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Ω = PER FOR GENERIC VECTOR FIELDS ON SOME OPEN SURFACES

Dedicated to the memory of Professor Roman Sikorski

In this paper we study the properties of nonwandering points of generic vector fields on some open surfaces. We extend the theorem of C. Pugh [6] i.e. we prove that on open surfaces of genus zero there exists a residual set of vector fields Y with $\Omega(Y) = \mathrm{Per}(Y)$. From this it follows that generically there are no oscillating orbits. Moreover together with the main results of my next paper (concerning the sufficiency of Ω-stability of vector fields on open manifolds) it implies that Ω-stability is a generic property for open surfaces of genus zero.

Let M be homeomorphic to S^2 without a countable number of points which form a closed subset of S^2. By E we denote the set of "infinities" of M.

$H^r(M)$ - the space of C^r vector fields on M which generate flows endowed with C^r-Whitney (strong) topology ($r \geq 1$).

$\Phi_Y: M \times R \to M$ the flow of Y.

We denote by $O_Y(x)$ an orbit of Y starting with x i.e. $\Phi_Y(x,0) = x$ and define the positive (resp. negative) semiorbit by

$$O^+_Y(x) = \{\Phi_Y(x,t):\ t>0\},$$

$$O^-_Y(x) = \{\Phi_Y(x,t):\ t<0\}.$$

Finally we denote by $O_Y[x,y]$ the closed Y-orbit segment from x to y.

We distinguish three kinds of asymptotic behavior for each semi-orbit:

(a) $O^{\pm}_Y(x)$ is bounded if it is contained in some compact set $K\subset M$,

(b) $O^{\pm}_Y(x)$ escapes to infinity if for each compact set $K\subset M$ there exists a point $y\in O^{\pm}_Y(x)$ for which $O^{\pm}_Y(y)\cap K = \emptyset$,

(c) $O^{\pm}_Y(x)$ oscillates if it is neither bounded nor escapes to infinity.

These kinds of behavior for $O^+_Y(x)$ (resp. $O^-_Y(x)$) can be also described in terms of the ω-limit (resp. α-limit) set of $x\in M$ under Φ_Y namely.

$$\omega(O^+_Y(x)) = \{y\in M:\ \exists t_n \to +\infty \ni .\ \Phi_Y(x,t_n)\to y\}$$

$$\alpha(O^-_Y(x)) = \{y\in M:\ \exists t_n \to -\infty \ni .\ \Phi_Y(x,t_n)\to y\}.$$

It is easy to see that we can distinguish the following cases:

(a) $O^+_Y(x)$ is bounded iff $\omega(O^+_Y(x))$ is compact (and non-empty)

(b) $O^+_Y(x)$ escapes to infinity iff $\omega(O^+_Y(x)) = \emptyset$,

(c) $O^+_Y(x)$ oscillates iff $\omega(O^+_Y(x))$ is a non-compact subset of M.

We extend the definition of ω-limit (resp. α-limit) set of $x\in M$ to ω^*-limit (resp. α^*-limit) set

$$\omega^*(O^+_Y(x)) = \{y\in M\cup E:\ \exists t_n \to +\infty \ni .\ \Phi_Y(x,t_n)\to y\}$$

$$\alpha^*(O^-_Y(x)) = \{y\in M\cup E:\ \exists t_n \to -\infty \ni .\ \Phi_Y(x,t_n)\to y\}.$$

Thus

(a) $O^+_Y(x)$ escapes to infinity iff there exists $P \in E$ such that $\omega^*(O^+_Y(x)) = \{P\}$,

(b) $O^+_Y(x)$ oscillates iff $\omega(O^+_Y(x)) \neq \emptyset$ and there exists $P \in E$ such that $P \in \omega^*(O^+_Y(x))$.

Let Per(Y), $\Omega(Y)$ denote respectively the periodic points and the non-wandering points of Y i.e.

$$\mathrm{Per}(Y) = \{x \in M: \Phi_Y(x,t) = x \quad \text{for some} \quad t > 0\}$$

$$\Omega(Y) = \{x \in M: \exists x_n \longrightarrow x,\ t_n \longrightarrow +\infty . \ni . \Phi_Y(x_n,t_n) \longrightarrow x\}.$$

By $H^r_o(M)$ we denote the subset of $H^r(M)$ such that every element of $H^r_o(M)$ has only hyperbolic restpoints and we denote by $H^r_{K-S}(M)$ the set of Kupka and Smale vector fields (for definition see [4]). In [5] it was proved that $H^r_o(M)$ is open and dense in $H^r(M)$ but $H^r_{K-S}(M)$ is residual in $H^r(M)$.

We say that vector fields X and Y defined respectively on manifolds N, M are topologically equivalent if there exists a homeomorphism $h : N \longrightarrow M$ taking X-orbits to Y-orbits and respecting time order.

At first we recall some properties of ω-limit (resp. α-limit) set of vector fields on S^2. All these properties were studied in [3].

Let Z be a continuous vector field on S^2 which generates a flow. We say that $O^+_Z(y)$ approaches an orbit $O_Z(x)$ if for any arbitrarilly small transversal section (a,b) at x there exists a sequence (x_n) of succesive common points of $O^+_Z(y)$ with (a,b) such that:

(i) all elements of (x_n) belong either to (a,x) or (x,b),

(ii) if $x_n \in (a,x)$ $(x_n \in (x,b))$, then $x_n \in (x_{n-1},x)$ $(x_n \in (x,x_{n-1}))$ for $n \in N$.

Let

$$\omega^s(O^+_Z(x)) = \{y \in \omega(O^+_Z(x)): Z(y) = 0\},$$

$$\omega^n(O_Z^+(x)) = \omega(O_Z^+(x)) - \omega^s(O_Z^+(x)).$$

Theorem 1. For $x \in S^2$ the set $\omega(O_Z^+(x))$ satisfies:

(a) $\omega(O_Z^+(x))$ is non-empty, closed and connected,

(b) $\omega(O_Z^+(x))$ is the boundary of a simply connected region $G \subset S^2$ (thus $\omega(O_Z^+(x))$ separates S^2),

(c) $\omega^n(O_Z^+(x)) = \omega(O_Z^+(x))$ iff $\omega^n(O_Z^+(x))$ is a periodic orbit,

(d) if $\omega^n(O_Z^+(y)) \subsetneq \omega(O_Z^+(x))$ and $O_Z^+(y) \subset \omega^n(O_Z^+(x))$ then $\omega(O_Z^+(y))$ (resp. $\alpha(O_Z^-(y))$) is contained in some component of $\omega^s(O_Z^+(x))$,

(e) $\omega^n(O_Z^+(x))$ consists of at most a countable number of orbits.

Let A be an arbitrary closed simply connected set in S^2. We say that $h : [0,1) \longrightarrow 2^A$ is a cyclic mapping of A if:

(i) $h(\tau)$ is a closed simply-connected subset of A for every τ, $0 \le \tau < 1$,

(ii) if $\tau = \lim \tau_n$, $\tau, \tau_n \in [0,1)$ then $\bigcap_{n=0}^{\infty} \left(\bigcup_{k \ge n} h(\tau_k)\right) \subseteq$ $\subseteq h(\tau)$ (the set $[0,1)$ we identify with a unit circle).

Theorem 2. The set $\omega(O_Z^+(x))$ for $x \in S^2$ has a cyclic mapping satisfying the following conditions:

(a) for any $\tau \in [0,1)$ $h(\tau)$ is either a regular point or a singular component of $\omega^s(O_Z^+(x))$,

(b) the set of τ's such that $h(\tau)$ are singular components of $\omega^s(O_Z^+(x))$ is closed and nowhere dense in $[0,1)$,

(c) $h^{-1}(\omega^n(O_Z^+(x))) = \bigcup_n (\tau_n, \tau_{n+1})$ and $h|\bigcup_n (\tau_n, \tau_{n+1})$ is a homeomorphism,

(d) suppose that $\tau_1 < \tau_2$ and U_1, U_2 are any non overlapping neighbourhoods of $h(\tau_1)$ and $h(\tau_2)$. Then there exists T dependent only on the choice of U_1 and U_2 such that for $t > T$ the semi-orbit $O_Z^+(x)$ crosses U_1 and U_2 alternately.

Now we define a cycle of vector field $Y \in H_0^r(M)$ and describe in Lemma 1 the relation between $\Omega(Y)$ and cycles of Y.

Next we state Lemma 2 and 3 which enables us to use the properties of ω-limit set of vector fields on S^2 to study the properties of cycles of vector fields.

For $x \in M$ which is not a restpoint of $Y \in H^r(M)$ by (a,x) (resp. (x,b)) we denote an open transversal interval with left end a and with right end x (resp. left end x and right end b).

D e f i n i t i o n 1. Let the sequence $(O_Y[x_n,\bar{x}_n])$ satisfy for any $n \in N$ one and only one of the following conditions:

(i_1) $x_n \in (a,x)$, $x_n \to x$ and the first intersection $\bar{x}_n$ of $O_Y^+(x_n)$ with (a,x) lies between x_n and x_{n+1};

(ii_1) $x_n \in (a,x)$, $x_n \to x$ and the first intersection $\bar{x}_n$ of $O_Y^+(x_n)$ with (a,x) lies between x_{n-1} and x_n;

(iii_1) $x_n \in (a,x)$, $x_n \to x$ and the first intersection x_n of $O_Y^+(x_n)$ with (a,x) satisfies $\bar{x}_n = x_n$;

(i_2)-(iii_2) are similar to (i_1)-(iii_1) using (x,b) instead of (a,x) then we will denote by $C_Y(x)$ the set $\{y \in M \cup E:\ y = \lim z_n \text{ and } z_n \in O_Y[x_n,\bar{x}_n]\}$, which is not a closed orbit of Y through x.

$C_Y^s(x) = \{y \in C_Y(x):\ Y(x) = 0 \vee y \in E\}$

$C_Y^n(x) = C_Y(x) - C_Y^s(x)$.

L e m m a 1. For $Y \in H_0^r(M)$ $x \in \Omega(Y) - \mathrm{Per}(y)$ iff there exists a cycle $C_Y(x)$ of Y through x.

P r o o f . The sufficiency of this condition is obvious. We prove its necessity. Let V_0 be a neighbourhood of x such that if $y \in V_0$ then $O_Y(y) \cap (a,b) \neq \emptyset$. By (V_n) we denote the sequence of neighbourhoods of x in (a,b) satisfying $\mathrm{cl}V_n \subset \mathrm{int}V_{n-1}$ and $\bigcap_{n=1}^{\infty} V_n = \{x\}$. Since $x \in \Omega(Y)$ then there exists $\bar{x}_n \in V_n$, $t_n > T > 0$ such that $\bar{x}_n = \phi_Y(x_n,t_n) \in V_n$ and $\bar{x}_n$ is the first common point of $O_Y^+(x_n)$ with V_n. Moreover if $x_n \in (a,x)$

($x_n \in (x,b)$) then $\bar{x}_n \in (a,x)$ ($x_n \in (x,b)$), $\bar{x}_n$ is also the first common point of $O_Y^+(x)$ with (a,b) and $\bar{x}_n$ tends to x. It is a consequence of the fact that genus of M is zero. Thus infinitely many elements of sequence (x_n) belong either to (a,x) or (x,b). We assume that they belong to (a,x). Because any pair of $x_n,\bar{x}_n \in (a,x)$ satisfies one of three possibilities: $\bar{x}_n \in (x_n,x)$, $\bar{x}_n \in (a,x_n)$, $\bar{x}_n = x_n$, then infinitely many elements of sequence (x_n) satisfy one of them. Suppose that there exists a subsequence of (x_n) which we denote also by (x_n) satisfying $\bar{x}_n \in (x_n,x)$ for any $n \in N$. Let $z_1 = x_1$. Then the first common point $\bar{z}_1$ of $O_Y^+(z_1)$ with (a,x) belongs to (z_1,x). Assume that the arcs $O_Y[z_n,\bar{z}_n]$ $1 \le n \le k$ were defined and they satisfy the condition (i_1). Let V_j will be an element of (V_n) such that $clV_j \cap (a,x) \subset (\bar{z}_k,x)$. Then for $z_{k+1} = x_j$, $z_{k+1} \in (\bar{z}_n,x)$, $\bar{z}_{k+1} = \bar{x}_j$, $\bar{z}_{k+1}$ denotes the first common point of $O_Y^+(z_k)$ with (a,x). From this it follows that there exists a sequence of arcs$(O_Y[z_n,\bar{z}_n])$ satisfying (i_1). In the other cases the proof is analogous.

L e m m a 2. For $Y \in H^r(M)$ there exists a C^r vector field Z on S^2 which generates a flow and a closed, countable set $K \subset S^2$ such that:

(a) $Z_{|S^2-K}$ and Y are topologically conjugated

(b) if $y \in K$ then $Z(y) = 0$.

P r o o f . We assume that M is homeomorphic to S^2 without a countable number of points which form a closed subset K of S^2. By [2] there exists a C^∞ diffeomorphism $f : M \to S^2$ with analogous properties. Then $Df(Y)$ is a C^r vector field on S^2-K for $Y \in H^r(M)$. Of course $Z_{|S^2-K}$ is topologically conjugated with Y. Let $\|\bullet\|$ denote complete C^∞ Riemannian metric on S^2 and $g : S^2 \to R$ be a C^∞ function satisfying: $g(y) = 0$ for $y \in K$, g positive on S^2-K and g sufficiently quickly going to zero as x escaping to K. Then $Z(x) = g(x)(1 + \|(Df(Y))(x)\|^2)^{1/2}(Df(Y))(x)$ is C^r vector field on S^2 topologically conjugated with Y on S^2-K and $Z(y) = 0$ if $y \in K$.

L e m m a 3. Let $C_Y(x)$ be a cycle of $Y \in H_0^r(M)$ through $x \in M$. Then there exists a continuous vector field Z on M and an orbit $O_Z(y)$ with properties:

(a) $\Phi_Z(z,t) = \Phi_Y(z,t)$ if $z \in C_Y(x)|_M$, $t \in R$

(b) $\omega^*(O_Z^+(y)) = C_Y(x)$ or $\alpha^*(O_Z^-(y)) = C_Y(x)$.

P r o o f . Let (a,b) be a transversal section of Y at x, and $(O_Y[x_n,\bar{x}_n])$ be a sequence of arcs with ends $x_n,\bar{x}_n \in (a,b)$ for which $C_Y(x)$ is the set of accumulation points. Suppose that the arcs$(O_Y[x_n,\bar{x}_n])$ satisfy condition (i_1) of Definition 1 i.e. $x_n \in (a,x)$, $\bar{x}_n \in (x_n,x_{n+1})$, $x_n \to x$, $\bar{x}_n \to x$, $\bar{x}_n$ denotes the first common point of $O_Y^+(x_n)$ and (a,x). We choose transversal sections S_1, S_2 respectively at $a_1 \in O_Y^+(x)$, $a_2 \in O_Y^-(x)$ such that $S_1 = \{\Phi_Y(x,t) : x \in (a,b) \text{ and } t = t_1 \text{ for some } t_1 > 0\}$, $S_2 = \{\Phi_Y(x,t) : x \in (a,b) \text{ and } t = t_1 \text{ for some } t_1 < 0\}$. Let x_n' be the first common point of $O_Y^+(x_n)$ with S_1, $\bar{x}_n'$ be the first common point of $O_Y^+(x_n)$ with S_2. Since genus of M is zero and $x \in \Omega(Y)$, $O_Y(x)$ has no common points with (a,b) besides x. Thus $O_Y(x)$ separates $C = O_Y^+(S_2) \cap O_Y^-(S_1)$ into two components. Let $\tilde{C}$ denote the closure of the component of $C - O_Y(x)$ which contains (a,x), V be a continuous vector field with support $\tilde{C}$ such that some subsequences (x'_{n_k}), $(\bar{x}'_{n_k})$ satisfy $x'_{n_{k+1}} \in O_Z^+(\bar{x}'_{n_k})$ for any n_k and $Z = Y + V$. Therefore $O_Z[x'_{n_k}, \bar{x}'_{n_k}] = O_Y[x'_{n_k},\bar{x}'_{n_k}]$ and there exists an orbit $O_Z(y) = O_Z(x_1')$ which contains the arcs $\left(O_Y[x'_{n_k},x'_{n_{k+1}}]\right)$. Moreover $\Phi_Z(z,t) = \Phi_Y(z,t)$ if $z \in C_Y(x) \cap M$, $t \in R$. It follows from the assumption about $(O_Y[x_n,\bar{x}_n])$ and the last two sentences that $\omega^*(O_Z^+(y)) = C_Y(x)$. If the sequence $(O_Y[x_n,\bar{x}_n])$ satisfies one of the other conditions of Definition 1 we can analogously prove that $C_Y(x) = \alpha^*(O_Z^-(y))$ or $C_Y(x) = \omega^*(O_Z^+(y))$.

The next theorem is a consequence of Lemmas 2, 3 and Theorem 1, 2.

Theorem 3. If $C_Y(x)$ is a cycle of $Y \in H_0^r(M)$ through $x \in M$ then:

a) $C_Y(x)|_M$ is an invariant set under Φ_Y and separates M

b) $C_Y^n(x)$ consists of at most a countable number of orbits

c) for any $O_Y(y) \subset C_Y^n(x)$ the set $\omega^*(O_Y^+(x)) = \{P\}$ (resp. $\alpha^*(O_Y^-(y) = \{Q\})$ and either $Y(P) = 0$ or $P \in E$ (either $Y(Q) = 0$ or $Q \in E$)

d) $C_Y(x)$ possesses a cyclic mapping satisfying properties described in Theorem 2 (replace in this theorem the sets $\omega^s(O_Z^+(x)), \omega^n(O_Z^+(x))$ by $C_Y^s(x)$, $C_Y^n(x)$)

e) any transversal section S of Y meets the cycle $C_Y(x)$ at most one point.

Now we shall prove that each cycle of a generic vector field $Y \in H^r(M)$ contains no point at infinity. This fact together with Lemma 1 implies that there exists a residual set of vector fields $Y \in H^r(M)$ satisfying $\Omega(Y) = \mathrm{Per}(Y)$.

In the sequel S^+ (resp. S^-) denotes the entrance (exit) set of flowbox F of vector field.

Lemma 4. Let a flowbox F for $Y \in H^r(M)$, a point $p \in \operatorname{int} S^+$ and a C^r neighbourhood U^* of Y be given. Then there exist a neighbourhood $\tilde{S}^+$ of p in S^+ and a flowbox $\tilde{F} \subset F$ with entrance set $\tilde{S}^+$ (and corresponding exit set $\tilde{S}^- \subset S^-$) such that for any pair of points $q^\pm \in \tilde{S}^\pm$ there exists a vector field Z satisfying the conditions:

(a) $Z \in U^*$

(b) $Z = Y$ off F

(c) $q^- \in O_Z^+(q^+)$ and $O_Z[q^+, q^-] \subset F$.

This lemma is proved in [1].

Lemma 5. Let $P \in E$, (a,b) be a transversal section of $Y \in H_0^r(M)$, $O_Y(x)$ be a cycle of Y through $x \in (a,b)$ and $P \in C_Y(x)$. Then for any neighbourhood $U^* \subset H_0^r(M)$ of Y there exists a vector field X satisfying:

(a) $X \in U^*$

(b) $\omega^*(O_X^+(x)) = \{P\}$

(c) $O_X^-(x)$ crosses (a,b) at $\bar{x} \neq x$.

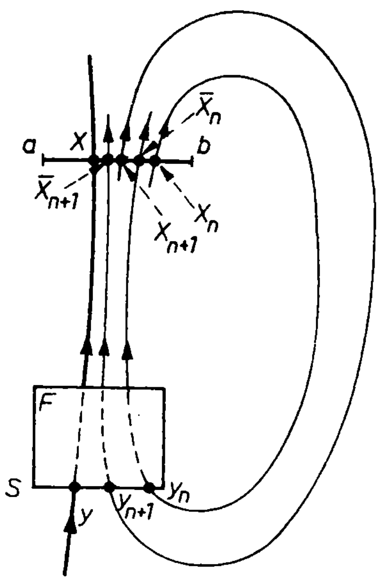

Fig.1

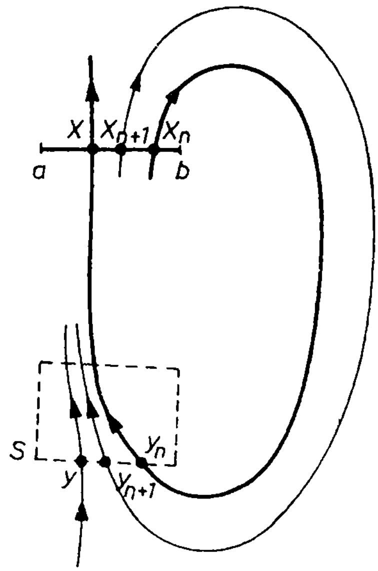

Fig.2

P r o o f . Let S be a transversal section of Y at $y \in O_Y^-(x)$ with $S \cap (a,b) = \emptyset$, and let F be a small enough flow-box with entrance set S (see Fig.1). Since $C_Y(x)$ is a cycle of Y through x then there exists a sequence of arcs$(O_Y[x_n,\bar{x}_n])$ for which $C_Y(x)$ is the set of accumulation points. We denote by y_n the common point of $O_Y[x_n,\bar{x}_n]$ with S. It follows from Lemma 4 that for any neighbourhood $U^* \subset H^r(M)$ of Y there exists a vector field $Z \in U^*$ satisfying: $Z(p) = Y(p)$ for $p \notin F$, and $y_k \in O_Z^-(x)$ for some k. Hence $O_Z^-(x)$ crosses (a,b) either at $x_k \neq x$ or $\bar{x}_k \neq x$ and $\omega^*(O_Z^+(x)) = \omega^*(O_Y^+(x))$ (see Fig.2).

By Theorem 3 $\omega^*(O_Y^+(x)) = \{Q\} = \omega^*(O_Z^+(x))$ and either $Z(Q) = 0$ or $Q \in E$ and there exist $\tau_0, \tau \in [0,1)$ such that $\tau_0 < \tau$,

$h(\tau_0) = \{x\}$, $h(\tau) = \{Q\}$. Moreover if $Q \neq P$ then one of the following conditions is satisfied: (i) there exist $\tau_n \in (0,1)$ $1 \le n \le n_0$ such that $\tau_0 < \tau_1 < \ldots < \tau_n$, $h(\tau_n) = z_n \in C^n_Y(x)$, $z_n \in O'_Z(z_m)$ if $n \neq m$, $\omega^*(O^+_Z(z_n)) \neq \{P\}$ if $n < n_0$, $\omega^*(O^+_Z(z_{n_0})) = \{P\}$, (ii) there exists an increasing sequence (τ_n), $\tau_n \in (\tau_0, \tau)$, $\tau_n \to \tau$, $h(\tau_n) = z_n \in C^n_Y(x)$, $z_n \notin O_Z(z_m)$ if $n \neq m$, $\omega^*(O^+_Z(z_n)) \neq \{P\}$ if $n \in N$. Suppose that $C^n_Y(x)$ satisfies (ii). By Theorem 3 if $\tau_n \to \tau$ then $\bigcap_{n=0}^{\infty} \left(\bigcup_{k \ge n} h(\tau_k)\right) \subseteq h(\tau)$. It implies that $\lim h(\tau_n) = \lim z_n = P$. Let $z_0 \in O^+_Z(x)$, $z_{1n} \in O^-_Z(z_n)$, $z_{2n} \in O^+_Z(z_n)$, $n \in N$. We choose small enough disjoint transversal sections S^+_0, S^{-1}_{1n}, S^+_{2n} in z_0, z_{1n}, z_{2n} and the flowboxes F_0, F_{1n}, F_{2n} such that S^+_0, S^+_{2n} are entrance sets of F_0, F_{2n}, S^-_{1n} is the exit set of F_{1n} and $F_0 \cap O^-_Z(x) = \emptyset$, $F_{in} \cap O^-_Z(x) = \emptyset$; $i = 1,2$, $n \in N$. Since $\lim z_n = P$ we can assume that P is exactly one accumulation point of the flowboxes F_{in}, $i = 1,2$ (see Fig.3). Let $\tilde F_0$, $\tilde F_{1n}$, $\tilde F_{2n}$ be the flowboxes with entrance sets $\tilde S^+_0$, $\tilde S^+_{1n}$, $\tilde S^+_{2n}$ and exit sets $\tilde S^-_0$, $\tilde S^-_{1n}$, $\tilde S^-_{2n}$ as in the conclusion of Lemma 1. For $\tilde S^-_0$ and $\tilde S^+_{11}$ and for any pair of $\tilde S^-_{2n}$, $\tilde S^+_{1n+1}$ there exists an arc $O_Z[y_n,\bar y_n] \subset O_Y[x_n,\bar x_n]$ crossing this pair of transversal sections. Since the genus of M is zero then $O_Z[y_n,\bar y_n]$ has at most one common point with $\tilde S^-_0$, $\tilde S^-_{2n}$, $\tilde S^+_{1n+1}$. Let $y_0 = O_Z[y_{n_0},\bar y_{n_0}] \cap \tilde S^-_0$, $y_{11} = O_Z[y_{n_0},\bar y_{n_0}] \cap \tilde S^+_{11}$, $y_{2n} = O_Z[y_n,\bar y_n] \cap \tilde S^-_{2n}$, $y_{1n+1} = O_Z[y_n,\bar y_n] \cap \tilde S^+_{1n+1}$. By Lemma 4 there exist vector fields Z_0, Z_{1n}, Z_{2n} satisfying: Z_0, Z_{1n}, $Z_{2n} \in U^*$, $Z(p) = Z_0(p)$ if $p \notin F_0$, $Z_{1n}(p) = Z(p)$ if $p \notin F_{1n}$, $Z_{2n}(p) = Z(p)$ if $p \notin F_{2n}$, $y_0 \notin O^+_{Z_0}(z_0)$, $O_{Z_0}[z_0,y_0] \subset F_0$, $y_{2n} \in O^+_{Z_{2n}}(z_{2n})$, $z_{1n+1} \in O^+_{Z_{1n+1}}(y_{1n+1})$, $O_{Z_{2n}}[z_{2n},y_{2n}] \subset F_{2n}$, $O_{Z_{1n+1}}[y_{1n+1},z_{1n+1}] \subset F_{1n+1}$. Thus the equations $X(p) = Z(p)$ if $p \in F \cup F_0 \cup \bigcup_{n=1}^{\infty} F_{1n} \cup$

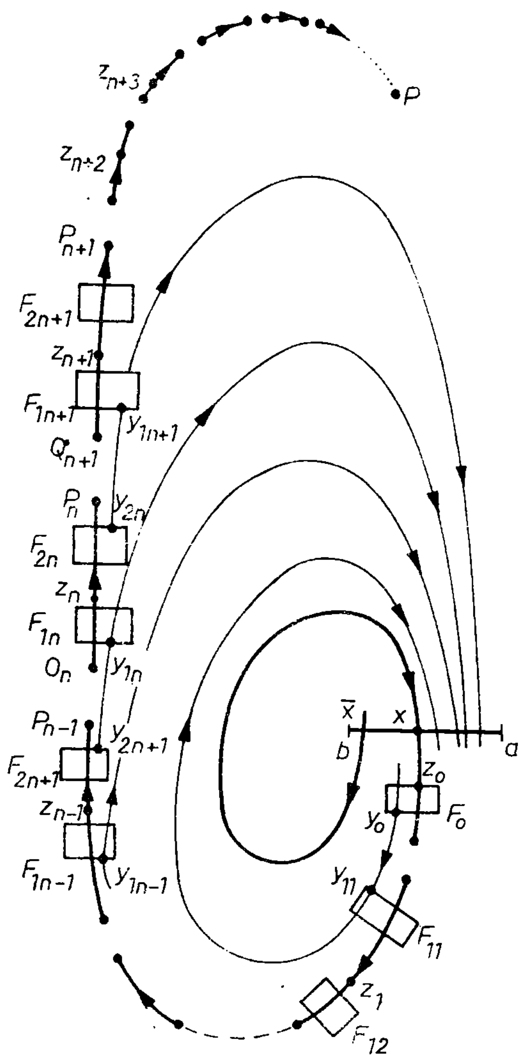

Fig.3

$\cup \bigcup_{n=1}^{\infty} F_{2n}$, $X(p) = Z_0(p)$, $p \in F_0$, $X(p) = Z_{in}(p)$, $p \in F_{in}$, $i = 1,2$, $m \in N$ define a vector field X with the properties: $X \in U^*$, $\omega^*(O_X^+(x)) = \{P\}$, $O_X^-(x)$ intersects (a,b) in $\bar{x} \neq x$. It is clear that $P \in \omega^*(O_X^+(x))$, but the equality is a consequence of assumption that the sequence (z_n) has no other accumulation points besides P. Now it is not difficult to see that the proof of condition (i) is analogous.

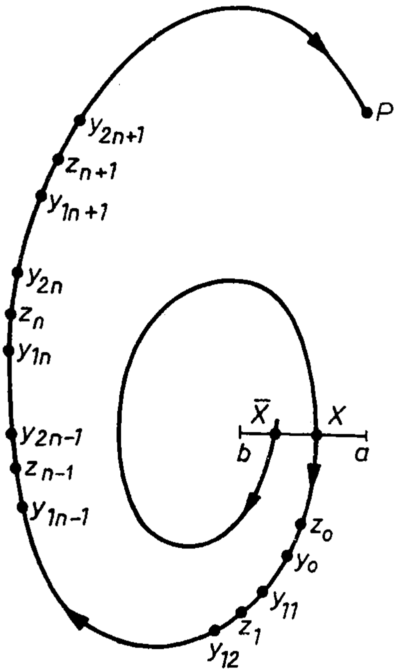

Fig.4

L e m m a 6. Let X be a vector field from the conclusion of Lemma 5 and assume that $\bar{x} \in (a,x)$. Then there is no cycle $C_X(y)$ of X which contains P and $y \in (a,x]$.

P r o o f . Suppose that there exists a cycle $C_X(y)$ of X with $P \in C_X(y)$ and $y \in (a,b)$. If $y = \bar{x}$ or $y = x$ then $O_X(y)$ crosses (a,b) in two different points contrary to Theorem 3(e). Since the genus of M is zero the unition of arc $O_X[x,\bar{x}]$ and $[x,\bar{x}] \subset (a,b)$ separates M into positive invariant component D^+ and negative invariant component D^-. This implies that for $y \in (a,\bar{x})$ either $C_X(y) \subset D^-$ or $C_X(y) \cap (\bar{x},x) \neq \emptyset$. In the first case $C_X(y)$ does not contain P, in the later $C_X(y)$ has two common points with (a,b). If $y \in (\bar{x},x)$, then $O_X^+(z) \subset D^+$, $O_X^-(z) \subset D^-$ for $z \in (\bar{x},x)$ contrary to the properties of the arcs($O_X[x_n,\bar{x}_n]$) assumed in the definition of a cycle. Hence there is no cycle $C_X(y)$ of X with $y \in (a,x]$ and $P \in C_X(y)$.

L e m m a 7. Let $P \in E$ and (a,b) be a transversal section of $Y \in H_0^r(M)$. Then for any neighbourhood $U^* \subset H_0^r(M)$ of there exists an open set $V^* \subset U^*$ such that for $Z \in V^*$ there is is no cycle $C_Z(x)$ which contains P and $x \in (a,b)$.

P r o o f . Let $U^* \subset H_0^r(M)$ be a neighbourhood of Y. Suppose that $C_Y(x)$ is a cycle of Y through $x \in (a,b)$ which contains P. By Lemma 5 there exists a vector field $X \in U^*$ such that $\omega^*(O_X^+(x)) = \{P\}$, $O_X^-(x)$ crosses (a,b) in $\bar{x} \neq x$. Assume that $\bar{x} \in (a,x)$. Let $\varepsilon = 1/2 \min\{\varrho(\bar{x},a),\varrho(\bar{x},x)\}$, ϱ is a metric induced by Riemannian metric on M. It follows from Proposition 4.5 in [1] that there exist an open set $U_1^* \subset U^*$ and a point $x^Z \in (a,b)$ satisfying $\omega^*(O_Z^+(x^Z)) = \{P\}$, $\varrho(x,x^Z) < \varepsilon$ for $Z \in U_1^*$. Moreover we can choose U_1^* such that $O_Z^-(x^Z)$ crosses (a,b) at $\bar{x}^Z \neq x^Z$, $\bar{x}^Z \in (a,x^Z)$ for $Z \in U_1^*$. By Lemma 6 there exists no cycle $C_Z(y)$ of $z \in V_1^*$ which contains P and $y \in (a,x^Z]$. Thus if U_1^* does not satisfy the conclusion of Lemma 7 then there exists $Y \in U_1^*$ with cycle $C_Y(y)$ such that: $P \in C_Y(y)$, $y \in (x^Y,b)$ and this cycle is the accumulation set of arcs($O_Y[y_n,\bar{y}_n]$) lying in the other component of $M - C_Y(y)$ than $O_Y(x^Y)$. Hence $y,y_n,\bar{y}_n \in (x^Y,b)$. Analogously like in the first step of this proof we can show that there exist open set $U_2^* \subset U_1^*$ and points $y^Z \in (x^Z,b)$ satisfying: $\omega^*(O_Z^+(y^Z)) = \{P\}$, $O_Z^-(y^Z)$ crosses (a,b) in $\bar{y}^Z \in (y^Z,b)$ and there is no cycle $C_Z(x)$ of Z with $x \in (a,x^Z] \cup [y^Z,b)$, $P \in C_Z(x)$ for $Z \in U_2^*$. It easy to see that if $x \in (x^Z,y^Z)$ then $O_Z^+(x)$ is contained in a region bounded by $O_Z^+(x^Z)$, $O_Z^+(y^Z)$ and $[x^Z,y^Z] \subset (a,b)$ so $O_Z^+(x) \cap [x^Z,y^Z] = \emptyset$ and $\Omega(Z) \cap [x^Z,y^Z] = \emptyset$. The last two sentences imply that U_2^* satisfies the conclusion of this lemma.

L e m m a 8. For $P \in E$ there exists a residual set $G_P^r(M) \subset H^r(M)$ of vector fields with no cycle which contains P.

P r o o f . Let (K_n) $n \in N$ be a covering of M by compact regions. We shall prove that for any K_n there exists an open and dense set $G_P^r(K_n) \subset H_o^r(M)$ such that if $Z \in G_P^r(K_n)$ and $x \in clK_n$ then Z has no cycle $C_Z(x)$ which contains P. Pick K_n and an open set $U^* \subset H^r(M)$. Because $H_o^r(M)$ is a dense and open subset of $H^r(M)$ then there exists an open set $V^* \subset U^* \subset H_o^r(M)$, the sets $S_1, \dots, S_k$ and points $p_1^Z, \dots, p_n^Z \in intK_n$ satisfying: $Z(p_i^Z) = 0$, S_i is an open transversal section of Z and for $x \in clK_n - \{p_1^Z, \dots, p_n^Z\}$ there exists S_i such that $O_Z(x) \cap S_i \neq \emptyset$ if $Z \in V^*$. It is a consequence of Lemma 4.3 and 4.4 in [1]. This implies that for any cycle $C_Z(x)$ of Z through $x \in clK_n$, $C_Z(x) \cap S_i \neq \emptyset$ for some S_i. Thus it is enough to prove that V^* contains an open set W^* of vector fields Z with no cycle $C_Z(x)$ such that $P \in C_Z(x)$ and $x \in \bigcup_{i=1}^{k} S_i$. Using succesively Lemma 7 to $S_1, \dots, S_k$ and V^* we get the open sets $W_j^* \subset W_{j-1}^* \subset \dots \subset W_1^* \subset V^*$ $1 \leq j \leq k$ such that if $Z \in W^*$ then there is no cycle $C_Z(x)$ of Z through $x \in \bigcup_{i=1}^{j} S_i$ which contains P. Thus it is enough to put $W^* = W_k^*$. So $G_P^r(K_n)$ contains an open and dense set and $G_P^r(M) = \bigcap_{n=1}^{\infty} G_P^r(K_n)$ contains a residual set.

T h e o r e m 4. The set $F^r(M) = \{Y \in H_{K-S}^r(M) : \Omega(Y)\} = Per(Y)$ is residual in $H^r(M)$.

P r o o f . For each $P \in E$ there exists a residual set $G_P^r(M)$ of vector fields Z with no cycle which contains P. Since E is a countable set, $G^r(M) = \bigcap_{P \in E} G_P^r(M)$ is residual in $H^r(M)$. We want to note that the assumption concerning countability of E is essentially used only in the proof of this Theorem. In Lemma 1 we proved that $\Omega(Y) = Per(Y)$ for $y \in H_o^r(M)$

iff there exists no cycle of Y. It is easy to see that any bounded cycle of $Y \in H_0^r(M)$ contains a saddle connection. Thus for $Y \in F^r(M) = G^r(M) \cap H^r_{K-S}(M)$ there are no bounded cycles ($H^r_{K-S}(M)$ denotes the Kupka and Smale vector fields). This together with Lemma 1 implies that $\Omega(Y) = Per(Y)$ if $Y \in F^r(M)$ and $F^r(M)$ is residual in $H^r(M)$.

C o r o l l a r y . There exists a residual set of vector fields in $H^r(M)$ without oscillating orbits.

P r o o f . Suppose that $O_Y^+(x)$ is an oscillating semi-orbit of vector field $Y \in F^r(M) = H^r_{K-S}(M) \cap G^r(M)$. In the first part of this paper we observed that $O_Y^+(x)$ oscillates iff $\omega(O_Y^+(x)) \neq \emptyset$ and there exists $P \in E$ such that $P \in \omega(O_Y^+(x))$. By Lemma 2 and Theorem 2 there exists a cycle $C_Y(z)$ of Y through $z \in \omega(O_Y^+(x))$ which contains P, contrary to $Y \in G^r(M)$. The proof is analogous when $O_Y^-(x)$ is an oscillating semi-orbit. This together with Theorem 4 implies that $F^r(M)$ is residual set of vector fields in $H^r(M)$ without oscillating orbits.

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