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CONCERNING A GOURSAT PROBLEM FOR SOME PARTIAL DIFFERENTIAL EQUATION OF ORDER 2p

Dedicated to the memory of Professor Roman Sikorski

1. In almost all papers devoted to Goursat problems for partial differential equations of order m greater than 2, the boundary conditions have been given either on two curves (surfaces) or on the characteristics of the equation considered or, finally, on the characteristics and one non-characteristic curve (see [6], [9] and the references in [3] and [12]). The case of a greater number of non-characteristic curves was examined in the papers [10] of O.Sjöstrand, [11] of Z.Szmydt and [3], [4] of the present author[1]. Let us note, however, that the method applied in [3] and [4] did not make possible finding formulas for the solutions nor proving the uniqueness of these solutions.

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[1] In [10] a problem with the boundary conditions set on four straight half-lines is considered in the class of analytic functions. In [11] the existence is proved of a solution of a non-linear problem for a system of equations of arbitrary order m with the boundary conditions given on m curves. Papers [3] and [4] concern an equation of order 2p, called a polyvibrating equation of D.Mangeron, and are devoted to Goursat problems with the boundary conditions of the type different from that in [11], given on 2p straight half-lines and p curves, respectively.

In paper [5] we formulated a theorem concerning the existence and the form of a solution of a Goursat problem for the polyvibrating equation of arbitrary even order 2p in a Banach space, with the boundary conditions given on a set of 2p curves emanating from a common point.

The aim of this paper is to prove this theorem.

Like in papers [3] and [4] we reduce the problem to a system of functional equations but then we examine this system in a way different from that in the said papers. The present method makes it possible both to find the solution in the form of a series and to prove its uniqueness.

2. Let Ω be the rectangle

$$\Omega = \left\{ (x,y) \in R^2 \ : \ 0 \leq x \leq A; \ 0 \leq y \leq B \right\},$$

where $0 < A, B < \infty$, and consider a system of 2p curves, where $p \geq 2$, given by the equations $y = f_i(x)$ and $x = h_i(y)$ ($f_i : \langle 0,A \rangle \to \langle 0,B \rangle$; $h_i : \langle 0,B \rangle \to \langle 0,A \rangle$ for $i = 1,2,\ldots,p$), respectively, passing through the origin $O(0,0)$ of the coordinates system and not intersecting elsewhere. In what follows Y denotes a Banach space with norm $\|\cdot\|$.

Let us consider the following partial differential equation

$$L^p u(x,y) = c(x,y) \tag{1}$$

(called the polyvibrating equation of D. Mangeron), where $L = \frac{\partial^2}{\partial x \partial y}$, $(x,y) \in \Omega$ and $c : \Omega \to Y$ is a given function.

By a solution of equation (1) in Ω we mean a function $u : \Omega \to Y$ that possesses continuous derivatives $D^\beta u$ (where $D^\beta = \frac{\partial^{|\beta|}}{\partial x^{\beta_1} \partial y^{\beta_2}}$; $|\beta| = \beta_1 + \beta_2$; $0 \leq \beta_1, \beta_2 \leq p$) in Ω and satisfies (1) at each point of Ω.

L e m m a 1. Let $c : \Omega \to Y$ be a continuous function. If $u : \Omega \to Y$ is of the form

$$(2) \quad u(x,y) = R(x,y) + \sum_{\alpha=1}^{p} \left[y^{\alpha-1}\varphi_\alpha(x) + x^{\alpha-1}\psi_\alpha(y)\right] + c_*$$

$((x,y)\in\Omega)$, where

$$(3) \quad R(x,y) = \left[(p-1)!\right]^{-2} \int_0^x \left\{ \int_0^y \left[(x-\xi)(y-\eta)\right]^{p-1} c(\xi,\eta)d\eta \right\} d\xi,$$

$\varphi_\alpha : \langle 0,A\rangle \to Y$ and $\psi_\alpha : \langle 0,B\rangle \to Y$ are functions of class C^p, and $c_* \in Y$ is a constant, then u is a solution of equation (1) in Ω. And conversely, for a given solution u of equation (1) in Ω there are functions $\varphi_\alpha : \langle 0,A\rangle \to Y$ and $\psi_\alpha : \langle 0,B\rangle \to Y$ $(\alpha = 1,2,\dots,p)$ of class C^p, fulfilling the conditions $\varphi_\alpha^{(m)}(0) = \psi_\alpha^{(m)}(0) = 0$ $(\alpha = 2,3,\dots,p;\ m = 0,1,\dots,\alpha-2)$, and a constant $c_* \in Y$ such that relation (2) is satisfied for $(x,y)\in\Omega$.

The proof of Lemma 1 is elementary.

We examine the following Goursat problem.

(G)-problem. Find a solution u of equation (1) in Ω satisfying the boundary conditions

$$(4) \quad \begin{cases} u\left[x,f_i(x)\right] = M_i(x) & \text{for } x \in \langle 0,A\rangle, \\ u\left[h_i(y),y\right] = N_i(y) & \text{for } y \in \langle 0,B\rangle \end{cases}$$

$(i = 1,2,\dots,p)$, where $M_i : \langle 0,A\rangle \to Y$ and $N_i : \langle 0,B\rangle \to Y$ are given functions.

Each function u having the aforesaid properties will be called a solution of the (G)-problem.

We make the following assumptions:

I. The functions $f_i : \langle 0,A\rangle \to \langle 0,B\rangle$ and $h_i : \langle 0,B\rangle \to \langle 0,A\rangle$ $(i = 1,2,\dots,p)$ are of class C^p, $f_i(0) = h_i(0) = 0$, and the inequalities

$$(5)\quad \begin{cases} \min(\overset{*}{f}_1,\overset{*}{h}_1) \geq 0, \\ \max(\overset{*}{f}_p,\overset{*}{h}_p) \leq 1; \quad g_0 := \overset{*}{f}_p\overset{*}{h}_p < p^{-2p/\varkappa_0}, \\ \min\limits_{2\leq i\leq p} (\overset{*}{f}_i-\overset{*}{f}_{i-1}) > [p(1+\varepsilon)]^{-1}\, \overset{*}{f}_p, \\ \min\limits_{2\leq i\leq p} (\overset{*}{h}_i-\overset{*}{h}_{i-1}) > [p(1+\varepsilon)]^{-1} \overset{*}{h}_p \end{cases}$$

are fulfilled, where $\overset{*}{f}_i = f_i'(0)$, $\overset{*}{h}_i = h_i'(0)$ $(i=1,2,\dots,p)$ and $\varkappa_0$ and ε are given numbers satisfying the conditions $\varkappa_0 \in (0,1\rangle$ and

$$(6)\qquad 0<\varepsilon < \left(p^p \sqrt{g_0}\right)^{1/1-p} - 1,$$

respectively. Moreover, f_i and h_i $(i=1,2,\dots,p)$ satisfy the conditions mentioned at the beginning of this section.

Let us note that although we assume the curves $y = f_i(x)$ and $x = h_i(y)$ $(i=1,2,\dots,p)$ to be sufficiently "flat" at the point O (see (5)), we do not make any assumptions concerning the slopes of these curves at the points of $\Omega\setminus\{0\}$.

II. The functions $M_i:\langle 0,A\rangle \to Y$ and $N_i:\langle 0,B\rangle \to Y$ $(i=1,2,\dots,p)$ are of class C^p and fulfil the conditions

$$(7)\qquad M_i(0) = N_j(0), \quad M_i^{(m)}(0) = N_i^{(m)}(0) = 0$$

$(i,j = 1,2,\dots,p;\ m = 1,2,\dots,p-1)$ and

$$(8)\qquad \|M_i^{(p)}(x)\| \leq K\, x^{p-1+\varkappa_0}, \quad \|N_i^{(p)}(y)\| \leq K\, y^{p-1+\varkappa_0}$$

$(i = 1,2,\dots,p;\ x\in\langle 0,A\rangle;\ y \in \langle 0,B\rangle)$, where K is a positive constant.

III. The function $c: \Omega \to Y$ is continuous.

R e m a r k 1. By Assumption II, we have

$$(9)\quad \begin{cases} \|(M_i(x) - a)^{(m)}\| \leq \dfrac{K}{(p-m)!}\, x^{2p-1-m+\varkappa_0}, \\ \|(N_i(y) - a)^{(m)}\| \leq \dfrac{K}{(p-m)!}\, y^{2p-1-m+\varkappa_0}, \end{cases}$$

$(i = 1,2,\dots,p$ and $m = 0,1,2,\dots,p)$, where $x \in \langle 0,A\rangle$; $y \in \langle 0,B\rangle$, $a = M_r(0) = N_s(0)$ $(r,s = 1,2,\dots,p)$.

We shall need the following lemma which is a direct consequence of Assumption I.

L e m m a 2. There is a positive number $\delta_1 \leq \min(A,B,1)$, such that the inequalities

$$(10)\qquad \begin{cases} f_i(x) - f_j(x) > [p(1+\varepsilon)]^{-1} f_p(x), \\ h_i(y) - h_j(y) > [p(1+\varepsilon)]^{-1} h_p(y), \end{cases}$$

$1 \leq j < i \leq p$, hold good for $x \in (0,\delta_1)$ and $y \in (0,\delta_1)$, respectively.

We shall also need the following lemma whose proof is straightforward.

L e m m a 3. There is a positive number $\delta_2 \leq \min(A,B,1)$, such that the inequalities

$$(11)\qquad \begin{cases} (1 - \varepsilon_0)\overset{*}{f}_1 < f_1'(x) < (1 + \varepsilon_0)\overset{*}{f}_1, \\ (1 - \varepsilon_0)\overset{*}{h}_1 < h_1'(y) < (1 + \varepsilon_0)\overset{*}{h}_1, \end{cases}$$

where

$$(11')\qquad 0 < \varepsilon_0 < 1 - \left[p^p\, g_0^{\varkappa_0/2}\right]^{1/6p-3+\varkappa_0}$$

are satisfied for $x \in (0,\delta_2)$ and $y \in (0,\delta_2)$, respectively.

In the sequel we shall use the notation $\delta = \min(\delta_1,\delta_2)$.

3. We now attempt to find a solution of the (G)-problem. Setting $c_* = a$ in (2) and imposing on function u the boundary conditions (4), we get the following system of functional equations[1)]

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2) Here and in the sequel, $\circ$ is the symbol of composition.

$$\sum_{\alpha=1}^{p} \left[(f_i(x))^{\alpha-1}\varphi_\alpha(x) + x^{\alpha-1}\psi_\alpha\circ f_i(x)\right] = M_i(x) - a - R\left[x,f_i(x)\right],$$

(12)

$$\sum_{\alpha=1}^{p} \left[y^{\alpha-1}\varphi_\alpha\circ h_i(y) + (h_i(y))^{\alpha-1}\psi_\alpha(y)\right] = N_i(y) - a - R\left[h_i(y),y\right]$$

$(x \in \langle 0,A\rangle;\ y \in \langle 0,B\rangle;\ i = 1,2,\dots,p)$.

By setting $i = 1,2,\dots,p$ in the first of equations (12), we obtain the following system of algebraic equations

$$\sum_{\alpha=1}^{p} (f_i(x))^{\alpha-1}\varphi_\alpha(x) = P_i(x) \tag{13}$$

with respect to $\varphi_1(x),\dots,\varphi_p(x)$, where

$$P_i(x) = M_i(x) - a - R\left[x,f_i(x)\right] - \sum_{\alpha=1}^{p} x^{\alpha-1}\psi_\alpha\circ f_i(x). \tag{14}$$

The system (13) is a Cramer's system for $x \in (0,A\rangle$, because the determinant of its coefficient matrix

$$D(x) = \prod_{1\leq\alpha<\beta\leq p} (f_\beta(x) - f_\alpha(x))$$

is different from zero $(x \in (0,A\rangle)$.

Using Cramer's formulas and the well known formulas concerning the Vandermonde determinants (see [7], pp. 70 and 236), we have

$$\varphi_\alpha(x) = (-1)^{\alpha-1}\sum_{i=1}^{p} \omega_i(x)e_i^\alpha(x)\ P_i(x), \tag{15}$$

where

$$\omega_i(x) = \left[\prod_{\substack{\beta=1\\ \beta\neq i}}^{p} (f_\beta(x) - f_i(x))\right]^{-1}, \tag{16}$$

$$e_i^1(x) = \prod_{\substack{\beta=1\\ \beta\neq i}}^{p} f_\beta(x), \tag{17}$$

$$e_i^\alpha(x) = \sum_{\substack{1\leq\beta_1<\dots<\beta_{p-\alpha}\leq p\\ \beta_1\neq i,\dots,\beta_{p-\alpha}\neq i}} f_{\beta_1}(x)\dots f_{\beta_{p-\alpha}}(x) \tag{18}$$

$(\alpha = 2,3,\dots,p-1)$ when $p\geqslant 3$ and

$$e_i^p(x) = 1, \tag{19}$$

$(i=1,2,\dots,p)$.

On joining (14) and (15), we get

$$\varphi_\alpha(x) = (-1)^{\alpha-1}\sum_{\nu=1}^{p} \omega_\nu(x)e_\nu^\alpha(x)(M_\nu(x) - a - R[x,f_\nu(x)]) +$$

$$+ (-1)^\alpha \sum_{\nu,k=1}^{p} \omega_k(x)e_k^\alpha(x)x^{\nu-1}\psi_\nu\circ f_k(x)$$

$(x\in(0,A\rangle;\ \alpha = 1,2,\dots,p)$.

Evidently, by setting $i = 1,2,\dots,p$ in the second of equations (12) and by using an argument analogous to that above, we obtain

$$\psi_\alpha(y) = (-1)^{\alpha-1}\sum_{\nu=1}^{p} \tilde\omega_\nu(y)\tilde e_\nu^\alpha(y)(N_\nu(y) - a - R[h_\nu(y),y]) +$$

$$+ (-1)^\alpha \sum_{\nu,k=1}^{p} \tilde\omega_k(y)\,\tilde e_k^\alpha(y)\,y^{\nu-1}\varphi_\nu\circ h_k(y),$$

($y \in (0,B\rangle$; $\alpha = 1,2,\dots,p$), where $\tilde{\omega}_\nu(y)$ and $\tilde{e}_\nu(y)$ are given by the formulas (16) and (17)-(19), respectively, with the replacement of f by h, x by y and i by ν.

Thus, finally, we can assert that for $x \in (0,A\rangle$, $y \in (0,B\rangle$ the system of functional equations (12) is equivalent to the following one

$$(20)\qquad \begin{cases} \varphi_\alpha(x) = V^\alpha(x) + \sum_{\nu,k=1}^{p} G^\alpha_{\nu k}(x)\, \psi_\nu \circ f_k(x), \\ \psi_\alpha(y) = \tilde{V}^\alpha(y) + \sum_{\nu,k=1}^{p} \tilde{G}^\alpha_{\nu k}(y)\, \varphi_\nu \circ h_k(y) \end{cases}$$

($x \in (0,A\rangle$; $y \in (0,B\rangle$; $\alpha = 1,2,\dots,p$), where

$$(21)\qquad \begin{cases} V^\alpha(x) = (-1)^{\alpha-1} \sum_{\nu=1}^{p} \omega_\nu(x) e^\alpha_\nu(x)(M_\nu(x)-a-R[x,f_\nu(x)]), \\ \tilde{V}^\alpha(y) = (-1)^{\alpha-1} \sum_{\nu=1}^{p} \tilde{\omega}_\nu(y) \tilde{e}^\alpha_\nu(y)(N_\nu(y)-a-R[h_\nu(y),y]), \end{cases}$$

$$(22)\qquad \begin{cases} G^\alpha_{\nu k}(x) = (-1)^\alpha \omega_k(x) e^\alpha_k(x)\, x^{\nu-1}, \\ \tilde{G}_{\nu k}(y) = (-1)^\alpha \tilde{\omega}_k(y) \tilde{e}^\alpha_k(y)\, y^{\nu-1}, \end{cases}$$

and φ_α and ψ_α ($\alpha = 1,2,\dots,p$) are the unknown functions.

Let us note that if $Y = R$, then the system (20) is contained by that examined in paper [1] but the assumptions made in the said paper are not satisfied (see [1], p.194).

Using classical methods of the theory of functional equations (see [8], Chapter VI), we are going to prove that, under the present assumptions, system (20) has a unique solution. First of all let us introduce the following notation

$$(23)\quad \begin{cases} z_{\vec{k}_{(2s)}}(x) = h_{k_{2s}} \circ f_{\vec{k}_{2s-1}} \circ z_{\vec{k}_{(2s-2)}}(x), \\ \tilde{z}_{\vec{k}_{(2s)}}(y) = f_{k_{2s}} \circ h_{k_{2s-1}} \circ \tilde{z}_{\vec{k}_{(2s-2)}}(y), \\ \overset{*}{z}_{\vec{k}_{(2s-1)}}(x) = f_{k_{2s-1}} \circ z_{\vec{k}_{(2s-2)}}(x), \\ \overset{**}{z}_{\vec{k}_{(2s-1)}}(y) = h_{k_{2s-1}} \circ \tilde{z}_{\vec{k}_{(2s-2)}}(y), \end{cases}$$

for s = 2,3...;

$$(23')\quad \begin{cases} z_{\vec{k}_{(2)}}(x) = h_{k_2} \circ f_{k_1}(x), \ \tilde{z}_{\vec{k}_{(2)}}(y) = f_{k_2} \circ h_{k_1}(y), \\ \overset{*}{z}_{\vec{k}_{(1)}}(x) = f_{k_1}(x), \ \overset{**}{z}_{\vec{k}_{(1)}}(y) = h_{k_1}(y), \end{cases}$$

where $\vec{k}_{(m)} = (k_1, k_2, \dots, k_m)$ for $m \in \mathcal{N}$, and $1 \leq k_i \leq p$ for $i \in \mathcal{N}$ ($\mathcal{N}$ denotes the set of all positive integers).

L e m m a 4. The sequences $\{z_{\vec{k}_{(2s)}}\}$ and $\{\tilde{z}_{\vec{k}_{(2s)}}\}$ (see (23), (23′)) tend uniformly to zero on $(0,A\rangle$ and $(0,B\rangle$, respectively, when $s \to \infty$.

Let us observe that Lemma 4 is a generalization of Lemma 3 in [2] and can be proved by an argument similar to that in [2].

L e m m a 5. If Assumptions I-III are satisfied, then system (20) has a solution given by the formulas

$$(24)\quad \begin{cases} \varphi_\alpha(x) = V^\alpha(x) + \sum_{n=1}^{\infty} a_n^\alpha(x), \\ \psi_\alpha(y) = \tilde{V}^\alpha(y) + \sum_{n=1}^{\infty} \tilde{a}_n^\alpha(y) \end{cases}$$

($x \in (0,A\rangle$; $y \in (0,B\rangle$; $\alpha = 1,2,\dots,p$), where

$$(25)\quad\begin{cases} a_n^{\alpha}(x) = \displaystyle\sum_{\nu_1,\dots,\nu_n=1}^{p}\ \sum_{k_1,\dots,k_n=1}^{p} \boxminus^{\alpha}_{\vec{\nu}_{(n)}\vec{k}_{(n)}}(x)\ \times \\ \qquad\qquad \times\, F^{\nu_n}\circ \mathfrak{z}_{\vec{k}_{(n)}}(x), \\ \tilde{a}_n^{\alpha}(y) = \displaystyle\sum_{\nu_1,\dots,\nu_n=1}^{p}\ \sum_{k_1,\dots,k_n=1}^{p} \tilde{\boxminus}^{\alpha}_{\vec{\nu}_{(n)}\vec{k}_{(n)}}(y)\ \times \\ \qquad\qquad \times\, \tilde{F}^{\nu_n}\circ \tilde{\mathfrak{z}}_{\vec{k}_{(n)}}(y) \end{cases}$$

with[3]

$$\boxminus^{\alpha}_{\vec{\nu}_{(n)}\vec{k}_{(n)}}(x) = G^{\alpha}_{\nu_1 k_1}(x)\left(\prod_{j=2}^{\lceil (n+1)/2\rceil} G^{\nu_{2j-2}}_{\nu_{2j-1}k_{2j-1}}\circ z_{\vec{k}(2j-2)}(x)\right)\left(\prod_{j=2}^{\lceil n/2+1\rceil} \tilde{G}^{\nu_{2j-3}}_{\nu_{2j-2}k_{2j-2}}\circ \overset{*}{z}_{\vec{k}_{(2j-3)}}(x)\right),$$

$$(26)\quad \tilde{\boxminus}_{\vec{\nu}_{(n)}\vec{k}_{(n)}}(y) = \tilde{G}^{\alpha}_{\nu_1 k_1}(y)\times\left(\prod_{j=2}^{\lceil (n+1)/2\rceil} \tilde{G}^{\nu_{2j-2}}_{\nu_{2j-1}k_{2j-1}}\circ \tilde{z}_{\vec{k}_{(2j-2)}}(y)\right)\times\left(\prod_{j=2}^{\lceil n/2+1\rceil} G^{\nu_{2j-3}}_{\nu_{2j-2}k_{2j-2}}\circ \overset{**}{z}_{\vec{k}_{(2j-3)}}(y)\right)$$

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[3] The symbol $\lceil x \rceil$ denotes the greatest integer not exceeding x.

$(\bar{\nu}_{(m)} = (\nu_1, \nu_2, \ldots, \nu_m)$ for $m \in \mathcal{N}$; $\alpha = 1,2,\ldots)$ and

(27)
$$\begin{cases} F^{\nu_n} = \begin{cases} V^{\nu_n} & \text{when } n \text{ is even}, \\ \tilde{V}^{\nu_n} & \text{when } n \text{ is odd}, \end{cases} \\ \tilde{F}^{\nu_n} = \begin{cases} \tilde{V}^{\nu_n} & \text{when } n \text{ is even}, \\ V^{\nu_n} & \text{when } n \text{ is odd}, \end{cases} \end{cases}$$

(28)
$$\begin{cases} \mathfrak{z}_{\vec{k}_{(n)}} = \begin{cases} z_{\vec{k}_{(n)}} & \text{when } n \text{ is even}, \\ \overset{*}{z}_{\vec{k}_{(n)}} & \text{when } n \text{ is odd}, \end{cases} \\ \tilde{\mathfrak{z}}_{\vec{k}_{(n)}} = \begin{cases} \tilde{z}_{\vec{k}_{(n)}} & \text{when } n \text{ is even}, \\ \overset{**}{z}_{\vec{k}_{(n)}} & \text{when } n \text{ is odd}. \end{cases} \end{cases}$$

This is the only solution of (20) in the class K of all systems of continuous functions $\varphi_\alpha : (0,A\rangle \to Y$ and $\psi_\alpha : (0,B\rangle \to Y$ $(\alpha = 1,2,\ldots,p)$ satisfying the inequalities

(29) $\|\varphi_\alpha(x)\| \leq C\, x^{2p-\alpha+\varkappa_0}$, $\|\psi_\alpha(y)\| \leq C\, y^{2p-\alpha+\varkappa_0}$,

respectively, where C is a positive constant.

Moreover, functions φ_α and ψ_α $(\alpha = 1,2,\ldots,p)$ given by (24) are of class C^p in the intervals $(0,A\rangle$ and $(0,B\rangle$, respectively.

P r o o f . First of all we are going to show that the series in (24) are uniformly convergent in the intervals $(0,A\rangle$ and $(0,B\rangle$, respectively. We will give the proof for the series $\sum_{n=1}^{\infty} a_n^{\alpha}(x)$; the argument for the series $\sum_{n=1}^{\infty} \tilde{a}_n^{\alpha}(y)$ is analogous.

It follows from Lemma 4 that there is a positive integer N such that for each $n>N$ and each $x \in (0,A\rangle$ the relation $\mathfrak{z}_{\vec{k}(n)}(x) \in (0,\delta)$ is valid.

Let us distinguish the following two cases:

1^0 $x \in (0,\delta)$ and 2^0 $x \in \langle \delta, A\rangle$; $n>N$, and begin with the first of them.

Observe that, in virtue of the relations (10) and (16), we have the estimates

$$|\omega_i(x)| \leqslant [p(1+\varepsilon)]^{p-1}(f_p(x))^{1-p} \tag{30}$$

$(i = 1,2,\ldots,p)$.

Further, by (10), (18) and Assumption I, we get the following sequence of inequalities.

$$|e_i^{\alpha}(x)| \leqslant (f_p(x))^{p-\alpha} \sum_{1\leqslant\beta_1<\ldots<\beta_{p-\alpha}\leqslant p} 1^{\beta_1}\ldots 1^{\beta_{p-\alpha}} \leqslant$$

$$\leqslant p^{p-2}(f_p(x))^{p-\alpha}$$

$(\alpha = 2,3,\ldots,p-1;\ i=1,2,\ldots,p;\ p\geqslant 3)$, whence and by (17) and (19) we can write

$$|e_i^{\alpha}(x)| \leqslant C_*(f_p(x))^{p-\alpha} \tag{31}$$

$(\alpha,\ i=1,2,\ldots,p;\ p\geqslant 2)$, where $C_* = p^{p-2}$.

It is clear that analogous estimates can be obtained for $\tilde{\omega}_i(y)$ and $\tilde{e}_i^{\alpha}(y)$.

From (22), (30) and (31) it follows that the functions $G_{\nu k}^{\alpha}$ and $\tilde{G}_{\nu k}^{\alpha}$ satisfy the inequalities

$$(32)\qquad \begin{cases} |G^{\alpha}_{\nu k}(x)| \leq C_* [p(1+\varepsilon)]^{p-1} (f_p(x))^{1-\alpha} x^{\nu-1}, \\ |\tilde{G}^{\alpha}_{\nu k}(y)| \leq C_* [p(1+\varepsilon)]^{p-1} (h_p(y))^{1-\alpha} y^{\nu-1} \end{cases}$$

$(\nu, k, \alpha = 1,2,\ldots,p)$, respectively.

Now, let us write the expression $a^{\alpha}_n(x)$ (see (25)) in the form[4)]

$$(33)\qquad a^{\alpha}_n(x) = \sum_{\nu_1,\ldots,\nu_n=1}^{p} \sum_{k_1,\ldots,k_n=1}^{p} G^{\alpha}_{\nu_1 k_1}(x)\, \tilde{G}^{\nu_1}_{\nu_2 k_2} \circ f_{k_1}(x)\, G^{\nu_2}_{\nu_3 k_3} \circ z_{\vec{k}_{(2)}}(x) \cdot \tilde{G}^{\nu_3}_{\nu_4 k_4} \circ \overset{*}{z}_{\vec{k}_{(3)}}(x)\, G^{\nu_4}_{\nu_5 k_5} \circ z_{\vec{k}_{(4)}}(x) \ldots$$
$$\ldots G^{\nu_{n-1}}_{\nu_n k_n} \circ z_{\vec{k}_{(n-1)}}(x) \cdot F^{\nu_n} \circ \overset{*}{z}_{\vec{k}_{(n)}}(x).$$

Basing on (33) and using the estimates (32) and (11) and the inequality

$$(34)\qquad \| F^{\nu_n} \circ \overset{*}{z}_{\vec{k}_{(n)}}(x) \| \leq \text{const} (\overset{*}{z}_{\vec{k}_{(n)}}(x))^{2p-\nu_n+\varkappa_0}$$

(see (9), (21) and (27)), we obtain the following sequence of inequalities[5)]

[4)] We assume that n is odd; for even values of n the last but one factor in (33) is $\tilde{G}^{\nu_{n-1}}_{\nu_n k_n} \circ z_{\vec{k}_{(n-1)}}(x)$.

[5)] Here and in the sequel, const denotes a positive constant.

(35)
$$\|a_n^{\alpha}(x)\| \leqslant$$

$$\leqslant \mathrm{const}\left\{C_*[p(1+\varepsilon)]^{p-1}\right\}^n \sum_{\nu_1,\dots,\nu_n=1}^{p} \sum_{k_1,\dots,k_n=1}^{p} \frac{x^{\nu_1-1}}{(f_p(x))^{\alpha-1}} \cdot$$

$$\cdot \frac{(f_{k_1}(x))^{\nu_2-1}}{(h_p\circ f_{k_1}(x))^{\nu_1-1}} \cdot \frac{\left(z_{\vec{k}(2)}(x)\right)^{\nu_3-1}}{\left(f_p\, z_{\vec{k}(2)}\circ(x)\right)^{\nu_2-1}} \cdots$$

$$\cdots \frac{\left(z_{\vec{k}(n-1)}(x)\right)^{\nu_n-1}}{\left(f_p\circ z_{\vec{k}(n-1)}(x)\right)^{\nu_{n-1}-1}} \left(\overset{*}{z}_{\vec{k}(n)}(x)\right)^{2p-\nu_n+\varkappa_0} \leqslant$$

$$\leqslant \mathrm{const}\left\{[p^2(1+\varepsilon)]^{p-1}(1-\varepsilon_0)^{2(1-p)}\right\}^n \cdot$$

$$\sum_{\nu_1,\dots,\nu_n=1}^{p} \sum_{k_1,\dots,k_n}^{p} \overset{*}{f}_p^{\,1-\alpha}(\overset{*}{h}_p\overset{*}{f}_{k_1})^{1-\nu_1}\cdot(\overset{*}{f}_p\overset{*}{h}_{k_2})^{1-\nu_2} \cdots$$

$$\cdots (\overset{*}{f}_p\overset{*}{h}_{k_{n-1}})^{1-\nu_{n-1}}\cdot\left(\overset{*}{z}_{\vec{k}(n)}(x)\right)^{2p-1+\varkappa_0}\cdot x^{1-\alpha} \leqslant$$

$$\leqslant \mathrm{const}\left\{p^{2p}(1+\varepsilon)^{p-1}(1+\varepsilon_0)^{2p-1+\varkappa_0}(1-\varepsilon_0)^{2(1-p)}\cdot\right.$$

$$\left.\cdot(\overset{*}{f}_p\overset{*}{h}_p)^{(1+\varkappa_0)/2}\right\}^n \cdot x^{2p-\alpha+\varkappa_0} \leqslant \mathrm{const}\left[p^p(1+\varepsilon)^{p-1}\sqrt{g_0}\right]^n \cdot$$

$$\cdot \left[p^p\, g_0^{\varkappa_0/2}(1-\varepsilon_0)^{3-4p-\varkappa_0}\right]^n,$$

where const is independent of n.

It follows from the choice of the parameters ε and ε_0 (see (6) and (11′)) that

$$p^p(1+\varepsilon)^{p-1}\sqrt{g_0}<1, \qquad p^p\, g_0^{\varkappa_0/2}(1-\varepsilon_0)^{3-4p-\varkappa_0}<1,$$

whence and by (35), we have

(36) $$\|a_n^\alpha(x)\| \leqslant \text{const}\; q^n x^{2p-\alpha+\varkappa_0},$$

($x\in(0,\delta)$; $\alpha=1,2,\dots,p$ and q is a number belonging to $(0,1)$).

In case 2° we estimate the first N factors of the product in (33) by a constant depending on N (but independent of n) and we proceed with the remaining factors in a way analogous to that in case 1° above. As a result we get

(37) $$\|a_n^\alpha(x)\| \leqslant \text{const}\; q^{n-N} x^{2p-\alpha+\varkappa_0} \leqslant \text{const}\; q^n x^{2p-\alpha+\varkappa_0},$$

($x\in\langle\delta, A\rangle$; $n>N$; $\alpha=1,2,\dots,p$; $0<q<1$), where const depends on N but is independent of n.

Thus, for $x\in(0,A\rangle$ and $n\in\mathcal{N}$, the following inequality

(38) $$\|a_n^\alpha(x)\| \leqslant \text{const}\; q^n x^{2p-\alpha+\varkappa_0}$$

($\alpha=1,2,\dots,p$; $0<q<1$) holds good, where const is independent of n and hence the series $\sum_{n=1}^{\infty} a_n^\alpha(x)$ is uniformly convergent in the interval $(0,A\rangle$. It is also clear (see (24) and (38)) that the functions φ_α are continuous in $(0,A\rangle$ and satisfy the inequality

(39) $$\|\varphi_\alpha(x)\| \leqslant \text{const}\; x^{2p-\alpha+\varkappa_0},$$

($x\in(0,A\rangle$; $\alpha=1,2,\dots,p$), whence we have

(40) $$\varphi_\alpha(0) := \lim_{x\to 0^+}\varphi_\alpha(x) = 0 \qquad (\alpha=1,2,\dots,p).$$

Evidently, all the above considerations concerning the functions φ_α $(\alpha=1,2,\dots,p)$ can be also performed for the functions ψ_α $(\alpha=1,2,\dots,p)$ given by the second of formulas (24) and lead to analogous results.

We proceed to verify that the functions φ_α and ψ_α $(\alpha=1,2,\dots,p)$ given by formulas (24) satisfy the system (20) for $x\in\langle 0,A\rangle$; $y\in\langle 0,B\rangle$.

To this end we consider the first of relations (24) and, using (25)-(27), write it in the form

$$\varphi_\alpha(x) = V^\alpha(x) + \sum_{\nu_1,k_1=1}^{p} G^\alpha_{\nu_1 k_1}(x)\Big\{\tilde{V}^{\nu_1}\circ f_{k_1}(x) +$$

$$+\sum_{n=2}^{\infty}\sum_{\nu_2,\dots,\nu_n=1}^{p}\sum_{k_2,\dots,k_n=1}^{p}\Bigg(\prod_{j=2}^{\lceil (n+1)/2\rceil} G^{\nu_{2j-2}}_{\nu_{2j-1}k_{2j-1}}\circ z_{\vec{k}_{(2j-2)}}(x)\Bigg)\cdot$$

$$\cdot\Bigg(\prod_{j=2}^{\lceil n/2+1\rceil}\tilde{G}^{\nu_{2j-3}}_{\nu_{2j-2}k_{2j-2}}\circ \overset{*}{z}_{\vec{k}_{(2j-3)}}(x)\Bigg)\cdot F^{\nu_n}\circ \mathfrak{z}_{\vec{k}_{(n)}}(x)\Big\}.$$

On setting $n-1=s$; $\nu_\gamma = l_{\gamma-1}$; $k_\gamma = q_{\gamma-1}$ $(\gamma=2,3,\dots,n)$, and on substituting $j-1=m$ in the second of the products $\prod$, we get

$$(41)\quad \varphi_\alpha(x) = V^\alpha(x) + \sum_{\nu_1,k_1=1}^{p} G^\alpha_{\nu_1 k_1}(x)\Big\{\tilde{V}^{\nu_1}\circ f_{k_1}(x) +$$

$$+\sum_{s=1}^{\infty}\sum_{l_1,\dots,l_s=1}^{p}\sum_{q_1,\dots,q_s=1}^{p}\tilde{G}^{\nu_1}_{l_1 q_1}\circ f_{k_1}(x)\cdot$$

$$\cdot\Bigg(\prod_{m=2}^{\lceil (s+1)/2\rceil}\tilde{G}^{l_{2m-2}}_{l_{2m-1}q_{2m-1}}\circ\tilde{z}_{\vec{q}_{(2m-2)}}\circ f_{k_1}(x)\Bigg)\cdot$$

$$\cdot\left(\prod_{j=2}^{\lceil s/2+1\rceil} G^{l_{2j-3}}_{l_{2j-2}q_{2j-2}} \circ \overset{**}{z}_{\bar{q}(2j-3)} \circ f_{k_1}(x)\right) \tilde{F}^{l_s} \circ \tilde{\mathfrak{z}}_{\bar{q}(s)} \circ f_{k_1}(x)\Big\}.$$

Equality (41) together with the second of relations (24) imply the relation

$$\varphi_\alpha(x) = V^\alpha(x) + \sum_{\nu_1,k_1=1}^{p} G^\alpha_{\nu_1 k_1}(x)\psi_{\nu_1} \circ f_{k_1}(x)$$

identical with the first of equations (20). In a similar way we verify that the functions φ_α and ψ_α ($\alpha=1,2,\dots,p$) given by formulas (24) satisfy the second of the said equations.

We are going to prove that the solution given by formulas (24) is the only solution of (20) in the class K (see p.11). To this purpose let us note that if some functions φ_α and ψ_α ($\alpha=1,2,\dots,p$) satisfy the system (20), then, for each positive integer m_0, the equality

$$\varphi_\alpha(x) = V^\alpha(x) + \sum_{n=1}^{m_0} a^\alpha_n(x) + r^\alpha_{m_0}(x) \tag{42}$$

($x \in (0,A>$; $\alpha=1,2,\dots,p$) holds good, where $a^\alpha_n(x)$ is given by (25) and

$$r^\alpha_{m_0}(x) = \sum_{\nu_1,\dots,\nu_{m_0+1}=1}^{p} \; \sum_{k_1,\dots,k_{m_0+1}=1}^{p} G^\alpha_{\nu_1 k_1}(x)\times \tag{43}$$

$$\times\left(\prod_{j=2}^{\lceil (m_0+1)/2\rceil} G^{\nu_{2j-2}}_{\nu_{2j-1}k_{2j-1}} \circ z_{\bar{k}(2j-2)}(x)\right)\cdot$$

$$\cdot\left(\prod_{j=2}^{\lceil m_0/2+1\rceil} \tilde{G}^{\nu_{2j-3}}_{\nu_{2j-2}k_{2j-2}} \circ \overset{*}{z}_{\bar{k}(2j-3)}(x)\right)\cdot$$

$$\cdot\ \overset{*}{G}{}^{\nu_{m_0}}_{\nu_{m_0+1}k_{m_0+1}} \circ \mathfrak{z}_{\bar{k}(m_0)}(x)\ \overset{*}{F}{}^{\nu_{m_0+1}} \circ \mathfrak{z}_{\bar{k}(m_0+1)}(x)$$

with

$$\overset{*}{G}{}^{\nu_{m_0}}_{\nu_{m_0+1}k_{m_0+1}} = \begin{cases} G^{\nu_{m_0}}_{\nu_{m_0+1}k_{m_0+1}} & \text{when } m_0 \text{ is even,} \\ \tilde{G}^{\nu_{m_0}}_{\nu_{m_0+1}k_{m_0+1}} & \text{when } m_0 \text{ is odd;} \end{cases}$$

$$\overset{*}{F}{}^{\nu_{m_0+1}} = \begin{cases} \psi_{\nu_{m_0+1}} & \text{when } m_0 \text{ is even} \\ \varphi_{\nu_{m_0+1}} & \text{when } m_0 \text{ is odd.} \end{cases}$$

Basing on the estimates (32) and the inequalities (11), and using an argument similar to that in the proof of (38), we obtain the following inequality

$$\|r^{\alpha}_{m_0}(x)\| \leqslant \text{const } q^{m_0} x^{2p-\alpha+\varkappa_0} \tag{44}$$

($x \in (0,A\rangle$; $\alpha = 1,2,\dots,p$), where $0<q<1$ and const is independent of m_0.

It follows directly from (42) - (44) that $\varphi_\alpha(x)$ ($x \in (0,A\rangle$; $\alpha = 1,2,\dots,p$) satisfy the first of equalities (24). In a similar way we show that $\psi_\alpha(y)$ ($y \in (0,B\rangle$; $\alpha=1,2,\dots,p$) satisfy the second of the said equalities.

Thus, in order to complete the proof of Lemma 5, it is enough to show that the functions φ_α and ψ_α ($\alpha=1,2,\dots,p$) given by (24) are of class C^p.

We first consider the case 1^0 (see p.12). Let us begin with some auxiliary estimates. We assert that the following relations

$$\left|\frac{d^s}{dx^s}\,\omega_k(x)\right| \leqslant C_s\,[p(1+\varepsilon)]^{p-1+s}(f_p(x))^{1-p-s} \tag{45}$$

and

$$\left|\frac{d^s}{dx^s}\,e^{\alpha}_{k}(x)\right| \leqslant \tilde{C}_s \begin{cases} (f_p(x))^{p-\alpha-s} & \text{for } 0 \leqslant s \leqslant p-\alpha \\ 1 & \text{for } p-\alpha < s \leqslant p \end{cases} \tag{46}$$

($x \in (0,A\rangle$; $s = 0,1,\dots,p$; $k, \alpha = 1,2,\dots,p$) hold good, where C_s and $\tilde{C}_s$ are positive constants depending on s.

We will give an inductive proof of (45) (the argument for (46) is similar).

If $s = 0$, then (45) is true (see (30)). Suppose that it is true for $s = 0,1,2,\dots,s_o$ $(0 \leq s_o \leq p-1)$ and observe that

$$\left|\frac{d^{s_o+1}}{dx^{s_o+1}}\omega_k(x)\right| \leq \sum_{\substack{j=1\\ j\neq k}}^{p}\left|\frac{d^{s_o}}{dx^{s_o}}\left[\frac{\omega_k(x)(f_j'(x)-f_k'(x))}{f_j(x)-f_k(x)}\right]\right| =$$

$$= \sum_{\substack{j=q\\ j\neq k}}^{q}\sum_{\beta=0}^{s_o}\binom{s_o}{\beta}\left|\frac{d^{s_o-\beta}}{dx^{s_o-\beta}}\omega_k(x)\right|\left|\frac{d^{\beta}}{dx^{\beta}}(f_j'(x)-f_k'(x))\cdot(f_j(x)-f_k(x))^{-1}\right|.$$

Hence (see (10)), we have

$$\left|\frac{d^{s_o+1}}{dx^{s_o+1}}\omega_k(x)\right| \leq$$

$$\leq [p(1+\varepsilon)]^{p+s_o}(f_p(x))^{-p-s_o}\cdot\sum_{\substack{j=1\\ j\neq k}}^{p}\sum_{\beta=0}^{s_o}\binom{s_o}{\beta}C_{s_o-\beta}\hat{C}_\beta \leq$$

$$\leq C_{s_o+1}[p(1+\varepsilon)]^{p-1+(s_o+1)}\cdot(f_p(x))^{1-p-(s_o+1)},$$

where $\hat{C}_\beta$ $(\beta = 0,1,\dots,s_o)$ and C_{s_o+1} are some positive constants, and as a consequence we can conclude that (45) is valid. Q.E.D.

Based on (22), (45) and (46), we have

$$\left|\frac{d^m}{dx^m}G^{\alpha}_{\nu k}(x)\right| = \sum_{\mu=0}^{\min(m,\nu-1)}\binom{m}{\mu}\binom{\nu-1}{\mu}\mu!\left|[\omega_k(x)e^{\alpha}_k(x)]^{(m-\mu)}\right|\cdot x^{\nu-1-\mu} \leq$$

$$\leq \sum_{\mu=0}^{\min(m,\nu-1)}\binom{m}{\mu}\binom{\nu-1}{\mu}\mu!\,x^{\nu-1-\mu}\left[\sum_{\gamma=0}^{\min(m-\mu,p-\alpha)}\binom{m-\mu}{\gamma}C_{m-\mu-\gamma}\tilde{C}_\gamma\right.$$

$$\cdot\,(1+\varepsilon)^{p-1+m-\mu-\gamma}(f_p(x))^{1-m+\mu-\alpha}+\sum_{\gamma=\min(m-\mu,p-\alpha)+1}^{m-\mu}\binom{m-\mu}{\gamma}C_{m-\mu-\gamma}\cdot$$

$$\cdot\,\tilde{C}_\gamma(1+\varepsilon)^{p-1+m-\mu-\gamma}(f_p(x))^{1-p-m+\mu+\gamma}\Big]$$

and hence

$$(47)\qquad \left|\frac{d^m}{dx^m}G^{\alpha}_{\nu k}(x)\right|\leqslant C^*_m[p(1+\varepsilon)]^{p-1+m}(f_p(x))^{1-\alpha-m}x^{\nu-1},$$

$(x\in(0,A\rangle;\ m,\alpha,\nu,k=1,2,\dots,p)$, where C^*_m is a positive constant depending on m.

In a similar way we obtain

$$(48)\qquad \left|\frac{d^m}{dy^m}\tilde{G}^{\alpha}_{\nu k}(y)\right|\leqslant C^{**}_m[p(1+\varepsilon)]^{p-1+m}(h_p(y))^{1-\alpha-m}y^{\nu-1}$$

$(y\in(0,B\rangle;\ m,\alpha,\nu,k=1,2,\dots,p)$, where C^{**}_m is a constant of the same type as C^*_m in (47).

In further reasoning we shall use the following formula for the m-th derivative of a composite function $H\circ z$ (where $m\in\mathcal{N}$, $z\in C^m(I,R)$, and $H\in C^m(z(I),E)$; $I\subset R$ being an interval and E a Banach space)

$$(49)\qquad (H\circ z)^{(m)}(x)=\sum_{i=1}^{m}A^m_i(x)(H^{(i)}\circ z)(x)$$

with

$$(50)\qquad A^{\nu}_i(x)=\sum_{r=i-1}^{\nu-1}\binom{\nu-1}{r}z^{(\nu-r)}(x)A^r_{i-1}(x)$$

for $i,\nu\in\mathcal{N}$; $2\leqslant i\leqslant\nu\leqslant m$; $A^{\nu}_1(x)=z^{(\nu)}(x)$ for $\nu\in\mathcal{N}$; $\nu\leqslant m$. We omit an inductive proof of this formula.

We shall also need the following estimate

$$(51)\qquad \left|\frac{d^{\nu}}{dx^{\nu}}z_{\bar{k}(2s)}(x)\right|\leqslant(2s)^{p(\nu-1)}\xi_{2s}$$

$(\nu = 1,2,\dots,p;\ s = 1,2,\dots)$, where

$$(52)\qquad \xi_{2s} = M_*(1+\varepsilon_0)^{2s}\left(\prod_{r=0}^{s-1} \overset{*}{f}_{k_{2r+1}}\right)\cdot\left(\prod_{r=1}^{s} \overset{*}{h}_{k_{2r}}\right)$$

with M_* being a positive constant not depending on n, which easily results from (23), (23′), Assumption I and Lemma 3.

Formulas (47), (49) and (51) will be applied to the estimates of the derivatives of $G^{\alpha}_{\nu_k} \circ z_{\vec{k}_{(2s)}}$. According to (49), we have

$$(53)\qquad \left|\frac{d^m}{dx^m} G^{\alpha}_{\nu_k} \circ z_{\vec{k}_{(2s)}}(x)\right| \leqslant \sum_{i=1}^{m} |A^m_i(x)|\cdot|G^{(i)}_{\nu k} \circ z_{\vec{k}_{(2s)}}(x)|$$

$(m = 1,2,\dots,p;\ s = 1,2,\dots)$, and based on (50), (51) and using mathematical induction we can prove that the inequality

$$(54)\qquad |A^m_i(x)| \leqslant \tilde{M}\,(2s)^{p(m-i)}(\xi_{2s})^i$$

$(i = 1,2,\dots,m)$ holds good, where $\tilde{M}$ is a positive constant independent of s.

Thus (see (47), (53) and (54)) we get

$$\left|\frac{d^m}{dx^m} G^{\alpha}_{\nu k} \circ z_{\vec{k}_{(2s)}}(x)\right| \leqslant \text{const}\, \mathcal{E}(x) \sum_{i=1}^{m} (2s)^{p(m-i)}\cdot(z_{\vec{k}_{(2s)}}(x))^{-i}\cdot$$

$$\cdot\ (1+\varepsilon_0)^{2si}\left[\left(\prod_{r=0}^{s-1} \overset{*}{f}_{k_{2r+1}}\right)\left(\prod_{r=1}^{s} \overset{*}{h}_{k_{2r}}\right)\right]^i$$

with

$$(55)\qquad \mathcal{E}(x) = [p(1+\varepsilon)]^{p-1}\left(f_p \circ z_{\vec{k}_{(2s)}}(x)\right)^{1-\alpha}\left(z_{\vec{k}_{(2s)}}(x)\right)^{\nu-1},$$

const being independent of n.

As a consequence, we obtain

$$(56)\quad \left|\frac{d^m}{dx^m} G^{\alpha}_{\nu k} \circ z_{\overline{k}_{(2s)}}(x)\right| \leqslant \text{const}\, \varepsilon(x)(2s)^{pm}(1-\varepsilon_0)^{-2ms}\cdot x^{-m},$$

$(m,\alpha,\nu,k = 1,2,\ldots p;\ s = 1,2,\ldots)$.

In a similar way one can derive the estimate

$$(57)\quad \left|\frac{d^m}{dy^m} \tilde{G}^{\alpha}_{\nu k} \circ z_{\overline{k}_{(2s-1)}}(x)\right| \leqslant$$

$$\leqslant \text{const}\, \tilde{\varepsilon}(x)(2s-1)^{pm}(1-\varepsilon_0)^{-m(2s-1)}\cdot x^{-m},$$

$(m,\alpha,\nu = 1,2,\ldots,p;\ s = 1,2,\ldots)$, where

$$(58)\quad \tilde{\varepsilon}(x) = [p(1+\varepsilon)]^{p-1}\left(h_p \circ \overset{*}{z}_{\overline{k}_{(2s-1)}}(x)\right)^{1-\alpha}\cdot\left(\overset{*}{z}_{\overline{k}_{(2s-1)}}(x)\right)^{\nu-1}.$$

Finally, by (21), (27), (49) and (51), we have the inequality[6)]

$$(59)\quad \left\|\frac{d^m}{dx^m} F^{\nu_n} \circ \overset{*}{z}_{\overline{k}_{(n)}}(x)\right\| \leqslant$$

$$\leqslant \text{const}\, n^{pm}\left(z_{\overline{k}_{(n)}}(x)\right)^{2p-\nu_n+\varkappa_0}(1-\varepsilon_0)^{-mn}x^{-m}$$

$(m = 1,2,\ldots,p)$, where const is a positive constant of the same type as those above.

Basing on relations (25), (56), (59) and using an argument analogous to that in the proof of (36), we obtain the following estimate

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6) From now on we assume that n is odd; the argument for even values of n is analogous.

(60) $$\left\|\frac{d^m}{dx^m}\cdot a_n^\alpha(x)\right\| \leq \text{const } n^{pm} x^{2p-\alpha-m+\varkappa_0}\cdot$$

$$\cdot\left[p^p(1+\varepsilon)^{p-1}\sqrt{g_0}\right]^n\cdot\left[p^p g_0^{\varkappa_0/2}(1-\varepsilon_0)^{3-6p-\varkappa_0}\right]^n$$

$(m = 1,2,\dots,p)$ and as (see (6) and (11′)) the equality

$$p^p(1+\varepsilon)^{p-1}\sqrt{g_0} < 1, \quad p^p g_0^{\varkappa_0/2}(1-\varepsilon_0)^{3-6p-\varkappa_0} < 1$$

hold good, we have

(61) $$\left\|\frac{d^m}{dx^m} a_m^\alpha(x)\right\| \leq \text{const } n^{pm} q^n x^{2p-\alpha-m+\varkappa_0}$$

$(\alpha, m = 1,2,\dots,p)$, where $x \in (0,\delta)$. The extension of this result to the case 2^0 is straightforward (see the substantiation of (37)).

Thus, the series obtained by the term by term differentiation of order m $(1 \leq m \leq p)$ of the first series in (24) are uniformly convergent in the interval $(0,A\rangle$ and hence the functions φ_α $(\alpha = 1,2,\dots,p)$ are of class C^p in this interval. It is also easily seen that

(62) $$\|\varphi_\alpha^{(m)}(x)\| \leq \text{const } x^{2p-\alpha-m+\varkappa_0}$$

$(x \in (0,A\rangle;\ \alpha, m = 1,2,\dots,p)$. Setting

(63) $$\varphi_\alpha^{(m)}(0) := \lim_{x\to 0^+} \varphi_\alpha^{(m)}(x) = 0$$

$(\alpha, m = 1,2,\dots,p)$ and using (40), we can assert that φ_α $(\alpha = 1,2,\dots,p)$ are of class C^p in the interval $\langle 0,A\rangle$.

By a similar argument one can show that the functions ψ_α, $(\alpha = 1,2,\dots,p$ (see (24))$)$ are of class C^p in the interval $\langle 0,B\rangle$ with $\psi_\alpha^{(m)}(0) := 0$ $(\alpha = 1,2,\dots,p;\ m = 0,1,\dots,p)$. This completes the proof of Lemma 5.

As a result of the aforegoing considerations we can formulate the following final theorem.

T h e o r e m . Under the Assumptions I - III, the Goursat problem (G) has a solution u given by formula (2) with $c_* = a$ (see p.5), where the functions φ_α and ψ_α ($\alpha = 1,2,\dots,p$), given by the series (24) for $x \in (0,A\rangle$, $y \in (0,B\rangle$ and by the equalities $\varphi_\alpha(0) = \psi_\alpha(0) := 0$, are of class C^p in the intervals $\langle 0,A\rangle$ and $\langle 0,B\rangle$, respectively. The said solution is the only solution of the (G)-problem in the set of all solutions of equation (1) (see Lemma 1) such that the system of functions φ_α, ψ_α ($\alpha = 1,2,\dots,p$) in formula (2) with $c_* = a$ belongs to the class K (see p.11).

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