}$$

for $f \in F$.

P r o o f . Let us take $f \in F$. We have then

$$D_f = f^{-1}[N] = f^{-1}\left[\bigcup_{\beta\in N} D_\beta\right] = \bigcup_{\beta\in N} D_{\beta\circ f} = \bigcup_{\alpha\in f^*[N]} D_\alpha.$$

So, $D_f \in \mathrm{Top}\, F^*N$. For any $\beta\in N$ we have $\beta\circ f \in f^*[N] \subset F^*N$. By Proposition 4 we get $\beta\circ f \in (F^*N)_{D_f}$. So,

(3) $$f: (F^*N)_{D_f} \longrightarrow N \quad \text{for} \quad f\in F.$$

To prove that (ii) is satisfied for $M = F^*N$ remark that for any $f\in F$ and $A\in \mathrm{Top}\, N$, by (3), we have $f^{-1}[A] \in \mathrm{Top}(F^*N)_{D_f}$. So, $f^{-1}[A]\in \mathrm{Top}\, F^*N$. Let $\alpha\in \bigcup_{f\in F} f^*[N]$ and B be open in R. We have then $\alpha = \beta\circ f$, where $\beta\in N$, $f\in F$, and $\alpha^{-1}[B] = f^{-1}[\beta^{-1}[B]] = f^{-1}[A]$, where $A\in \mathrm{Top}\, N$. So, $\mathrm{Top}\, F^*N = \mathrm{Top} \bigcup_{f\in F} f^*[N]$ coincides with the induced topology to the set $\underline{F^*N}$ from $\mathrm{Top}\, N$ by all functions $f\in F$.

Assume that a continuous mapping $c: \mathrm{Top}\, L \longrightarrow \mathrm{Top}\, f^*N$ gives smooth mappings (2) for $f\in F$. Taking any $\beta\in N$ and $f\in F$ we have, by Proposition 4, $\beta\circ f\circ c \in L_{c^{-1}[D_f]} \subset L$. Hence, by Proposition 1, it follows that

(4) $$c: L \longrightarrow F^*N.$$

Now, assuming (4), by Proposition 2, we obtain

$$c|c^{-1}[D_f] : L_{c^{-1}[D_f]} \longrightarrow (F^*N)_{D_f} \quad \text{for} \quad f\in F.$$

So, by (3), we get (2).

To complete the proof of Theorem 1 take any a.p. M satisfying (i) - (iii). By (ii) we have $\mathrm{Top}\, M = \mathrm{Top}\, F^*N$. Setting $L = M$ and $c = id_{\underline{M}}$ from (iii) we conclude that $f: M_{D_f} \longrightarrow N$ for $f\in F$. So $f^*[N] \subset M_{D_f} \subset M$ for $f\in F$. Thus, $F^*N \subset M$. On the other hand, (3) yields (2) for $f\in F$ where $c = id_{\underline{M}}$ and $L = F^*N$. From (iii) it follows that $id_{\underline{M}}: F^*N \longrightarrow M$. So, $M\subset F^*N$. Q.E.D.

2. The a.p. coinduced by a set of mappings

Let M be a.p. and F be any set of functions defined on $\underline{M}$. The a.p. generated by the set of all real functions β with domains contained in $\bigcup_{f \in F} f[\underline{M}]$ and such that $\beta \circ f \in M$ for $f \in F$ will be called the a.p. coinduced from M by the set F of mappings and denoted by FM. The following theorem gives the characterization of FM.

T h e o r e m 2. The a.p. coinduced from M by the set of functions defined on $\underline{M}$ is the exactly one a.p. N satisfying the following conditions:

(iv) $f: M \to N$ for $f \in F$;

(v) $\underline{N} = \bigcup_{f \in F} f[\underline{M}]$;

(vi) for any a.p. P and any function $g: \underline{N} \to \underline{P}$ we have the smooth mapping $g: N \to P$ if and only if

$$g \circ f: M \to P \quad \text{for} \quad f \in F. \tag{5}$$

P r o o f . For N = FM (v) directly follows from the definition of FM. For N = FM (iv) follows from Proposition 1.

Let us take $g: \underline{FM} \to \underline{P}$ such that (5) holds. Taking any $\gamma \in P$ we have $\gamma \circ g \circ f \in M$ for $f \in F$. Thus, $\gamma \circ g \in FM$. So, $g: FM \to P$.

To end the proof assume that N is any a.p. fulfilling (iv) - (vi). Taking $g = id_{\underline{N}}$ and $P = N$, by (vi), we get (5). Hence it follows that $id_{\underline{N}}: FM \to N$. Thus, $N \subset FM$. From the fact that $f: M \to FM$ for $f \in F$ it follows that

$$id_{\underline{N}} \circ f: M \to FM \quad \text{for} \quad f \in F.$$

This yields $id_{\underline{N}}: N \to FM$. Therefore, $FM \subset N$. Q.E.D.

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