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ON JETS IN DIFFERENTIAL SPACES

Dedicated to the memory of Professor Roman Sikorski

0. Introduction

In 1967 R. Sikorski [6] introduced the concept of differential space as a generalization of C^{∞}-differentiable manifold. Independently, S. MacLane [4] introduced the same concept of differential space in his lectures on modern theoretical mechanics. W. Waliszewski in [9] introduced the concept of jet in the category of differential spaces. The present paper contains two other definitions of jets based on some concepts of tangency of mappings at a point. There are examined some connections among the above definitions.

1. Terminology and notation

Let S be any set and let C be a set of real functions defined on S. Denote by τ_C the weakest topology on S such that all functions of C are continuous. Let C_A, $A \subset S$, be the set of all real functions α defined on A such that for any $p \in A$ there exists a set $B \in \tau_C$ with $p \in B$ and a function $\beta \in C$ satisfying the equality $\alpha | A \cap B = \beta | A \cap B$. We say (cf. [4]) that C is closed with respect to localization if $C_S = C$. We denote the set of all functions of the form

$$S \ni p \longmapsto \varphi(\alpha_1(p), \alpha_2(p), \ldots, \alpha_s(p)),$$

where s is any positive integer, $\alpha_1, \alpha_2, \ldots, \alpha_s \in C$ and $\varphi \in C^\infty(R^s)$. Any set C satisfying $scC = C$ is said to be closed with respect to superposition with all real C^∞-functions (cf. [8]). For an arbitrary set C of real functions defined on S we have $\tau_{scC} = \tau_C = \tau_{C_S}$. Moreover for any $A \subset S$ the topology τ_{C_A} is the restriction of τ_C to A.

C is said to be a differential structure on the set S iff C is non-empty and $scC = C = C_S$. Any pair (S,C), where S is a set and C is a dfiferential structure on S we call a differential space. For any set C of real functions defined on S the set $(scC)_S$ is the smallest set containing C and closed with respect to superposition with all C^∞-functions and localization. If C is non-empty then the pair $(S,(scC)_S)$ forms a differential space. $(scC)_S$ is called the structure generated by C. By a subspace of a differential space (S,C) we mean any differential space of the form (A,C_A), where A is a subset of S.

Let us consider a differential space M of the form (S,C). The set S will be also denoted by PointsM and called the set of all points of M. Similarly, the set C is called the set of all real smooth functions on M and denoted by $F(M)$.

A mapping f:PointsM $\longrightarrow$ PointsN is said to be a smooth mapping from M to N iff for any $\beta \in F(N)$ we have $\beta \circ f \in F(M)$. This fact is denoted in the form $f: M \longrightarrow N$.

Let us take any point p of M. The union of all sets of the shape C_U, where $p \in U \in \tau_C$, $C = F(M)$, will be denoted by $F(M,p)$. The set of all derivations (see [5], [7]) on $F(M,P)$ together with natural operations of addition and multiplication by reals is called the tangent space to M at p and denoted by T_pM.

By $[\alpha, p]$, where p is a point of M and $\alpha \in F(M,P)$, we denote the germ of α at p. By $m(M,p)$ we denote the algebra of all germs of functions from $F(M,p)$, whose values at p are zero. By $m^k(M,p)$, where k is a positive integer, we denote the ideal of $\mathfrak{m}(M,p)$ generated by all germs of the form $g_1g_2\ldots g_k$, where $g_1,g_2,\ldots,g_k$ are elements of $\mathfrak{m}(M,p)$. It is well known (see e.g. [1]) that the mapping defined by the formula $v \longmapsto ([\alpha, p] + \mathfrak{m}^2(M,p) \longmapsto v(\alpha))$ is correctly defined isomorphism of T_pM onto $(\mathfrak{m}(M,p)/\mathfrak{m}^2(M,p))^*$.

By the tangent bundle to the differential space M (cf. [5]) we mean the differential space TM which structure is generated by the set $\{\alpha \circ \pi_M;\ \alpha \in F(M)\} \cup \{d_M\alpha;\ \alpha \in F(M)\}$ of real functions defined on the set PointsTM of all pairs (p,v) with p in M and v in T_pM, where π_M and $d_M\alpha$ are the mappings $\mathrm{PointsTM} \ni (p,v) \longmapsto p \in \mathrm{PointsM}$ and $\mathrm{PointsTM} \ni (p,v) \longmapsto v(\alpha) \in R$ respectively. We will identify $(p,v) \in \mathrm{PointsTM}$ with the vector v tangent to M at p.

We have $\pi_M: TM \longrightarrow M$ and for any $\alpha \in F(M)$ $d_M\alpha: TM \longrightarrow R$, where R denotes the differential space $(R, C^\infty(R))$.

Let M, N be two differential spaces and $f: M \longrightarrow N$. Then we have $Tf: TM \longrightarrow TN$, where Tf is the mapping such that for any v $Tf(v)$ is of the shape $F(N,f(p)) \ni \beta \longmapsto v(\beta \circ f)$ and the following condition $\pi_N \circ Tf = f \circ \pi_M$ is fulfilled. By T_pf we mean the restriction of Tf to T_pM.

By $\mathfrak{X}(M)$ we denote the set of all smooth mappings X from M into TM satisfying the condition $\pi_M \circ X = id_M$. Such mappings are called smooth vector fields, shortly: vector fields, tangent to M. If $X \in \mathfrak{X}(M)$ and $\alpha \in F(M)$ then by $\partial_X\alpha$ we denote the function $\mathrm{PointsM} \ni p \longmapsto X(p)(\alpha) \in R$.

Put $T^0M = M$, $\pi_M^0 = id_M$, $T^0f = f$ and $d_M^0\alpha = \alpha$, where M is a differential space, f is a smooth mapping defined on M into a differential space and $\alpha \in F(M)$. For any positive integer k and M, f, α as above put $T^kM = T(T^{k-1}M)$, $\pi_M^k = \pi_M^{k-1} \circ \pi_{T^{k-1}M}$, $T^kf = T(T^{k-1}f)$ and $d_M^k\alpha = d_{T^{k-1}M}(d_M^{k-1}\alpha)$. Let for p in M $T_p^kM = (\pi_M^k)^{-1}(p)$ and $T_p^kf = (T^kf)|T_p^kM$.

It is easy to verify the following four propositions.

1.1. Proposition. Let M be a differential space, $\alpha \in F(M)$ and $X \in \mathfrak{X}(M)$. Then we have the equality $\partial_X\alpha = d_M\alpha \circ X$.

1.2. P r o p o s i t i o n . If $f: M \to N$ and $\beta \in F(N)$ then $d^k_M(\beta \circ f) = d^k_N \beta \circ T^k f$, where k is any positive integer.

1.3. P r o p o s i t i o n . Let $f: M \to N$ and $g: N \to P$. Then for any positive integer k we have $T^k(g \circ f) = T^k g \circ T^k f$.

1.4. P r o p o s i t i o n . For any differential space M, $\alpha \in F(M)$ and $X_1, X_2, \dots, X_k \in \mathfrak{X}(M)$ we have

$$\partial_{X_1} \partial_{X_2} \dots \partial_{X_k} \alpha = d^k_M \alpha \circ T^{k-1} X_k \circ \dots \circ T^1 X_2 \circ X_1 .$$

2. The first kind of tangency of mappings

The following definition extends to the category of differential spaces the concept of tangency of smooth mappings of differentiable manifolds (see [2]).

2.1. D e f i n i t i o n : Let

$$f: M \to N, \quad g: M \to N \tag{2.1}$$

and p be a point of M. We say that f is tangent to g of order 1 at p (in the first sense), what we write in the form $f \doteq_{1,p} g$ iff

$$f(p) = g(p) \tag{2.2}$$

and $T_p f = T_p g$. For any integer $k > 1$ we write

$$f \doteq_{k,p} g \tag{2.3;k}$$

iff (2.2) and for v in $T_p M$ $Tf \doteq_{k-1,v} Tg$. We say that f is tangent to g of order k at p (in the first sense).

2.2. L e m m a : Let us take mappings (2.1) and p in M. If (2.3;k) holds, where k is a positive integer, then for any $l \leq k$ (2.3;l).

P r o o f . To prove the lemma it sufficies to consider the case $l = k-1$. We use the induction with respect to k.

Let $k = 2$. If (2.3;2) then according to Definition 2.1 for any v in $T_p M$ we have $Tf \doteq_{1,v} Tg$, which implies $T_v Tf = T_v Tg$. Hence for any w in $T_v TM$ we have

$$Tf(v) = Tf(\pi_{TM}(w)) = \pi_{TN}(TTf(w)) =$$

$$= \pi_{TN}(T_v Tf(w)) = \pi_{TN}(T_v Tg(w)) = Tg(v).$$

Hence $T_p f = T_p g$, which gives (2.3;1).

Now let $k \geq 2$ be such integer that for any mappings (2.1) and p in M the condition (2.3;k) implies (2.3;k-1). From (2.3;k+1) by Definition 2.1 it follows that

(2.4;k) $$Tf \doteq_{k,v} Tg$$

for v in $T_p M$, what, as we have assumed, implies (2.4;k-1) for v in $T_p M$. Then (2.3;k). QED.

2.3. Theorem. For any mappings (2.1) and any positive integer k the condition (2.3;k) is equivalent to the formula

(2.5;k) $$T_p^k f = T_p^k g.$$

Proof. For k = 1 the proof by a direct verification.

Let k be such positive integer that (2.3;k) is equivalent to (2.5;k) for any M, N, f, g, p as above. For any v in $T_p M$ the condition (2.3;k+1) implies (2.4;k), which is by the induction hypothesis equivalent to the condition

(2.6) $$T_v^k Tf = T_v^k Tg.$$

Since $T_v^k Tf$ is the restriction $T^k Tf | T_v^k TM = T^{k+1} f | T_v^k TM$ and $\bigcup\{T_v^k TM ; v \in T_p^1 M\} = T_p^{k+1} M$ we get $T^{k+1} f | T_p^{k+1} M = T^{k+1} g | T_p^{k+1} M$, which gives the formula (2.5;k+1). Thus we have proved that (2.3;k+1) implies (2.5;k+1). To complete the proof of the theorem it sufficies to prove the inverse statement.

Assume that f, g, p satisfy (2.5;k+1). For any v in $T_p M$ we have (2.6). Hence (2.4;k). Moreover

$$\pi_N^{k+1} \circ T_p^{k+1} f(T_p^{k+1} M) = \pi_N^{k+1} \circ T^{k+1} f(T_p^{k+1} M) =$$

$$= f \circ \pi_M^{k+1}(T_p^{k+1} M) = \{f(p)\}.$$

Similarly, setting g instead of f, from (2.5;k+1) we get (2.2). Now, by Definition 2.1 we have (2.3;k+1).

3. The second kind of tangency of mappings

3.1. Definition (see [9]). Let us take any mappings (2.1), p in M, and let k be a positive integer. We say that f is tangent to g of order k at the point p (in the second sense), what we write in the form

$$f \mathrel{\hat{=}}_{k,p} g, \tag{3.1}$$

iff (2.2) holds and for any differential space L, $\beta \in F(N)$, $\varphi : L \longrightarrow M$, $X_1, X_2, \ldots, X_k \in \mathfrak{X}(L)$, $t \in \text{Points}L$ such that $\varphi(t) = p$ and for any positive integer $l \leq k$

$$\partial_{X_1}\partial_{X_2}\ldots\partial_{X_l}(\beta\circ f\circ\varphi)(t) = \partial_{X_1}\partial_{X_2}\ldots\partial_{X_l}(\beta\circ g\circ\varphi)(t). \tag{3.2}$$

3.2. Theorem. For any mappings (2.1), any point p of M and positive integer k, satisfying (2.3;k) we have (3.1).

Proof. Let us suppose the condition (2.3;k). By Definition 2.1 we have (2.2). Moreover, by Theorem 2.3, we have (2.5;k). Let $L, \beta, \varphi, X_1, X_2, \ldots, X_k$, t, l be as in Definition 3.1. We have to prove the condition (3.2). By Lemma 2.2 and Theorem 2.3 we have (2.5;l). Moreover, by Proposition 1.4, we have

$$\partial_{X_1}\partial_{X_2}\ldots\partial_{X_l}(\beta\circ f\circ\varphi)(t) = d_L^l(\beta\circ f\circ\varphi)\circ T^{l-1}X_1\circ\ldots\circ T^1X_2\circ X_1(t),$$

which is equal to

$$d_N^l\beta\circ T^lf\circ T^l\varphi\circ T^{l-1}X_l\circ\ \ldots\ \circ T^1X_2\circ X_1(t). \tag{3.3}$$

Since

$$\pi_M^l\circ T^l\varphi\circ T^{l-1}X_l\circ\ \ldots\ \circ T^1X_2\circ X_1(t) =$$

$$= \varphi\circ\pi_L^l\circ T^{l-1}X_l\circ\ \ldots\ \circ T^1X_2\circ X_1(t) = \varphi(t) = p$$

we have $T^l\varphi \circ T^{l-1}X_l \circ \dots \circ T^1X_2 \circ X_1(t) \in T^l_pM$. Hence T^lf may be replaced in (3.3) by T^l_pf, which is equal to T^l_pg. Now we obtain (3.2).

The following example shows that the inverse of Theorem 3.2 is not true.

3.3. Example. Let M be the differential subspace of R such that $\mathrm{Points}M = \{1/n; n \in N\} \cup \{0\}$. Put for $x \in \mathrm{Points}M$ $f(x) = 0$ and $g(x) = x$. It is easy to verify that $f: M \to R$ and $g: M \to R$.

Because $v(f) = 0 \neq 1 = v(g)$, where $v(h) = \lim_{n\to\infty} n\cdot(h(1/n)-h(0))$ for $h \in F(M,0)$, we remark that $f \doteq_{1,0} g$ is not fulfilled.

We will show that $f \hat{=}_{1,0} g$. It suffices to show that for any differential space L, $\gamma \in F(R)$, $\varphi: L \to M$, $X \in \mathfrak{X}(L)$ and t in L such that $\varphi(t) = 0$ we have $\partial_X(\gamma \circ g \circ \varphi)(t) = 0$. Let L, γ, φ, X, t be as above. We have $\partial_X(\gamma \circ g \circ \varphi)(t) =$
$= (d_R\gamma \circ Tg \circ T\varphi \circ X)(t) = (d_R\gamma \circ Tg)((T\varphi \circ X)(t))$. We shall prove that

$$(3.4) \qquad (T\varphi \circ X)(t) = 0.$$

Let us suppose that $(T\varphi \circ X)(t) \in T^1_0M \setminus \{0\}$. We have $d_Mg \in F(TM)$ and $(d_Mg)^{-1}(R \setminus \{0\}) = T^1_0M \setminus \{0\}$. Hence, $T^1_0M \setminus \{0\}$ is open in TM. Let us take neighbourhood U of t in L such that $(T\varphi \circ X)(x) \in T^1_0M \setminus \{0\}$ for $x \in U$. Hence $\varphi(x) = 0$ for $x \in U$. This implies $T_t\varphi = 0$, which ends the proof of (3.4).

4. k-derivations

4.1. Definition. By a k-derivation ($k \in N$) on a differential space M at a point p of M we mean any mapping $F: F(M,p) \to R$ such that for any $\alpha, \beta, \alpha_1, \alpha_2, \dots, \alpha_{k+1} \in F(M,p)$ and $a \in R$ we have

$$(4.1) \qquad F(\alpha+\beta) = F(\alpha) + F(\beta),\ F(a\alpha) = aF(\alpha),$$

(4.2) $$F(\alpha_1\cdot\alpha_2\cdot\ldots\cdot\alpha_{k+1}) =$$

$$= \sum_{\emptyset\neq A\subsetneq N_{k+1}} (-1)^{1+\#A}F\Big(\prod_{i\in N_{k+1}\setminus A}\alpha_i\Big)\cdot\prod_{i\in A}\alpha_i(p),$$

where $N_{k+1} = \{1,2,\ldots,k+1\}$ and # A stands for the cardinal number of the set A.

According to above definition by 1-derivations on M at p we call vectors tangent to M at p, because for k = 1 (4.2) can be written as $F(\alpha_1\cdot\alpha_2) = F(\alpha_1)\cdot\alpha_2(p) + F(\alpha_2)\cdot\alpha_1(p)$.

4.2. Remark. Let F be a k-derivation on M at p. For any $\alpha\in F(M,p)$ equal to 0 on a neighbourhood U of p we have $F(\alpha) = 0$.

To prove the remark it suffices to put in (4.2) $\alpha_1 = \alpha$ and $\alpha_2 = \alpha_3 = \ldots = \alpha_{k+1} = 1_U$.

Denote by $F_0(M,p)$ the set $\{\alpha\in F(M,p);\alpha(p) = 0\}$.

4.3. Theorem. Let the mapping $F:F(M,p)\to R$ satisfy the condition (4.1) for $\alpha,\beta\in F(M,p)$, $a\in R$. Then F is a k-derivation on M at p iff $F(1_{\text{PointsM}}) = 0$ and for any $\alpha_1,\alpha_2,\ldots,\alpha_{k+1}\in F_0(M,p)$ $F(\alpha_1\alpha_2\ldots\alpha_{k+1}) = 0$.

Proof by an easy verification.

4.4. Corollary. If $k\leq 1$ and F is a k-derivation on M at p then F is an 1-derivation on M at p.

The set of k-derivations on M at p is a linear space with respect to addition and multiplication defined as follows $(F+G)(\alpha) = F(\alpha) + G(\alpha)$, $(aF)(\alpha) = a(F(\alpha))$ for $\alpha\in F(M,p)$ and $a\in R$. This space we will denote by D^k_pM. According to Remark 4.2 and Theorem 4.3 D^k_pM is canonically isomorphic to $(m(M,p)/m^{k+1}(M,p))^*$. The isomorphism is given by the formula $F\longmapsto([\alpha,p]+m^{k+1}(M,p)\longmapsto F(\alpha))$.

Then, it is related to the concept of k-jets used in the theory of singularities of smooth mappings (see e.g. [10]).

4.5. Definition. Let $f:M\to N$ and k be a positive integer. By D^k_pf we denote the linear mapping of

$D^k_p M$ into $D^k_{f(p)} N$ defined by the following formula $D^k_p f(F)(\beta) = F(\beta \circ f)$ for F in $D^k_p M$ and $\beta \in \mathcal{F}(N, f(p))$.

5. The third kind of tangency of mappings

5.1. **Definition.** Let us take any mappings (2.1), a point p in M and a positive integer k. We say that f is tangent to g of order k at p (in the third sense), what we write in the form

$$f \equiv_{k,p} g, \tag{5.1}$$

iff (2.2) holds and $D^k_p f = D^k_p g$.

5.2. **Corollary.** We have $T_p M = D^1_p M$ and $T_p f = D^1_p f$. Thus tangency of order 1 (in the first sense) is equivalent to tangency of order 1 (in the third sense).

5.3. **Proposition.** For any $[\alpha, p]$ in $m^k(M,p)$, where $k \geqslant 2$, and any v in $T_p M$ the germ $[d_M \alpha, v]$ is an element of $m^{k-1}(TM, v)$.

Proof by an easy verification.

5.4. **Corollary.** Let p be a point of a differential space M and k be a positive integer. For any v in $T^k_p M$ the mapping $\bar{v}: \mathcal{F}(M,p) \longrightarrow R$ defined by the formula $\bar{v}(\alpha) = d^k_M \alpha(v)$ is a k-derivation on M at p.

5.5. **Lemma.** Let M be a differential space, which structure is generated by a set $C \subset R^{\mathrm{Points}M}$. Then we have

$$\mathcal{F}(TM) = (\mathrm{sc}(\{d_M \alpha;\, \alpha \in C\} \cup \{\alpha \circ \pi_M;\, \alpha \in \mathcal{F}(M)\}))_{\mathrm{Points}TM}. \tag{5.2}$$

If, moreover, k is a positive integer then

$$\mathcal{F}(T^k M) = (\mathrm{sc}(\{d^k_M \alpha;\, \alpha \in C\} \cup \bigcup_{i=0}^{k-1} \{d^{k-1-i}_{T^{i+1}M}(\alpha \circ \pi_{T^i M};\, \alpha \in \mathcal{F}(T^i M)\}))_{\mathrm{Points}T^k M}. \tag{5.3}$$

Proof of (5.2) by an easy verification. (5.3) we obtain from (5.2) by induction.

5.6. Proposition. Let M be a differential space, which structure is generated by a set $C \subset R^{\mathrm{Points}M}$. For any p in M and any v, w in T_pM such that $v(\alpha) = w(\alpha)$ for $\alpha \in C$ we have $v = w$.

Proof by an easy verification.

5.7. Theorem. Let f, g be mappings (2.1) and p be a point of M. For any positive integer k the condition (5.1) implies (2.3;k).

Proof. Theorem holds for $k = 1$ by Corollary 5.2. Let us take any integer $k > 1$ and suppose that theorem holds for $k - 1$. For any v in T_p^kM, $i \in \{0,1,2,\dots,k-2\}$ and $\beta \in \mathcal{F}(T^iN)$ we have

$$T^kf(v)\left(d^{k-2-i}_{T^{i+1}N}(\beta \circ \pi_{T^iN})\right) = T^kf(v)\left(d^{k-2-i}_{T^iN}\beta \circ T^{k-2-i}\pi_{T^iN}\right) =$$

$$= v\left(d^{k-2-i}_{T^iN}\beta \circ T^{k-2-i}\pi_{T^iN} \circ T^{k-1}f\right) =$$

$$= v\left(d^{k-2-i}_{T^iN}\beta \circ T^{k-2-i}\left(\pi_{T^iN} \circ T^{i+1}f\right)\right) =$$

$$= v\left(d^{k-2-i}_{T^iN}\beta \circ T^{k-2-i}\left(T^if \circ \pi_{T^iM}\right)\right) =$$

$$= v\left(d^{k-2-i}_{T^iN}\beta \circ T^{k-2}f \circ T^{k-2-i}\pi_{T^iM}\right) =$$

$$= T^{k-1}f\left(T^{k-1-i}\pi_{T^iM}(v)\right)\left(d^{k-2-i}_{T^iN}\beta\right) =$$

$$= T^{k-1}g\left(T^{k-1-i}\pi_{T^iM}(v)\right)\left(d^{k-2-i}_{T^iN}\beta\right).$$

The last equality follows from the induction hypothesis $T_p^{k-1}f = T_p^{k-1}g$. If moreover (5.1) holds, we have for $\beta \in \mathcal{F}(N)$ $T^kf(v)\left(d_N^{k-1}\beta\right) = d_N^k\beta\left(T^kf(v)\right) = d_N^k\beta \circ T^kf(v) = d_M^k(\beta \circ f)(v) =$ $= \bar{v}(\beta \circ f) = D_p^kf(\bar{v})(\beta) = D_p^kg(\bar{v})(\beta) = T^kg(v)\left(d_N^{k-1}\beta\right)$. Moreover,

$T^k f(v)$ is an element of $T_{T_p^{k-1} f(\pi(v))} T^{k-1}N$, which is equal to $T_{T_p^{k-1} g(\pi(v))} T^{k-1}N$. Now, by Lemma 5.5 and Proposition 5.6 we have $T^k f(v) = T^k g(v)$ for v in $T_p^k M$, which ends the proof of (2.3;k). For k=1, by Corollary 5.2, the inverse of Theorem 5.7 holds. The following example shows that for k=2 the inverse of Theorem 5.7 is not true.

5.8. Example. Let M, f and v be such as in Example 3.3. Define the mapping $h: M \to R$ by the formula $h(x) = x^2$. We have Tf = Th. Hence $T_0^2 f = T_0^2 h$ which yields $f \doteq_{2,0} h$.

However we have $f \not\equiv_{2,0} h$, because

$$D_0^2 f(F)(id_R) = F(f) = 0 \neq 1 = F(h) = D_0^2 h(F)(id_R),$$

where $F(g) = \lim_{n\to\infty} n^2(g(1/n)-g(0)-v(g))$ for $g \in \mathcal{F}(M,0)$.

6. The case where T_pM is smooth of finite dimension

6.1. Proposition. Let W be a linear space of finite dimension and let V be a subspace of W. Let $e_1, e_2, \dots, e_n$ be a basis of V and let $f_1, f_2, \dots, f_m$ be such elements of W that $f_1+V, f_2+V, \dots, f_m+V$ form a basis of W/V. Then the elements $e_1, e_2, \dots, e_n, f_1, f_2, \dots, f_m$ form a basis of W.

Proof by a direct verification.

6.2. Proposition. Let p be a point of a differential space M and k be a positive integer. Let Z be such set of elements of $\mathfrak{m}(M,P)$, that the set $\{x+\mathfrak{m}^2(M,p); x \in Z\}$ generates $\mathfrak{m}(M,p)/\mathfrak{m}^2(M,p)$. Then the set $\{x_1 x_2 \dots x_k + \mathfrak{m}^{k+1}(M,p); x_1, x_2, \dots, x_k \in Z\}$ generates $\mathfrak{m}^k(M,p)/\mathfrak{m}^{k+1}(M,p)$.

Proof by an easy verification.

6.3. Definition. By a smooth basis of tangent space to a differential space M at a point p we mean any system of vector fields on neighbourhoods of p, such that their values at p form a basis of T_pM. We say that

T_pM is smoothly of finite dimension iff it has a smooth finite basis.

6.4. Lemma. Let $(X_1,X_2,\dots,X_n)$ be a smooth basis of T_pM and let $k>1$. Consider the set of mappings $X_{i_1 i_2\dots i_k}$ of the form $F(M,p)\ni\alpha\longmapsto \partial_{X_{i_1}}\partial_{X_{i_2}}\dots\partial_{X_{i_k}}\alpha(p)$, where $1\leqslant i_1\leqslant i_2\leqslant\dots\leqslant i_k\leqslant n$. Then the mappings $X_{i_1 i_2\dots i_k}$ are k-derivations on M at p and the elements $X_{i_1 i_2\dots i_k}+D_p^{k-1}M$, $1\leqslant i_1\leqslant i_2\leqslant\dots\leqslant i_k\leqslant n$, form a basis of the space $D_p^kM/D_p^{k-1}M$.

Proof. It is easy to verify that $X_{i_1 i_2\dots i_k}$ is a k-derivation on M at p.

The space D_p^kM/D_p^{k-1} is canonically isomorphic to $\left(\mathfrak{m}^k(M,p)/\mathfrak{m}^{k+1}(M,p)\right)^*$. The isomorphism is given by the formula $F+D_p^{k-1}M\longmapsto\left(\alpha+\mathfrak{m}^{k+1}(M,p)\longmapsto F(\alpha)\right)$. We may identify those spaces without anxiety of misunderstanding.

By assumption, $X_1(p),X_2(p),\dots,X_n(p)$ form a basis of D_p^1M. Let $\left(\alpha_i+\mathfrak{m}^2(M,p);\ i=1,2,\dots,n\right)$ be the dual basis in $\mathfrak{m}(M,p)/\mathfrak{m}^2(M,p)$. Then, by Proposition 6.2, the set $B=\left\{\alpha_{i_1}\alpha_{i_2}\dots\alpha_{i_k}+\mathfrak{m}^{k+1}(M,p);\ 1\leqslant i_1\leqslant i_2\leqslant\dots\leqslant i_k\leqslant n\right\}$ generates $\mathfrak{m}^k(M,p)/\mathfrak{m}^{k+1}(M,p)$. Moreover, for $1\leqslant i_1\leqslant i_2\leqslant\dots\leqslant i_k\leqslant n$ and $1\leqslant j_1\leqslant j_2\leqslant\dots\leqslant j_k\leqslant n$ we have

$$X_{i_1 i_2\dots i_k}(\alpha_{j_1}\alpha_{j_2}\dots\alpha_{j_k})=$$

$$=\partial_{X_{i_1}}\partial_{X_{i_2}}\dots\partial_{X_{i_k}}\alpha_{j_1}\alpha_{j_2}\dots\alpha_{j_k}(p)=\sum_{\sigma\in S_k}\prod_{l=1}^{k}\left(\partial_{X_{i_l}}\alpha_{j_{\sigma(l)}}(p)\right)=$$

$$=\begin{cases}0 & \text{for } (i_1,i_2,\dots,i_k)\neq(j_1,j_2,\dots,j_k)\\ \gamma_{i_1 i_2\dots i_k}=\prod_{l=1}^{n}\left(\left(\#\left\{m\in N_k; i_m=l\right\}\right)!\right) & \\ \quad\text{for } i_1=j_1,\ i_2=j_2,\dots,i_k=j_k,\end{cases}$$

where S_k denotes the set of all permutations of the set N_k. Hence the elements of B are linearly independent and B is a basis of $m^k(M,p)/m^{k+1}(M,p)$. Its dual basis is $\left(X_{i_1 i_2 \dots i_k}/\gamma_{i_1 i_2 \dots i_k} + D_p^{k-1}M;\ 1 \leq i_1 \leq i_2 \ \dots \leq i_k \leq n\right)$. Thus, so $\left(X_{i_1 i_2 \dots i_k} + D_p^{k-1}M;\ 1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n\right)$ is a basis of $D_p^k M/D_p^{k-1}M$.

Applying Proposition 6.1 to Lemma 6.4 we get

6.5. Corollary. The mappings $X_{i_1 i_2 \dots i_l} = \left(\alpha \longmapsto \partial_{X_{i_1}} \partial_{X_{i_2}} \dots \partial_{X_{i_l}} \alpha(p)\right)$, $1 \leq l \leq k$, $1 \leq i_1 \leq i_2 \leq \dots \leq i_l \leq n$ form a basis of $D_p^k M$.

6.6. Theorem. Consider mappings (2.1) and let us suppose that p is such point of M that T_pM is smoothly of finite dimension. For any positive integer k the condition (3.1) implies (5.1).

Proof. Let $(X_1, X_2, \dots, X_n)$ be a smooth basis of T_pM. Assume (3.1). By Definition 3.1, for any positive integer $l \leq k, \leq 1 \leq i_1 \leq i_2 \leq \dots \leq i_l \leq n$, and $\beta \in N(N)$ we have

$$\partial_{X_{i_1}} \partial_{X_{i_2}} \dots \partial_{X_{i_l}} (\beta \circ f)(p) = \partial_{X_{i_1}} \partial_{X_{i_2}} \dots \partial_{X_{i_l}} (\beta \circ g)(p).$$

Hence we have $D_p^k f\left(X_{i_1 i_2 \dots i_l}\right) = D_p^k g\left(X_{i_1 i_2 \dots i_l}\right)$ when $1 \leq i_1 \leq i_2 \leq \dots \leq i_l \leq n$. The mappings $D_p^k f$ and $D_p^k g$ are equal on the basis listed in Corollary 6.5. Hence we have $D_p^k f = D_p^k g$, which ends the proof.

The hypothesis in Theorem 6.6 that T_pM is smooth of finite dimension is satisfied in the case where M is of finite dimension (see [7]). The example of A. Kowalczyk (see [3]) indicates that there exists a rather wide class of differential spaces of finite dimension which are not differentiable manifolds. Then, by Theorem 6.6 the three concepts of tangency of mappings at the point for spaces of finite dimension are equivalent. Of course, each differentiable C^{∞}-manifold is a differential space of finite dimension, too.

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