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Zbigniew Lonc, Mirosław Truszczyński

DECOMPOSITION OF GRAPHS INTO GRAPHS WITH BOUNDED MAXIMUM DEGREES

Dedicated to the memory of Professor Roman Sikorski

1. In general we follow the terminology of Harary [6]. In particular, by a graph we always mean a finite, simple graph and we denote by $V(G)$, $E(G)$ and $e(G)$ the vertex set, the edge set and the size (the number of edges) of a graph G, respectively. By $d_G(x)$ we denote the degree of a vertex x in G and by $\Delta(G)$ we denote the maximum vertex degree in G. If $X \subseteq V(G)$ then by $G[X]$ we denote the subgraph of G induced by the vertices in X.

In this paper we deal with problems related to graph decompositions. By a decomposition of a graph G we mean a family of edge disjoint subgraphs of G whose union is G. We shall write $G = H_1 \dot{\cup} H_2 \dot{\cup} \ldots \dot{\cup} H_n$ to denote that $\{H_1, H_2, \ldots, H_n\}$ is a decomposition of G. For a positive integer p, we define a p-decomposition to be a decomposition into subgraphs with maximum degree less than or equal to p. Y. Caro [3] and Bialostocki and Roditty [2] solved for r = 2 and 3 the following decomposition problem: determine all those graphs G which have a 1-decomposition consisting of isomorphic copies of rK_2 (by rK_2 we mean the disjoint union of r copies of K_2).

Clearly, the following two conditions are necessary for such a decomposition to exist

(1) $$e(G) \equiv 0 \pmod r$$

(2) $$\Delta(G) \leqslant e(G)/r.$$

Let us call a graph which satisfies (1) and (2) and do not have any 1-decomposition into isomorphic copies of rK_2 an exception. Caro [3] determined the set of exceptions for $r=2$ (there is exactly one exception in this case) and Bialostocki and Roditty [2] determined the set of exceptions for $r = 3$ (there are 26 of them). Those results were an inspiration for our investigations. Since the number of exceptions increases rapidly with r, determining sets of exceptions for $r \geqslant 4$ seems to be hopeless. Therefore we asked easier question:

Is it true that for every r the set of exceptions is finite? To answer it we consider in the paper the following more general question:

Given positive integers $r_1, \ldots, r_k$, does a graph G have a p-decomposition $G = H_1 \dot{\cup} \ldots \dot{\cup} H_k$ such that $e(H_i) = r_i$, $1 \leqslant i \leqslant k$.

Partial answer to this question is given by Theorems 2 and 3, proved in Section 2, which contains some sufficient conditions for such a p-decomposition to exist. It turns out that this condition depends on the value $\chi'_p(G)$ being the minimum number of graphs in a p-decomposition of G. Let us note that $\chi'_1(G)$ coincides with $\chi'(G)$, the chromatic index of G. Moreover, it follows easily from celebrated Vizing's Theorem that

(3) $$\lceil \Delta(G)/p \rceil \leqslant \chi'_p(G) \leqslant \lceil (\Delta(G)+1)/p \rceil.$$

In the last part of Section 2 we determine $\chi'_p(G)$ for a certain class of graphs and we use this result, together with Theorems 2 and 3 to prove that only finitely many graphs satisfying conditions (1) and

(2') $$\Delta(G) \leqslant p \cdot e(G)/r$$

(which is an extension of the condition (2) to the case of arbitrary p) do not have any p-decomposition into graphs of size r. In particular, for p = 1 this answers positively our initial question.

Recently, we have learnt that the case p = 1 was solved independently by Alon [1]. However, as our results concern a more general situation, we decided to publish them. It should be noted here that in the proof of Lemma 4 we follow [1]. Our original proof of this lemma was much longer.

2. We start with an observation that each p-decomposition of a graph G corresponds to a p-bounded colouration of edges of G (but not conversely), the notion introduced and investigated by de Werra [7]. Hence some of methods he used can almost literally be applied in our case. First we state a technical lemma.

Lemma 1. Let A and B be edge disjoint graphs with sizes a and b, respectively, having maximum degrees less than or equal to p. Suppose that $a + 1 < b$ and let $G = A \cup B$. Then for every d, $0 \leq d \leq b - a$, G has a decomposition into graphs A' and B' with sizes $a + d$ and $b - d$, respectively, and having maximum degrees less than or equal to p. (Note that a can be equal to 0 i.e. A can have size 0).

Proof. Clearly it suffices to prove the assertion for d = 1. By an alternating walk we shall mean, as usual a walk whose edges are alternatively in A and B. We shall decompose the graph $A \cup B$ into alternating walks as follows. For an arbitrary edge u of $A \cup B$ let W be a maximal alternating walk in $A \cup B$ containing u. We remove the edges of W from G and if there are any edges left we repeat this procedure as many times as necessary. Let $\mathcal{W}$ be the collection of walks obtained in this way. Since $a + 1 < b$ there is a walk W' in $\mathcal{W}$ which starts and ends with edges from B. Let x and y be its end vertices. It follows from the construction that there is no walk in $\mathcal{W}$ which starts in x with the edge from A.

Hence $d_A(x) < d_B(x)$ and similarly $d_A(y) < d_B(y)$. Let W_A (respectively W_B) be the set of those edges of W' which are in A (respectively B). We define A' (respectively B') to be the graph spanned in $A \cup B$ by the edges in $(E(A) \setminus W_A) \cup W_B$ (respectively $(E(B) \setminus W_B) \cup W_A$). Clearly $e(A') = a + 1$, $e(B') = b - 1$, $\Delta(A') \leq p$ and $\Delta(B') \leq p$.

Now, we are ready to give a sufficient condition for the existence of a p-decomposition of a graph G into graphs having prescribed sizes.

Suppose $s_1 \geq \ldots \geq s_t > 0$. We call a sequence $(s_1, \ldots, s_t)$ a p-feasible sequence for a graph G if G has a p-decomposition $\{G_1, \ldots, G_t\}$ such that $e(G_i) = s_i$, for $1 \leq i \leq t$. Let $r_1 \geq \ldots \geq r_k > 0$ and $s_1 \geq \ldots \geq s_t > 0$ be two sequences of integers. We say that $(r_1, \ldots, r_k)$ is less than or equal to $(s_1, \ldots, s_t)$ ($(r_1, \ldots, r_k) \leq (s_1, \ldots, s_t)$ in short) if the following conditions hold:

$$\sum_{i=1}^{k} r_i = \sum_{i=1}^{t} s_i, \tag{4}$$

$$\sum_{i=1}^{\min(k,n)} r_i \leq \sum_{i=1}^{n} s_i, \quad \text{for every} \quad 1 \leq n < t. \tag{5}$$

It could be noted here that the relation introduced is indeed a partial ordering. Let us observe also that (4) and (5) imply that $k \geq t$.

T h e o r e m 2. Let $(s_1, \ldots, s_t)$ be a p-feasible sequence for a graph G and let $(r_1, \ldots, r_k) \leq (s_1, \ldots, s_t)$. Then $(r_1, \ldots, r_k)$ is a p-feasible sequence for G. (It should be noted that in the case p = 1 this assertion appeared already in the paper by Folkman and Fulkerson [5]).

P r o o f . First we shall prove the assertion in the case when $s_1 - s_t \leq 1$. So, let $\{H_1, \ldots, H_t\}$ be a p-decomposition of G such that $e(H_i) = s_i$, for $1 \leq i \leq t$. Clearly, (5) implies that

$$r_i \leq s_i \quad \text{for every} \quad 1 \leq i \leq t. \tag{6}$$

Let G_1 be an arbitrary subgraph of H_1 which has size r_1 (such a graph exists since $r_1 \leq s_1$) and let A_1 be the graph obtained from H_1 by removing the edges of G_1. Define moreover $n_1 = 1$. Assume that we have already defined graphs G_i and A_i and a positive integer n_i, $i < k$, such that the following conditions hold

(7) $$\Delta(G_i) \leq p,$$

(8) $$e(G_i) = r_i,$$

(9) $$\Delta(A_i) \leq p,$$

(10) $$n_i \leq t,$$

(11) $$\{E(G_1), \ldots, E(G_i), E(A_i)\} \text{ is a partition of } E(H_1) \cup \ldots \cup E(H_{n_i}).$$

Clearly, G_1, A_1 and n_1 satisfy all these conditions. We define G_{i+1}, A_{i+1} and n_{i+1} as follows. If $r_{i+1} \leq e(A_i)$, then we take for G_{i+1} an arbitrary subgraph of A_i which has size r_{i+1} and define A_{i+1} to be the graph obtained from A_i by removing all edges of G_{i+1}. Finally we put $n_{i+1} = n_i$. If $r_{i+1} > e(A_i)$ then it follows from (11) that $r_1 + \ldots + r_{i+1} > s_1 + \ldots + s_{n_i}$. This implies that $n_i < t$. Moreover, by (6) we have $r_1 + \ldots + r_{i+1} > r_1 + \ldots + r_{n_i}$, which yields $i+1 > n_i$. Hence $s_{n_i+1} \geq s_{i+1} \geq r_{i+1}$. By Lemma 1 it follows that there are edge-disjoint graphs A' and B' such that $A' \cup B' = A_i \cup H_{n_i+1}$, $\Delta(A') \leq p$, $\Delta(B') \leq p$, $e(A') = r_{i+1}$ and $e(B') = e(A_i) + e(H_{n_i+1}) - r_{i+1}$. We define $G_{i+1} = A'$, $A_{i+1} = B'$, $n_{i+1} = n_i + 1$. One can readily verify that G_{i+1}, A_{i+1} and n_{i+1} satisfy (7)-(11). So, by (7), (8) and (11) $(r_1, \ldots, r_k)$ is a p-feasible sequence for G, as claimed.

Let us also observe that the assertion is trivially true when $t = 1$. So, now let us assume that the theorem fails for some graph G. Let $s = (s_1,\dots,s_t)$ and $r = (r_1,\dots,r_k)$ be sequences such that $r \leqslant s$, s is p-feasible and r is not. We can assume that G, r and s are chosen so that t is the least possible. Moreover, we can assume that r and s minimize $|\{q : r \leqslant q \leqslant s\}|$. Clearly we have $r \neq s$, $t \geqslant 2$ and $s_1 - s_t \geqslant 2$. Let $\{H_1,\dots,H_t\}$ be a p-decomposition of G such that $e(H_i) = s_i$, for $1 \leqslant i \leqslant t$. Suppose first that there is n, $1 \leqslant n < t$, such that $\sum_{i=1}^{n} r_i = \sum_{i=1}^{n} s_i$. Then $(s_1,\dots,s_n)$ is p-feasible for $H_1 \cup \dots \cup H_n$, $(r_1,\dots,r_n) \leqslant (s_1,\dots,s_n)$ and $n < t$, hence $(r_1,\dots,r_n)$ is p-feasible for $H_1 \cup \dots \cup H_n$. Similarly $(r_{n+1},\dots,r_k)$ is p-feasible for $H_{n+1} \cup \dots \cup H_t$, consequently r is p-feasible for G, which contradicts the way s and r were chosen. So, for every n, $1 \leqslant n \leqslant t$, we have $\sum_{i=1}^{n} r_i < \sum_{i=1}^{n} s_i$. Consider a sequence $s_1-1, s_2,\dots,s_{t-1}, s_t+1$ and relabel it $s'_1,\dots,s'_t$ in an nonincreasing order and put $s' = (s'_1,\dots,s'_t)$. Clearly $r \leqslant s' < s$. Moreover s' is p-feasible which follows from Lemma 1 applied to H_1 and H_t. But this again contradicts the way the sequences r and s were chosen. Hence the proof is complete.

Now it is clear that to exhibit large sets of p-feasible sequences one needs to have p-feasible sequences which lie high in the partial order of sequences introduced above. It seems to be difficult to determine all maximal p-feasible sequences. The theorem below gives only very poor answer to the problem outlined.

T h e o r e m 3. Let $t = \chi'_p(G)$ and let $m \equiv e(G) \pmod t$, $0 \leqslant m < t$. Put $s_1 = \dots = s_m = \lceil e(G)/t \rceil$ and $s_{m+1} = \dots = s_t = \lfloor e(G)/t \rfloor$. Then $(s_1,\dots,s_t)$ is p-feasible.

P r o o f . The assertion follows easily by repeated application of Lemma 1.

Let us return to p-decompositions into graphs of equal size. To this end let

$$\mathcal{G}_{r,p} = \{G : \Delta(G) = kp \text{ and } e(G) \leqslant kr, \text{ for some positive integer } k\}.$$

L e m m a 4. For every pair of integers $p,r \geqslant 1$, there are only finitely many graphs in $\mathcal{G}_{r,p}$ satisfying $\chi'_p(G) > \lceil \Delta(G)/p \rceil$.

P r o o f . Consider $G \in \mathcal{G}_{r,p}$. If $k \geqslant 3r/p^2$ then $e(G) < \frac{1}{8}(3(\Delta(G))^2 + 6\Delta(G) - 1)$. So, $\chi'(G) = \Delta(G)$ (see [4], p.119) which implies $\chi'_p(G) = \lceil \Delta(G)/p \rceil$.

R e m a r k 5. It should be noted that if p is even then stronger result holds, namely $\chi'_p(G) = \lceil \Delta(G)/p \rceil$ for every graph G. It follows easily from the fact that every 2d-regular graph has a 2-factorization.

As a corollary of these results we obtain the following theorem which gives the positive answer to our initial question about finiteness of the set of exceptions.

T h e o r e m 6. The set of graphs which satisfy (1) and (2′) and do not have a p-decomposition into graphs of size r is finite. Moreover, if p is even, then this set is empty.

P r o o f . By (3) and Lemma 4 it follows that there are only finitely many graphs G satisfying (1) and (2′) such that $\chi'_p(G) > e(G)/r$. So, the first part of the assertion follows from Theorems 2 and 3. The second part follows from Remark 5 and Theorems 2 and 3 again.

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INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW,
00-661 WARSZAWA
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