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THE APPLICATION OF THE FINITE ELEMENTS METHOD TO THE INITIAL PROBLEMS OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

Dedicated to the memory of Professor Roman Sikorski

The method of finite elements generally is used for the boundary value problems. In some papers concerned with its application to technical problems this method is also used for initial value problems but without any theoretical explanations of the convergence of the approximative solutions to the exact one (see e.g. [2], [3]). To this aim the latter problem will be studied in this paper. The results of this work generalise to the nonlinear case the results of [4], [5].

1. Let $T>0$ be a fixed constant and $f: \langle 0,T\rangle \times R^2 \longrightarrow R$ be a Lipschitz continuous function, i.e. there exists some $L>0$, such that for any $t_1,t_2 \in \langle 0,T\rangle$ and for any $x_1,x_2,y_1,y_2 \in R$

$$(1)\quad |f(t_1,x_1,y_1) - f(t_2,x_2,y_2)| \leqslant L(|t_1-t_2|+|x_1-x_2|+|y_1-y_2|).$$

For any $x_0,x_1 \in R$ let us consider the Cauchy problem (for $t \in \langle 0,T\rangle$):

$$(2) \qquad \ddot{x} = f(t,x,\dot{x})$$

and

$$(3) \qquad x(0) = x_0, \ \dot{x}(0) = x_1.$$

By assumptions made on the function f problem (2), (3) has exactly one solution existing on $\langle 0,T\rangle$.

2. Lemma. For any $t \in \langle 0,T\rangle$ and any $\gamma \in R$ such that $t+\gamma \in \langle 0,T\rangle$ the solution x of (2), (3) satisfies

$$(4) \qquad x(t+\gamma) = x(t) + \gamma\dot{x}(t) + \frac{1}{2}\gamma^2\ddot{x}(t) + O(\gamma^3),$$

Proof. The solution x of the (2), (3) is of the class $C^2(\langle 0,T\rangle)$. Thus the functions $\dot{x}$ and $\ddot{x}$ are bounded on $\langle 0,T\rangle$. Let

$$K_1 = \max_{\langle 0,T\rangle} |\dot{x}(t)|, \quad K_2 = \max_{\langle 0,T\rangle} |\ddot{x}(t)|.$$

By Taylor's rule

$$(5) \qquad x(t+\gamma) = x(t) + \gamma\ \dot{x}(t) + \frac{1}{2}\gamma^2\ddot{x}(t+\theta\gamma)$$

for some $\theta \in (0,1)$. For the solution x we have (for $\delta = \theta\gamma$) by (2) and (1)

$$|\ddot{x}(t+\delta) - \ddot{x}(t)| = |f(t+\delta, x(t+\delta), \dot{x}(t+\delta)) - f(t, \dot{x}(t), \dot{x}(t))| \leq$$

$$\leq L(|\delta| + |x(t+\delta) - x(t)| + |\dot{x}(t+\delta) - \dot{x}(t)|) =$$

$$= L(|\delta| + |\dot{x}(t+\theta_1\delta)||\delta| + |\ddot{x}(t+\theta_2\delta)||\delta|) \leq$$

$$\leq L(1+K_1+K_2)|\delta| \leq L(1+K_1+K_2)|\gamma|$$

for some $\theta_1, \theta_2 \in (0,1)$. Then $\ddot{x}(t+\theta\gamma) = \ddot{x}(t) + O(\gamma)$ and then from (5) we obtain (4).

3. Now let us introduce the finite elements method by the standard procedure. Choose any natural integer N, let $h = \frac{T}{N}$. Let φ_p^h be the functions given for p=0,1,...,N by

$$(6) \qquad \varphi_p^h(t) = \begin{cases} \frac{t}{h} - p + 1 & \text{for} \quad t \in (ph-h,\ ph\rangle \\ -\frac{t}{h} + p + 1 & \text{for} \quad t \in (ph,\ ph+h), \\ 0 & \text{for} \quad t \in \langle 0,T\rangle \setminus (ph-h, ph+h), \end{cases}$$

For p = 0 and p = N the functions φ_p^h are considered only on $\langle 0,T\rangle$. The functions φ_p^h are differentiable for all $t \in \langle 0,T\rangle$ except some of the points ph, p = 1,2,...,N-1, where they have the left and right derivatives.

We will look for approximate solution x^h of the form:

$$(7) \qquad x^h(t) = \sum_{k=0}^{N} \alpha_k^h \varphi_k^h(t),$$

where $\alpha_k^h \in R$, k = 0,1,2,...,N are unknown constants. Let us notice that for any $\alpha_k^h \in R$ the function x^h is differentiable for all $t \in \langle 0,T\rangle$ except perhaps for points kh, k = 1,2,...,N-1 where it has left and right derivatives. From now on $\dot{x}^h(t)$ will denote the right derivative in t. Observe that

$$(8) \qquad \alpha_k^h = x^h(kh).$$

The function x^h of the form (7) is said to be an approximate solution if it satisfies the initial conditions

$$(9) \qquad x^h(0) = x_0, \quad \dot{x}^h(0) = x_1$$

and the "Galerkin rule" of the form

$$(10) \quad \int_0^T \left[\dot{x}^h(t)\dot{\varphi}_l(t) + f(t, x^h(t), \dot{x}^h(t))\varphi_l(t)\right] dt = 0, \quad l=1,2,\ldots,N-1.$$

The first condition gives $\alpha_o^h = x_o, \alpha_o^h\dot{\varphi}_o^h(0) + \alpha_1^h\dot{\varphi}_1^h(0) = x_1$ and thus

$$\alpha_o^h = x_o, \quad \frac{\alpha_1^h - \alpha_o^h}{h} = x_1. \tag{11}$$

Let us examine the equations (10). The functions φ_l have support on $\langle lh-h,\ lh+h\rangle$, and thus

$$\int_{lh-h}^{lh+h} \left[\dot{x}^h(t)\dot{\varphi}_l(t) + f(t,x^h(t),\dot{x}^h(t))\varphi_l(t)\right] dt = 0, \tag{12}$$
$$l = 1,2,\dots,N-1.$$

Inserting x^h from (7) into (12) and taking into account that in the interval $\langle lh-h,\ lh+h\rangle$ only three functions: $\varphi_{l-1}, \varphi_l, \varphi_{l+1}$ are not identically equal to zero we obtain

$$\int_{lh-h}^{lh+h} \left[\sum_{k=l-1}^{l+1} \alpha_k^h\dot{\varphi}_k^h(t)\dot{\varphi}_l^h(t) + \right. \tag{13}$$
$$\left. + f\left(t,\ \sum_{k=l-1}^{l+1} \alpha_k^h\varphi_k^h(t),\ \sum_{k=l-1}^{l+1} \alpha_k^h\dot{\varphi}_k^h(t)\right)\varphi_l^h(t)\right] dt = 0,$$
$$l = 1,2,\dots,N-1.$$

Dividing the interval of integration into two parts: $\langle lh-h, lh\rangle$ and $\langle lh, lh+h\rangle$ and using (6) we get

$$\frac{\alpha_{l+1}^h - 2\alpha_l^h + \alpha_{l-1}^h}{h} = \tag{14}$$
$$= \int_{lh-h}^{lh} f\left(t, \alpha_{l-1}^h\varphi_{l-1}^h(t) + \alpha_l^h\varphi_l^h(t), \alpha_{l-1}^h\dot{\varphi}_{l-1}^h(t) + \right.$$
$$\left. + \alpha_l^h\dot{\varphi}_l^h(t)\right)\varphi_l^h(t)dt +$$

$$+\int_{lh}^{lh+h} f\left(t,\alpha_l^h\varphi_l^h(t)+\alpha_{l+1}^h\varphi_{l+1}^h(t),\alpha_l\dot{\varphi}_l^h(t)+\alpha_{l+1}^h\dot{\varphi}_{l+1}^h(t)\right)\varphi_l^h(t)dt,$$

$$l = 1,2,\dots,N-1.$$

To the integral given in (14) we apply the trapezoidal rule of integration, hence

$$\frac{\alpha_{l+1}^h - 2\alpha_l^h + \alpha_{l-1}^h}{h^2} = \tag{15}$$

$$= \frac{1}{2}\left\{ f\left(lh,\alpha_l^h, \frac{1}{h}(\alpha_l^h-\alpha_{l-1}^h)\right) + f\left(lh,\alpha_l^h, \frac{1}{h}(\alpha_{l+1}^h-\alpha_l^h)\right)\right\},$$

$$l = 1,2,\dots,N-1.$$

The received system (15) with the conditions (11) will be treated as difference schema for the problem (2), (3). For the system (15) and (11) we will prove that for sufficiently small h all the coefficients α_l^h, $l = 0,1,\dots,N$ can be found, and that the corresponding approximate solutions (7) converge to the solution of (2), (3).

4. The proof of solvability of the system (11), (15) (for sufficiently small h) will be presented for the more general case. Let $\varepsilon_k \in R$, $k = 0,1,\dots,N$, and consider the system

$$\beta_0 = x_0 + \varepsilon_0, \quad \frac{\beta_1 - \beta_0}{h} = x_1 + \varepsilon_1,$$

$$\frac{\beta_{l+1} - 2\beta_l + \beta_{l-1}}{h^2} = \tag{16}$$

$$= \frac{1}{2}\left\{ f\left(lh,\beta_l, \frac{1}{h}(\beta_l-\beta_{l-1})\right) + f\left(lh,\beta_l, \frac{1}{h}(\beta_{l+1}-\beta_l)\right)\right\} + \varepsilon_{l+1},$$

$$l = 1,2,\dots,N-1.$$

The system (11), (15) can be obtained from the system (16) by taking $\varepsilon_k = 0$, $k = 0,1,\dots,N$. For the proof we use the method of mathematical induction. Suppose that we have already found $\beta_0, \beta_1, \ldots, \beta_l$. For β_{l+1} we have the equation

$$(17) \qquad \beta_{l+1} = 2\beta_l - \beta_{l-1} + \frac{h^2}{2} f\Big(lh, \beta_l, \frac{1}{h}(\beta_l - \beta_{l-1})\Big) + \frac{h^2}{2} f\Big(lh, \beta_l, \frac{1}{h}(\beta_{l+1} - \beta_l)\Big) + h^2 \varepsilon_{l+1}.$$

Consider the operator $\mathcal{T} : R \longrightarrow R$ given by

$$\mathcal{T}\gamma = 2\beta_l - \beta_{l-1} + \frac{h^2}{2} f\Big(lh, \beta_l, \frac{1}{h}(\beta_l - \beta_{l-1})\Big) + \frac{h^2}{2} f\Big(lh, \beta_l, \frac{1}{h}(\gamma - \beta_l)\Big) + h^2 \varepsilon_{l+1}.$$

For two given numbers γ and $\tilde{\gamma}$ we have

$$|\mathcal{T}\gamma - \mathcal{T}\tilde{\gamma}| = \frac{h^2}{2} \Big| f\Big(lh, \beta_l, \frac{1}{h}(\gamma - \beta_l)\Big) - f\Big(lh, \beta_l, \frac{1}{h}(\tilde{\gamma} - \beta_l)\Big)\Big| \le \frac{h^2}{2} L \frac{1}{h} |\gamma - \tilde{\gamma}| = \frac{h}{2} L |\gamma - \tilde{\gamma}|.$$

If we assume that $h < \frac{2}{L}$ then the operator $\mathcal{T}$ will be contractive, hence the existence of a unique solution of the equation $\mathcal{T}\gamma = \gamma$, i.e. a unique solution β_{l+1} of the equation (17) results from the Banach principle.

5. To prove that the received approximate solutions converge to the solution of (2), (3) we show that the corresponding interative schema (see (8), (11) and (15))

$$(18) \qquad x^h(0) - x_0 = 0, \qquad \frac{x^h(h) - x^h(0)}{h} - x_1 = 0$$

and

$$\frac{x^h((l+1)h) - 2x^h(lh) + x^h((l-1)h)}{h^2} - \tag{19}$$

$$- \frac{1}{2}\left\{ f\left(lh,\ x^h(lh),\ \frac{1}{h}\left(x^h(lh) - x^h((l-1)h)\right)\right) + \right.$$

$$\left. + f\left(lh,\ x^h(lh),\ \frac{1}{h}\left(x^h((l+1)h) - x^h(lh)\right)\right)\right\} = 0,$$

$$l = 1,2,\ldots,N-1,$$

approximates the problem (2), (3) and is stable (see [1]).

3. Inserting to the absolute values of the left sides of the equations (18) the values of the solution x of the problem (2), (3) instead of x^h, we obtain

$$|x(0) - x_0| = 0,$$

$$\left|\frac{x(h)-x(0)}{h} - x_1\right| = \left|\frac{1}{h}\,\dot{x}(\theta h)h - \dot{x}(0)\right| = \left|\ddot{x}(\theta_1 h)\theta h\right| \leq K_2 h,$$

where $\theta, \theta_1 \in (0,1)$ and K_2 is the constant from the section 2. Denote by A the corresponding expression obtained from the equations from (19) (with x instead of x^h), i.e.

$$A := \left| \frac{x((l+1)h) - 2x(lh) + x((l-1)h)}{h^2} - \right.$$

$$- \frac{1}{2}\left\{ f\left(lh,\ x(lh),\ \frac{1}{h}\,(x(lh) - x((l-1)h))\right) + \right.$$

$$\left.\left. + f\left(lh,\ x(lh),\ \frac{1}{h}\,(x((l+1)h) - x(lh))\right)\right\}\right| .$$

By the lemma we have

$$\frac{x((l+1)h) - 2x(lh) + x((l-1)h)}{h^2} =$$

$$= \ddot{x}(lh) + O(h) = f(lh,\ x(lh),\ \dot{x}(lh)) + O(h)$$

hence

$$A\leq \frac{1}{2}\left| f(lh,x(lh),\dot{x}(lh)) - f\left(lh,x(lh), \frac{1}{h}(x(lh)-x((l-1)h))\right)\right| +$$

$$+\frac{1}{2}\left|f(lh,x(lh),\dot{x}(lh))-f\left(lh,x(lh), \frac{1}{h}(x((l+1)h)-x(lh))\right)\right| +O(h)\leq$$

$$\leq \frac{1}{2} L\left|\dot{x}(lh) - \frac{x(lh)-x((l-1)h)}{h}\right| +$$

$$+\frac{1}{2} L\left|\dot{x}(lh) - \frac{x((l+1)h)-x(lh)}{h}\right| + O(h) =$$

$$= \frac{1}{2} L|\dot{x}(lh)-\dot{x}(lh-\Theta_1 h)| + \frac{1}{2} L|\dot{x}(lh)-\dot{x}(lh+\Theta_2 h)| + O(h) =$$

$$= \frac{1}{2} L\, h|\ddot{x}(lh-\Theta_3 h)| + \frac{1}{2} Lh|\ddot{x}(lh+\Theta_4 h)| + O(h)\leq K_2 Lh+O(h) = O(h)$$

for some Θ_1, Θ_2, Θ_3, $\Theta_4 \in (0,1)$ and K_2 from the lemma. Thus we have verified that the iterative schema (18), (19) approximates the problem (2), (3) with the order 1.

7. Now we want to show that the schema (18), (19) is stable. Together with the schema (18), (19) we shall examine the perturbate schema of the form

$$(20)\quad \begin{cases} \tilde{x}^h(0) - x_0 = \varepsilon_0, \\ \dfrac{\tilde{x}^h(h) - \tilde{x}^h(0)}{h} - x_1 = \varepsilon_1, \\ \dfrac{\tilde{x}^h((l+1)h) - 2\tilde{x}^h(lh) + \tilde{x}^h((l-1)h)}{h^2} - \\ -\dfrac{1}{2}\Big\{ f\left(lh,\tilde{x}^h(lh), \frac{1}{h}(\tilde{x}^h(lh) - \tilde{x}^h((l-1)h))\right) + \\ + f\left(lh,\tilde{x}^h(lh), \frac{1}{h}(\tilde{x}^h((l+1)h) - \tilde{x}^h(lh))\right)\Big\} = \varepsilon_{l+1}, \\ \qquad\qquad l = 1,2,\ldots,N-1. \end{cases}$$

From the section 4 we know that from the system (20) we can find all values $\tilde{x}^h(kh)$, $k = 0,1,\dots,N$.

Let $c = 4eLT(T+1)$. In the following we restrict ourselves to integers $N \geq c$. In R^{N+1} we introduce a special norm: if $y = (y_0,y_1,\dots,y_N) \in R^{N+1}$ then

$$\|y\| = \max_{k=0,1\dots N} (|y_k|\varphi(k))$$

where

$$\varphi(k) = e^{-\frac{k}{N}c} .$$

To show that the schema (18), (19) is stable we must prove that there exists such $K \in R$ that for all $N \geq c$ and for the corresponding vectors

$$x^h = (x^h(0),x^h(h),\dots,x^h((N-1)h),x^h(T)),$$

$$\tilde{x}^h = (\tilde{x}^h(0),\tilde{x}^h(h),\dots,\tilde{x}^h((N-1)h),\tilde{x}^h(T))$$

(where $h = \frac{T}{N}$) from (18), (19), (20) and for $\varepsilon = (\varepsilon_0,\varepsilon_1,\dots,\varepsilon_N)$ we have $\|x^h - \tilde{x}^h\| \leq K\|\varepsilon\|$.

For simplicity we introduce new variables γ_l, $\tilde{\gamma}_l$, $l = 0,1,\dots,N$, taking $\gamma_0 = 0$, $\tilde{\gamma}_0 = 0$ and

$$\text{(21)} \qquad \begin{cases} \gamma_l := \dfrac{x^h(lh) - x^h((l-1)h)}{h}, \\ \tilde{\gamma}_l := \dfrac{\tilde{x}^h(lh) - \tilde{x}^h((l-1)h)}{h}, \\ l = 1,2,\dots,N. \end{cases}$$

Then

$$\text{(22)} \qquad x^h(lh) = x^h(0) + h\sum_{p=1}^{l} \gamma_p, \quad \tilde{x}^h(lh) = \tilde{x}^h(0) + h\sum_{p=1}^{l} \tilde{\gamma}_p.$$

In these new variables the schemas (18), (19) and (20) have the forms

$$(23)\quad \begin{cases} \gamma_0 = 0, \\ \gamma_1 = x_1, \\ \gamma_{k+1} = \gamma_k + \frac{1}{2}h\left\{ f\left(kh, x_0 + h\sum_{l=1}^{k}\gamma_l, \gamma_k\right) + \right. \\ \qquad \left. + f\left(kh, x_0 + h\sum_{l=1}^{k}\gamma_l, \gamma_{k+1}\right)\right\} \end{cases}$$

and

$$(24)\quad \begin{cases} \tilde{\gamma}_0 = 0, \\ \tilde{\gamma}_1 = x_1 + \varepsilon_1, \\ \tilde{\gamma}_{k+1} = \tilde{\gamma}_k + \frac{1}{2}h\left\{ f\left(kh, x_0 + \varepsilon_0 + h\sum_{l=1}^{k}\tilde{\gamma}_l, \tilde{\gamma}_k\right) + \right. \\ \qquad \left. + f\left(kh, x_0 + \varepsilon_0 + h\sum_{l=1}^{k}\tilde{\gamma}_l, \tilde{\gamma}_{k+1}\right)\right\} + \varepsilon_{k+1}h. \end{cases}$$

By summing up the last equations in (23) and (24) in k from 1 to $p-1$ we receive

$$\begin{cases} \gamma_0 = 0, \qquad \gamma_1 = x_1, \\ \gamma_p = \gamma_1 + \frac{h}{2}\sum_{k=1}^{p-1} f\left(kh, x_0 + h\sum_{l=1}^{k}\gamma_l, \gamma_k\right) + \\ \qquad + \frac{h}{2}\sum_{k=1}^{p-1} f\left(kh, x_0 + h\sum_{l=1}^{k}\gamma_l, \gamma_{k+1}\right) \end{cases}$$

and

$$(26)\quad \begin{cases} \tilde{\gamma}_0 = 0, \qquad \tilde{\gamma}_1 = x_1+\varepsilon_1, \\ \tilde{\gamma}_p = \tilde{\gamma}_1 + \frac{h}{2}\sum_{k=1}^{p-1} f\left(kh, x_0+\varepsilon_0+h\sum_{l=1}^{k}\tilde{\gamma}_l, \tilde{\gamma}_k\right) + \\ \quad + \frac{h}{2}\sum_{k=1}^{p-1} f\left(kh, x_0+\varepsilon_0+h\sum_{l=1}^{k}\tilde{\gamma}_l, \tilde{\gamma}_{k+1}\right) + h\sum_{k=1}^{p-1}\varepsilon_{k+1}. \end{cases}$$

For $p = 2,3,\dots,N$ we have

$$|\tilde{\gamma}_p-\gamma_p| \leqslant |\tilde{\gamma}_1-\gamma_1| + \frac{h}{2}\sum_{k=1}^{p-1}\left| f\left(kh, x_0+\varepsilon_0+h\sum_{l=1}^{k}\tilde{\gamma}_l, \tilde{\gamma}_k\right) - \right.$$

$$- f\left(kh, x_0+h\sum_{l=1}^{k}\gamma_l, \gamma_k\right)\Bigg| + \frac{h}{2}\sum_{k=1}^{p-1}\left| f\left(kh, x_0+\varepsilon_0+h\sum_{l=1}^{k}\tilde{\gamma}_l, \tilde{\gamma}_{k+1}\right) - \right.$$

$$- f\left(kh, x_0+h\sum_{l=1}^{k}\gamma_l, \gamma_{k+1}\right)\Bigg| + h\sum_{k=2}^{p}|\varepsilon_k|.$$

By the Lipschitz condition we obtain

$$|\tilde{\gamma}_p-\gamma_p| \leqslant |\tilde{\gamma}_1-\gamma_1| + hL\sum_{k=1}^{p-1}\left(|\varepsilon_0| + h\sum_{l=1}^{k}|\tilde{\gamma}_l-\gamma_l|\right) +$$

$$+ \frac{h}{2}L\sum_{k=1}^{p-1}|\tilde{\gamma}_k-\gamma_k| + \frac{h}{2}L\sum_{k=1}^{p-1}|\tilde{\gamma}_{k+1}-\gamma_{k+1}| + h\sum_{k=2}^{p}|\varepsilon_k| =$$

$$= hL(p-1)|\varepsilon_0| + |\varepsilon_1| + h^2L\sum_{k=1}^{p-1}\sum_{l=1}^{k}|\tilde{\gamma}_l-\gamma_l| + \frac{h}{2}L\sum_{k=1}^{p-1}|\tilde{\gamma}_k-\gamma_k| +$$

$$+ \frac{h}{2}L\sum_{k=2}^{p-1}|\tilde{\gamma}_k-\gamma_k| + h\sum_{k=2}^{p}|\varepsilon_k| + \frac{h}{2}L|\tilde{\gamma}_p-\gamma_p|.$$

But $N \geq c \geq TL$, then for $h = \frac{T}{N}$ we have $h \leq \frac{1}{L}$ and $1 - \frac{h}{2} L \geq \frac{1}{2}$, thus

$$|\tilde{\gamma}_p - \gamma_p| \leq 2\Big\{ hL(p-1)\,|\varepsilon_0| + |\varepsilon_1| + h^2 L \sum_{k=1}^{p-1} \sum_{l=1}^{k} |\tilde{\gamma}_l - \gamma_l| +$$

$$+ \frac{h}{2} L \sum_{k=1}^{p-1} |\tilde{\gamma}_k - \gamma_k| + \frac{h}{2} L \sum_{k=2}^{p-1} |\tilde{\gamma}_k - \gamma_k| + h \sum_{k=2}^{p} |\varepsilon_k| \Big\}.$$

But

$$\sum_{k=1}^{p-1} \sum_{l=1}^{k} |\tilde{\gamma}_l - \gamma_l| = \sum_{l=1}^{p-1} (p-1-l)\,|\tilde{\gamma}_l - \gamma_l| \leq (p-1) \sum_{l=1}^{p-1} |\tilde{\gamma}_l - \gamma_l|,$$

hence

$$|\tilde{\gamma}_p - \gamma_p| \leq$$

$$\leq 2\Big\{ hL(p-1)\,|\varepsilon_0| + |\varepsilon_1| + \left(h^2 L(p-1) + hL\right) \sum_{l=1}^{p-1} |\tilde{\gamma}_l - \gamma_l| + h \sum_{k=2}^{p} |\varepsilon_k| \Big\}.$$

If $p-1 \leq N$ and $hN = T$ we receive

$$|\tilde{\gamma}_p - \gamma_p| \leq 2\Big\{ LT\,|\varepsilon_0| + |\varepsilon_1| + hL(T+1) \sum_{l=1}^{p-1} |\tilde{\gamma}_l - \gamma_l| + h \sum_{k=2}^{p} |\varepsilon_k| \Big\},$$

$$p = 2,3,4,\ldots,N.$$

Multiplying the inequalities by $\varphi(p)$ and using the norm for the vectors $\tilde{\gamma} = (\tilde{\gamma}_0, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_N)$, $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_N)$, $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N)$ we observe that

$$|\tilde{\gamma}_p - \gamma_p|\varphi(p) \leq 2\Big\{ LT\,|\varepsilon_0|\varphi(p) + |\varepsilon_1|\varphi(p) +$$

$$+ hL(T+1) \sum_{l=1}^{p-1} \frac{\varphi(p)}{\varphi(l)} |\tilde{\gamma}_l - \gamma_l|\varphi(l) + h \sum_{k=2}^{p} \frac{\varphi(p)}{\varphi(k)} |\varepsilon_k|\varphi(k) \Big\} \leq$$

$$\leq 2\Big\{ LT\varphi(p)\|\varepsilon\| + \frac{\varphi(p)}{\varphi(1)}\|\varepsilon\| + hL(T+1) \sum_{l=1}^{p-1} \frac{\varphi(p)}{\varphi(l)} \|\tilde{\gamma} - \gamma\| + h \sum_{k=2}^{p} \frac{\varphi(p)}{\varphi(k)} \|\varepsilon\| \Big\}.$$

Observing that $\varphi(p)\leqslant\varphi(1)\leqslant 1$ and that

$$\sum_{l=1}^{p-1}\frac{\varphi(p)}{\varphi(l)}=\sum_{l=1}^{p-1}e^{-(p-l)c/N}=\sum_{l=1}^{p-1}e^{-l\,\frac{c}{N}}\leqslant\frac{1}{1-e^{-\frac{c}{N}}}$$

we obtain

$$|\tilde{\gamma}_p-\gamma_p|\varphi(p)\leqslant 2\left\{LT\|\varepsilon\|+\|\varepsilon\|+\frac{h}{1-e^{-\frac{c}{N}}}\|\varepsilon\|+hL(T+1)\frac{1}{1-e^{-\frac{c}{N}}}\|\tilde{\gamma}-\gamma\|\right\},$$

$$p=2,3,\dots,N.$$

Now we can see that this inequality holds also for $p=0,1$. Then $\left(h=\frac{T}{N}\right)$

$$\|\tilde{\gamma}-\gamma\|\leqslant 2\left\{LT+1+\frac{T}{N\left(1-e^{-\frac{c}{N}}\right)}\right\}\|\varepsilon\|+\frac{2L(T+1)T}{N\left(1-e^{-\frac{c}{N}}\right)}\|\tilde{\gamma}-\gamma\|.$$

Accordingly $1-e^{-x}\geqslant\frac{1}{e}x$ for $x\in(0,1>$; by our proposition we have $N\geqslant c$, then $\frac{c}{N}\leqslant 1$ and $1-e^{-\frac{c}{N}}\geqslant\frac{1}{e}\frac{c}{N}$, $\frac{1}{N\left(1-e^{-\frac{c}{N}}\right)}\leqslant\frac{e}{c}$.

Thus,

$$\|\tilde{\gamma}-\gamma\|\leqslant 2(LT+1+e\,\tfrac{T}{c})\,\|\varepsilon\|+2e\,\frac{LT(T+1)}{c}\|\tilde{\gamma}-\gamma\|=$$

$$=2(LT+1+e\,\tfrac{T}{c})\,\|\varepsilon\|+\tfrac{1}{2}\|\tilde{\gamma}-\gamma\|$$

and

$$\|\tilde{\gamma}-\gamma\|\leqslant 4(LT+1+e\,\tfrac{T}{c})\|\varepsilon\|\,.$$

But $\tilde{x}^h(lh)-x^h(lh)=\tilde{x}^h(0)-x^h(0)+h\sum_{p=1}^{l}(\tilde{\gamma}_p-\gamma_p)=$

$=\varepsilon_0+h\sum_{p=1}^{l}(\tilde{\gamma}_p-\gamma_p)$, then

$$|\tilde{h}^h(lh)-x^h(lh)|\varphi(l) \leqslant |\varepsilon_0|\varphi(l) + h\sum_{p=1}^{l} |\tilde{\gamma}_p-\gamma_p|\varphi(l) \leqslant$$

$$\leqslant \|\varepsilon\| + h\sum_{p=1}^{l} \frac{\varphi(l)}{\varphi(p)}\|\tilde{\gamma}-\gamma\| \leqslant$$

$$\leqslant \|\varepsilon\| + \frac{h}{1-e^{-\frac{c}{N}}}\|\tilde{\gamma}-\gamma\| \leqslant \|\varepsilon\| + e^{\frac{T}{c}}\|\tilde{\gamma}-\gamma\| \leqslant (4LT+4+5e^{\frac{T}{c}})\|\varepsilon\|$$

for $l = 0,1,\dots,N$. Thus, for $K = 4LT + 4 + 5e^{\frac{T}{c}} = 4(LT + 1) + \frac{5}{4L(T+1)}$

$$\|\tilde{x}^h - x^h\| \leqslant K\|\varepsilon\| ,$$

what we wanted to prove.

8. Remark. If the initial problem (2), (3) is given for $f: \langle 0,\infty)\times R^2 \longrightarrow R$ and f satisfies the inequality (1) for all $t_1, t_2 \in \langle 0,\infty)$ we can introduce the functions φ_k^h, (as in the section 3), but for $k \in \mathcal{N}\cup\{0\}$ and approximate solutions of the form $x^h(t) = \sum_{k=0}^{\infty}\alpha_k^h\varphi_k^h(t)$ (in "Galerkin rule" we take $\int_0^\infty$). Observe that for any $\alpha_k^h \in R$ the given series converge for any $t\in\langle 0,\infty)$. In this case we have the same conditions (8), (11), (15) for the constants α_l^h (the conditions (15) are given for $l \in \mathcal{N}$). For the system (11), (15) for $l \in \mathcal{N}$ the same proof as in the section 4 shows that it is possible to find all α_l^h, $l \in \mathcal{N}\cup\{0\}$. The proofs given in sections 5-7 show that approximate solutions x^h tend to the solution of (2), (3) on every interval $\langle 0,T\rangle$. Thus we have the convergence on full interval $\langle 0,\infty)$.

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