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CONVEX CONGRUENCES ON BCK-ALGEBRAS

Dedicated to the memory of Professor Roman Sikorski

1. Introduction

In this note we shall prove that all those (and only those) congruence relations on BCK-algebras are convex which keep their factor algebras being again BCK-algebras, and that any convex congruence class V of BCK-algebras has congruence extension property if and only if S(V) is again a convex congruence class; in particular each variety of BCK-algebras does have congruence extension property.

An algebra $\underline{A} = (A; \cdot, 0)$ of type (2,0) is said to be a BCK-algebra (see e.g. K. Iseki [4]) provided, for all x,y,z in A (we put dot only to avoid first-order brackets and use juxtaposition in other cases):

BCK1. $(xy \cdot xz) \cdot zy = 0$, BCK2. $(x \cdot xy)y = 0$,

BCK3. $xx = 0$, BCK4. $0x = 0$,

BCK5. $xy = 0 = yx$ implies $x = y$.

The underlying set A is partially ordered by the relation $x \leq y$ if and only if $xy = 0$. If the set $\{x \in A: xa \leq b\}$ has a largest element, for some $a,b \in A$, it is denoted by a+b and called a sum. If the sum a+b exists for all a,b in A, we say that the BCK-algebra $\underline{A}$ has a sum (see e.g. K. Iseki, [5]).

The following notation will be useful in the sequel

$$xy^0 = x \quad \text{and} \quad xy^{n+1} = xy^n \cdot y \quad \text{for } n=0,1,\ldots$$

where x,y are in a BCK-algebra.

2. Convex congruence relations of BCK-algebras

The notion of convex congruence relation is a useful tool in studying partially ordered algebraic systems (see e.g. L. Fuchs, [1], p.18).

Definition. Let $\underline{A} = (A; F)$ be an algebra with a partial ordering relation $\leqslant$ defined on the set A. A congruence relation Θ on $\underline{A}$ is said to be convex, if for all a,b,c in A the following implication holds

$$(1) \qquad a \leqslant b \leqslant c \quad \text{and} \quad a \sim c(\Theta) \quad \text{imply} \quad a \sim b(\Theta).$$

$\underline{A}$ is called a convex congruence algebra provided each congruence relation on $\underline{A}$ is convex. A class W of algebras is called convex congruence provided each member of W is a convex congruence algebra.

Of course, every lattice is a convex congruence algebra. However congruence relations on BCK-algebras may not be necessarily convex. Non-convex congruence relations have been used in order to show that neither the class of all BCK-algebras nor even the class of BCK-algebras with a sum is a variety (see A. Wroński [7] and D. Higgs [3]).

Lemma 1. For every BCK-algebra $\underline{A}$, $x,y \in A$, and for all positive integers m,n, we have

$$(2) \qquad [y(xy)^m](yx)^n \leqslant [x(xy)^m](yx)^{n-1} \leqslant [y(xy)^{m-1}](yx)^{n-1}.$$

Proof. These inequalities follow, by induction, from the well known fact that in every BCK-algebra $\underline{A}$

$$(3) \qquad y \cdot yx \leqslant x \quad \text{for all} \quad x,y \in A.$$

T h e o r e m 1. Let θ be a congruence relation on a BCK-algebra $\underline{A} = (A; \cdot, 0)$. Then the following conditions are equivalent:

(4) θ is convex;

(5) $xy \sim 0(\theta)$ and $yx \sim 0(\theta)$ together imply $x \sim y(\theta)$, for all $x, y \in A$;

(6) $\underline{A}/\theta$ is a BCK-algebra.

P r o o f . Conditions (5) and (6) were proved to be equivalent by H. Yutani (see [9], Proposition 3). Now let (4) hold and suppose that $xy \sim 0(\theta)$ and $yx \sim 0(\theta)$. Then, obviously,

$$(y \cdot xy) \cdot yx \sim y \cdot xy \sim y(\theta).$$

Therefore, by the convexity of θ, we get from (2) (for m=1, n=1) that

$$(y \cdot xy) \cdot yx \sim x \cdot xy(\theta), \quad \text{i.e.} \quad x \sim y(\theta).$$

Conversely, let (5) hold, and let $a \leqslant b \leqslant c$ and $a \sim c(\theta)$ for any a,b,c in A. Then $ba \sim bc(\theta)$ and $bc = 0$. Hence $ba \sim 0(\theta)$. Since also $ab \sim 0(\theta)$, we get $a \sim b(\theta)$ by (5), and then (4) follows.

C o r o l l a r y 1. If V is a quasi-variety of BCK-algebras, which is determined by a set of equations together with the quasi-equation BCK5, then V is a variety if and only if it is a convex congruence class.

C o r o l l a r y 2. Every variety of BCK-algebras is a subclass of the class K of all convex congruence BCK-algebras.

T h e o r e m 2. A subclass $V \subseteq K$ has the congruence extension property if and only if $S(V) \subseteq K$.

P r o o f . Let $\underline{B}$ be a subalgebra of an algebra $\underline{A}$ in V. Let us suppose that $S(V) \subseteq K$ and let θ be a congruence relation on $\underline{B}$. Let Δ stand for the ideal of $\underline{A}$, which is generated by the equivalence class $[0]_{\Theta}$, and take Φ to be the uniquely determined congruence relation on A such that

$\Delta = [0]_{\Phi}$ (see H. Yutani, [9], Theorem 3). Then, again by Yutani's Theorem 3, the intersection $\Phi \cap B^2$ is equal to the unique congruence relation θ on $\underline{B}$ such that $[0]_{\Theta} = \Delta \cap B$, i.e. Φ is an extension of θ.

Conversely, if V has the congruence extension property then for every congruence relation θ on $\underline{B}$ there is a congruence relation Φ on $\underline{A}$ such that $\theta = \Phi \cap B^2$. Since Φ is convex, so is θ, which proves that $S(V) \subseteq K$.

A. Wroński and J.K. Kabziński proved in [8] that there is no greatest variety of BCK-algebras. Therefore the class K is not a variety. So the following problem arises.

Problem. Prove or disprove that the class K has the congruence extension property.

An application. The definition of a convex congruence relation, as given by the formula (1), makes sense in a more general case when A is an algebra with a quasi-ordering relation $\leq$. For example, let V be a variety of algebras of type (2,0) which is based by the following identities:

(7) $$xy \cdot zy = xz \cdot yz,$$

(8) $$xx = 0,$$

(9) $$x0 = x,$$

(10) $$zx \cdot y = zy \cdot x.$$

A quasi-ordering relation is imposed by setting

$$x \leq y \quad \text{if and only if} \quad xy = 0.$$

It is easy to check that identities (7), (8), (9), (10) imply BCK1, BCK2, BCK3, BCK4, and that conditions (4), (5), (6) of Theorem 1 remain pairwise equivalent for every A in V. Furthermore, the following relation θ on A

$$x \sim y(\theta) \quad \text{if and only if} \quad xy = 0 = yx$$

is a convex congruence relation. An easy proof that θ is a congruence relation one can find in B.Bosbach paper [10], p.445; convexity follows from (7). The factor algebra $\underline{A}/\theta$ is a BCK-algebra.

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REFERENCES

[1] L. F u c h s : Partially ordered algebraic systems. Oxford, 1963.

[2] G. G r a t z e r : Universal algebra. New York, Heidelberg, Berlin, 1979.

[3] D. H i g g s : Dually residuated commutative monoids with identity element do not form an equational class, Math. Japon. 19 (1984), 69-75.

[4] K. I s e k i : On bounded BCK-algebras, Math. Sem. Notes, Kobe Univ. 3 (1975) III

[5] K. I s e k i : A special class of BCK-algebras, these NOTES, 5 (1977) 191-198.

[6] K. I s e k i , S. T a n a k a : An introduction to the theory of BCK-algebras, Math. Japon., 23 (1978) 1-26.

[7] A. W r o ń s k i : BCK-algebras do not form a variety, Math. Japon., 28 (1983) 211-213.

[8] A. W r o ń s k i , J.K. K a b z i ń s k i : There is no largest variety of BCK-algebras, submitted to Math. Japon.

[9] H. Y u t a n i : Quasi-commutative BCK-algebras and congruence relations, Math. Sem. Notes, Kobe Univ. 5 (1977) 469-480.

[10] B. B o s b a c h : Residuation groupoids and lattices, Studia Sc. Math. Hungarica 13 (1978) 433-451.

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