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FUNCTIONS OF BOUNDED φ-VARIATION AND SOME RELATED OPERATORS

Dedicated to the memory of Professor Roman Sikorski

1.1. The notion of variation with p-th power, called p-variation was first introduced by N. Wiener in [14]. Next, L.C. Young developed this idea in [15] and other papers and defined the φ-variation of a real-valued function of real argument. By φ we understand here a continuous, non-decreasing function $\varphi(u) : \langle 0,\infty) \to \langle 0,\infty)$, taking 0 only for $u = 0$ and tending to ∞ when $u \to \infty$. Any such φ will be referred to in this paper as a φ-function. These are the ones that gave rise to Orlicz spaces. Let $x(t) : \langle a,b\rangle \to R$. Throughout the paper it is assumed that x takes finite values and $x(a) = 0$.

Let π be a fixed partition of $\langle a,b\rangle$, $\pi : a = t_0 < t_1 < \ldots < t_n = b$. We call the variational sum on π

$$\sigma_\varphi(x,\pi) = \sum_{i=1}^{n} \varphi(|x(t_i) - x(t_{i-1})|).$$

The number

$$v_\varphi(x) = \sup_\pi \sigma_\varphi(x,\pi)$$

with supremum taken over all partitions π of the interval $\langle a,b\rangle$, is called the φ-variation of x. Letter I is going to denote an interval $I = \langle \alpha, \beta\rangle \subset \langle a,b\rangle$ and $v_\varphi(x)$ calculated over I will be denoted $v_\varphi(x,I) = v_\varphi(x,\alpha,\beta)$. When $v_\varphi(x) < \infty$ the function x is said to have bounded (or finite) φ-variation or to be of φ-BV. These functions found numerous applications and proved to be of independent interest. φ-BV functions are bounded, have both one-sided limits at each point of (a,b) and right-sided at a, left-sided at b.

1.2. For a φ-function φ the following condition will be needed:

φ is said to satisfy condition Δ_2 (for small u) if there exist $k > 1$, $u_0 > 0$ such that

$$\varphi(2u) \leqslant k\varphi(u) \quad \text{for} \quad 0 \leqslant u \leqslant u_0.$$

An equivalent formulation of condition Δ_2 is the following:

for every $c > 0$ there exists a constant $d_c > 1$ such that

$$\varphi(cu) \leqslant d_c\varphi(u) \quad \text{for} \quad 0 \leqslant u \leqslant u_0.$$

We can express the condition Δ_2 analogously for large values of u and all u. If φ satisfies the condition Δ_2 for $0 \leqslant u \leqslant u_0$ it satisfies it also for any other finite u_0, with the constant k suitably changed.

1.3. Consider the vector space

$\mathcal{V}_\varphi^* = \{x: x(a) = 0,\ v_\varphi(\lambda x) < \infty \text{ for some constant } \lambda > 0\}$.

$\mathcal{V}_\varphi^*$ is called the space of functions of φ-BV.

$\mathcal{V}_\varphi = \{x: x(a) = 0,\ v_\varphi(x) < \infty\}$ is called the class of functions of φ-BV. Whenever in the definition of $\mathcal{V}_\varphi^*$ we take $v_\varphi(x,\alpha,\beta)$ instead of $v_\varphi(x,a,b) = v_\varphi(x)$, this fact is denoted $\mathcal{V}_\varphi^*\langle\alpha,\beta\rangle$. $v_\varphi(x)$ is a modular and for φ-convex φ function $\mathcal{V}_\varphi^*$ is a modular space in Musielak-Orlicz sense [9], [10].

For $\varphi(u) = u^p$, $p \geqslant 1$ we get the Wiener-Young classes. It is easy to see that functions satisfying the Hölder condition with $\frac{1}{p}$, $p \geq 1$ are properly included in $\mathcal{V}_p$. Riazanov in [13] established a theorem being to some extent the converse of this fact. Theorem 6 of our paper is a generalization of Riazanov's result.

1.4. Let φ, ψ be arbitrary φ-functions, A a nonempty subset of $\langle a,b \rangle$. We say x is of class $H_\psi^\varphi(A,K)$ if the following inequality is satisfied

$$(*)\quad \varphi(|x(t') - x(t'')|) \leq K\psi(|t' - t''|) \quad \text{for } t', t'' \in A.$$

When $\varphi(u) = u$ we write $H_\psi^1(A,K)$ instead of $H_\psi^\varphi(A,K)$. Other symbols are changed similarly. If $\varphi(u) = u$, $\psi(u) = u^p$, $0 < p \leq 1$, then $H_\psi^\varphi(A,K)$ is replaced by $H_p^1(A,K)$ and we are getting the Hölder classes, $p = 1$ - the Lipschitz classes. If $A = \langle a,b \rangle$ we write short $H_\psi^\varphi(K)$, $H_p^1(K)$ to denote the classes introduced above.

Functions of class $H_\psi^\varphi(A,K)$ are bounded on A. Let us assume φ is strictly increasing and its inverse function φ_{-1} satisfies condition Δ_2 for small u. Then, with some constant $\overline{K}$

$$|x(t_2) - x(t_1)| \leq \overline{K}\gamma(|t_2 - t_1|) \text{ where } \gamma = \varphi_{-1}(\psi)$$

i.e. $x \in H_\gamma^1(K)$.

Note also that if $\gamma(u)/u \to 0$, $u \to 0$ x has in $\langle a,b \rangle$ the derivative $=0$, so in this case $H_\gamma^1(K)$ consists only of constant functions, for example $H_p^1(K)$, $p > 1$.

2. $(**)$ Let $A \subset \langle a,b \rangle$ be a nonempty closed set containing a,b. Let (a_i, b_i) denote disjoint intervals with endpoints $a_i, b_i \in A$, such that $\bigcup_1^\infty (a_i, b_i) = \langle a,b \rangle \setminus A$.

Define the following function F

$$F(t) = \begin{cases} x(t) & \text{for } t \in A \\ \text{linear on every } \langle a_i, b_i \rangle. \end{cases}$$

From now on $F(t)$ is called the function corresponding to x and the set A. Two theorems follow.

Theorem 1. Let φ, ψ satisfy the conditions:

$$(\alpha) \qquad \varphi(uv) \leqslant c_1 v^{\gamma}\varphi(u) \quad 0 \leqslant v \leqslant 1,$$

$$(\beta) \qquad \psi(uv)c_2 \geqslant v^{\gamma_1}\psi(u) \quad \gamma_1 > 0, \quad 0 \leqslant v \leqslant 1,$$

$$(\gamma) \qquad \gamma \geqslant \gamma_1$$

$$(\delta) \qquad \varphi(3u) \leqslant c_3\varphi(u) \quad \text{for} \quad u \geqslant 0 \quad (\text{i.e. for all } u).$$

(It follows $c_1, c_2, c_3 \geqslant 1$).

Under these assumptions, if $x \in H_{\psi}^{\varphi}(A,K)$, where the set A satisfies $(**)$, then for the corresponding function $F \in H_{\psi}^{\varphi}(C)$, where $C = 3Kc_1c_2c_3$.

Proof. First, let us show: if $t' \neq t'' \in \langle a_i, b_i \rangle$ then

$$(1) \qquad \frac{\varphi(|F(t'') - F(t')|)}{\psi(|t'' - t'|)} \leqslant Kc_1c_2c_3.$$

We have

$$\frac{\varphi(|F(t'') - F(t')|)}{\psi(|t'' - t'|)} =$$

$$= \varphi\left(\frac{|t'' - t'|}{b_i - a_i}\,|F(b_i) - F(a_i)|\right)\Big/\psi(|t'' - t'|) \leqslant$$

$$\leqslant c_1\left(\frac{|t'' - t'|}{b_i - a_i}\right)^{\gamma}\frac{\psi(b_i - a_i)}{\psi(|t'' - t'|)}\,\frac{\varphi(|F(b_i) - F(a_i)|)}{\psi(b_i - a_i)} \leqslant$$

$$\leqslant c_1\left(\frac{|t'' - t'|}{b_i - a_i}\right)^{\gamma_1}\frac{\psi(b_i - a_i)}{\psi(|t'' - t'|)}.$$

However, from (α) we have

$$\left(\frac{|t'' - t'|}{b_i - a_i}\right)^{\gamma_1}\psi(b_i - a_i) \leqslant c_2\psi(|t'' - t'|).$$

So, inequality (1) is verified.

We are going to consider the following three possible situations:

1°. $t_1, t_2 \in A$ then $\varphi(|F(t_2)-F(t_1)|) \leq K\psi(|t_2-t_1|)$ from the definition of x and F,

2°. $t_1 \in A$, $t_2 \varphi(a_i,b_i)$, $t_1 < a_i$. Then

$$\varphi(|F(t_2)-F(t_1)|) \leq \varphi(|F(t_2)-F(a_i)|+|F(a_i)-F(t_1)|) \leq$$
$$\leq \varphi(2|F(t_2)-F(a_i)|)+\varphi(2|F(a_i)-F(t_1)|) \leq$$
$$\leq c_3(\varphi(|F(t_2)-F(a_i)|)+\varphi(|F(a_i)-F(t_1)|)) \leq$$
$$\leq Kc_1c_2c_3\psi(t_2-a_i) + K\psi(a_i-t_1) \leq$$
$$\leq c_3K \sup(1,c_1c_2)(\psi(t_2-a_i)+\psi(a_i-t_1)) \leq$$
$$\leq C_1\psi(|t_2-t_1|),$$

where $C_1 = 2K_3\sup(1,c_1c_2) \leq 2Kc_3c_2c_1$ since $c_1 \geq 1$, $c_2 \geq 1$. The same estimate is obtained in the case $t_2 \in A$, $t_2 \geq b_i$, $t_1 \in (a_i,b_i)$.

3°. $t_1 \in (a_i,b_i)$, $t_2 \in (a_j,b_j)$, $b_i \leq a_j$. Then, in view of (1)

$$\varphi(|F(t_2)-F(t_1)|) \leq \varphi(|F(b_i)-F(t_1)| + |F(a_j)-F(b_i)| +$$
$$+ |F(t_2)-F(a_j)|) \leq \varphi(3|F(b_i)-F(t_1)|) +$$
$$+ \varphi(3|F(a_j)-F(b_i)|) + \varphi(3|F(t_2)-F(a_j)|) \leq$$
$$\leq Kc_1c_2c_3\psi(b_i-t_1) + Kc_3\psi(a_j-b_i) +$$
$$+ Kc_1c_2c_3\psi(t_2-a_j) \leq$$
$$\leq K \sup(c_3,c_1c_2c_3)\left\{\psi(b_i-t_1)+\psi(a_j-b_i) +\right.$$
$$\left.+ \psi(t_2-a_j)\right\} \leq C_2\psi(t_2-t_1),$$

where $C_2 = 3Kc_3\sup(1,c_1c_2) = 3Kc_3c_2c_1$.

4^o. $t_1, t_2 \in \langle a_i, b_i \rangle$ then from (1) we have $\varphi(|F(t_2)-F(t_1)|) \leq C_3\psi(t_2-t_1)$ where $C_3 = Kc_1c_2$. If $\varphi(u) = u^{\gamma}$, $\psi(u) = u^{\gamma_1}$, $\gamma \geq \gamma_1$ one can assume $c_1 = c_2 = 1$, $c_3 = 3^{\gamma}$, so, one can put $C = K3^{\gamma+1}$. In the limiting case i.e. $\varphi(u) = \psi(u) = u$ it is possible to take $C = K$. Indeed, if $t', t'' \in \langle a_i, b_i \rangle$ then $|F(t'') - F(t')| \leq K|t'' - t'|$. We get the same estimates when $t_1, t_2 \in A$. If $t_1 \in A$, $t_2 \in \langle a_i, b_i \rangle$, $t_1 < a_i$ then it is easy to see that the coefficient of direction of the straight line connecting points $(t_1, F(t_1))$, $(t, F(t))$ where $a_i \leq t \leq b_i$, this is to say the quotient $\frac{F(t) - F(t_1)}{t - t_1}$ is of the form $\frac{pt + q}{rt + s}$. The denominator of this homographic function is $\neq 0$ in $\langle a_i, b_i \rangle$. So, it assumes its extremal values at the endpoints of this interval. Since $\left|\frac{F(b_i) - F(t_1)}{b_i - t_1}\right| \leq K$, $\left|\frac{F(a_i) - F(t_1)}{a_i - t_1}\right| \leq K$ so $\left|\frac{F(t_2) - F(t_1)}{t_2 - t_1}\right| \leq K$. We obtain the same estimate when $t_2 \in A$, $t_2 > b_i$. If $t_1 \in \langle a_i, b_i \rangle$, $t_2 \in \langle a_j, b_j \rangle$, $b_i < a_j$ then

$$|F(t_2)-F(t_1)| \leq |F(b_i)-F(t_1)| + |F(t_2)-F(b_i)| \leq$$

$$\leq K((b_i-t_1) + (t_2-b_i)) = K(t_2-t_1).$$

T h e o r e m 2. Let φ be a convex φ-function, F - the function corresponding to x and some closed set A. Then

$$(*) \qquad v_{\varphi}(F,a,b) = v_{\varphi}(x,A).$$

P r o o f . In the case when $\varphi(u) = u^p$, $p \geq 1$, A is finite - this theorem can be found in Musielak-Semadeni [11], in the case when φ - is a convex and A finite - in Leśniewicz-Orlicz [6] the case of arbitrary A, $\varphi(u) = u^p$, $p \geq 1$ is in Riazanov [13].

We are going to carry out the proof for the general case and arbitrary convex φ along the same lines as in Musielak--Semadeni.

Let $\pi : a = t_0 < t_1 < \ldots < t_n = b$ be an arbitrary partition. Let us set up a new partition $\pi^1 : a = \tau_0 < \tau_1 < \ldots < \tau_m = b$, whose division points are some from A and t_i except for one t_i. Let t_j be the smallest index such that t_j does not belong to A. Consider two cases:

1^0. $(F(t_j) - F(t_{j-1}))(F(t_{j+1}) - F(t_j)) \geq 0$.
Since φ is superadditive, it follows that

$$(1) \qquad \varphi(|F(t_j)-F(t_{j-1})|) + \varphi(|F(t_{j+1})-F(t_j)|) \leq$$
$$\leq \varphi(|F(t_{j+1})-F(t_{j-1})|).$$

2^0. $(F(t_j)-F(t_{j-1}))(F(t_{j+1})-F(t_j)) < 0$.
Consider for instance the situation $F(t_j) > F(t_{j-1})$, $F(t_j) > F(t_{j+1})$. Since t_j does not belong to A so it is an interior point of some interval $\langle \tau', \tau \rangle$, $t_{j-1} \leq \tau'$, on which F is linear. If $F(\tau') > F(\tau'')$, then $F(\tau') > F(t_j)$ and

$$(2) \qquad F(\tau')-F(t_{j-1}) \geq F(t_j)-F(t_{j-1}).$$

We have also

$$(3) \qquad F(\tau')-F(t_{j+1}) > F(t_j)-F(t_{j+1}).$$

If $F(\tau') \leq F(\tau'')$, (we allow $\tau' = t_{j-1}$) then there cannot be $\tau'' > t_{j+1}$ because the straight line between points $(\tau', F(\tau'))$, $(\tau'', F(\tau''))$ is increasing and the point $(t_{j+1}, F(t_{j+1}))$ does not lie on the segment. So we have $t_j < \tau'' < t_{j+1}$,

$$(4) \qquad F(\tau'')-F(t_{j-1}) \geq F(t_j)-F(t_{j-1}),$$

$$(5) \qquad F(\tau'')-F(t_{j+1}) \geq F(t_j)-F(t_{j+1}).$$

So, in both cases we have for some $\tau \in A$,

(6) $\varphi(|F(\tau)-F(t_{j-1})|) \geq \varphi(|F(t_j)-F(t_{j-1})|)$,

(7) $\varphi(|F(\tau)-F(t_{j+1})|) \geq \varphi(|F(t_j)-F(t_{j+1})|)$.

We are going to construct a new partition π_1 out of π in the following way.

In the case 1° we leave out the point t_j and let the remaining ones t_i stand without change. In the case 2° we leave out t_j and replace it with τ appearing in (6), (7) without changing the remaining points t_i. It can be seen that in both cases we get a partition π_1 such that

$$\sigma_\varphi(F,\pi) \leq \sigma_\varphi(F,\pi_1).$$

Then, we proceed with π_1 analogously to π and delete from the partition another point t_i, getting in this way the next partition π_2, such that

$$\sigma_\varphi(F,\pi_1) \leq \sigma_\varphi(F,\pi_2).$$

Proceeding analogously in this way we get finally a partition π_k such that $\sigma_\varphi(F,\pi) \leq \sigma_\varphi(F,\pi_k)$ with all points of π_k belonging to A. So, there is $\sigma_\varphi(F,\pi) \leq v_\varphi(F,A)$ for arbitrary π and in consequence

$$v_\varphi(F,a,b) \leq v_\varphi(x,A).$$

Since the opposite inequality is obvious, the proof is complete.

R e m a r k . The corresponding function F(t) defines a linear operator acting from the space of functions x such that with some $\lambda > 0$ and constant K both depending on x, $\lambda x \in H_\psi^\varphi(A,K)$ into the space $H_\psi^\varphi(C)$ (with a suitably chosen norm). The function F(t) can also be regarded to define analogously a linear operator acting from the space of functions of φ-BV on a set A into v_φ^*.

3.1. Let f be a function defined on some interval $\langle\alpha,\beta\rangle$. In this section we shall make use of the essential supremum of f on $\langle\alpha,\beta\rangle$, denoted either $\sup^*_{\langle\alpha,\beta\rangle} f$ or $\operatorname{supess}_{\langle\alpha,\beta\rangle} f$. It is well known from the definition that $\sup^*_{\langle\alpha,\beta\rangle} f = \inf(\sup_e f)$ where infimum is taken over all subsets of $\langle\alpha,\beta\rangle$ of full measure. $\sup^*_{\langle\alpha,\beta\rangle} f$ is always assumed on a set e_0, $\mu(e_0) = \beta-\alpha$. For two functions $f = f_1$ almost everywhere on $\langle\alpha,\beta\rangle$ $\sup^*_{\langle\alpha,\beta\rangle} f = \sup^*_{\langle\alpha,\beta\rangle} f_1$. $\sup^*$ on an open or halfopen interval is defined analogously.

In this section functions are assumed to be defined on some interval $\langle 0,\tau)$ (not always the same) and extended periodically on the whole real line with period τ. Let x be such a function and moreover $x \in v_\varphi\langle 0,\tau\rangle$. We introduce here for x its x^*-starred x. It is defined on $\langle 0,2\tau)$ as $x^*(t) = x(t+)$. It has the following properties:

(i) x^* is right continuous for $0 \le t < 2\tau$, $v_\varphi(x,0,2\tau) < \infty$.

(ii) $x^*(t-) = x(t-)$ for $0 \le t \le \tau$.

Define

$$(*)\qquad L^{*\varphi}_{\psi}(x,t) = \lim_{\delta\to 0}\ \sup^*_{0<h\le\delta} \frac{\varphi(|x(t+h)-x(t)|)}{\psi(h)}.$$

If in the definition $(*)$ we replace $\sup^*$ by $\sup$ we write

$$(**)\qquad L^{\varphi}_{\psi}(x,t) = \lim_{\delta\to 0}\ \sup_{0<h\le\delta} \frac{\varphi(|x(t+h)-x(t)|)}{\psi(h)} =$$

$$= \limsup_{h\to 0^+} \frac{\varphi(|x(t+h)-x(t)|)}{\psi(h)}.$$

It is plain $L^{*\varphi}_{\psi}(x,t) \le L^{\varphi}_{\psi}(x,t)$. For $\varphi(u) = u^p$ we write $L^{p}_{\psi}(x,t)$ and for $\psi(u) = u^p$ – $L^{\varphi}_{p}(x,t)$. If x is continuous then $L^{*\varphi}_{\psi}(x,t) = L^{\varphi}_{\psi}(x,t)$.

3.2. L e m m a 1. Let $x(t)$ be defined on $\langle a,b\rangle$. If $L^{\varphi}_{\psi}(x,t) < \infty$ for all $t \in \langle a,b\rangle$ off a set of measure 0 then x is continuous for almost all t and measurable.

P r o o f . If $L^{\varphi}_{\psi}(x,t)<\infty$ for $t\in e$, $\mu(e)$ = b-a then it is plain that at every such point x is continuous and measurability follows.

L e m m a 2. If $L^1_1(x,t)<\infty$ for all t but a set of measure 0 then $L^1_1(x,t)$ is a measurable function.

P r o o f . Denote $L^+(x,t)$, $L_+(x,t)$ the upper and lower right Dini derivatives of x at the point t. Let $a_1=\{t\in\langle a,b\rangle : -k<L^+(x,t)<k\}$ analogously a_2 and $a=\{t\in\langle a,b\rangle : L^1_1(x,t)<k\}$. There holds $a=a_1\cap a_2$. According to Lemma 1, x is measurable. By a Banach theorem [2] Dini derivatives are measurable functions. So, a is measurable with any $k>0$ and thus $L^1_1(x,t)$ is measurable.

T h e o r e m 3. If $L^1_1(x,t)\leq M$ for every $t\in\langle a,b\rangle$ then $x\in H^1_1(M)$, i.e. x satisfies in $\langle a,b\rangle$ the Lipschitz condition with a constant M.

Proof is based on a classical theorem of Dini which we quote here according to [5]:

Let x be continuous in $\langle a,b\rangle$, $L^+(x,t)$, $L_+(x,t)$, $L^-(x,t)$, $L_-(x,t)$ be its Dini derivatives at $t\in\langle a,b\rangle$. All the four derivatives assume in $\langle a,b\rangle$ the same supremum K and infimum k. K is at the same time equal to the supremum of difference quotients $\dfrac{x(t_2)-x(t_1)}{t_2-t_1}$ for $t_1,t_2\in\langle a,b\rangle$ and k - the infimum of these quotients.

Let us remark here that there does not hold a theorem in a way analogous to Th.1 with $L^1_1(x,t)$ replaced by $L^1_{\gamma}(x,t)$ and Lipschitz condition by Hölder one.

T h e o r e m 4. For arbitrary $0<\gamma<1$ there exists a function x, with $v_p(x,0,1)=\infty$ for any $p>0$ and in consequence belonging to no Hölder class, such that $L^1_{\gamma}(x,t)=0$ for $t\in\langle 0,1\rangle$.

P r o o f . The function x is supposed to be defined on $\langle 0,1\rangle$. First let us construct in the interval $\langle\frac{1}{2^n},\frac{1}{2^{n-1}}\rangle$, $n=1,2,\dots$ a function in the following way. We choose a sequence of powers $p_n\to\infty$ and divide $\langle\frac{1}{2^n},\frac{1}{2^{n-1}}\rangle$ with points

$\frac{1}{2^n} = t_0 < t_1 < \dots < t_{k_n} = \frac{1}{2^{n-1}}$ into k_n equal subintervals $\langle t_{i-1}, t_i \rangle$, so as to have at the same time $k_n \geq \frac{1}{2}\, 2^{p_n}\, 2^{(n-1)p_n}\, n$.

Define the functions x_n, by taking

$$x_n(t) = \begin{cases} 0 & t = t_i, \quad i = 0,1,\dots,k_n \\ \frac{1}{2^{n-1}} & t = t_i^o \text{ for } t_i^o = \frac{t_{i-1} + t_i}{2}, \quad i = 0,1,\dots,k_n \\ \text{completed to linear function in} & \langle t_{i-1}, t_i^o \rangle \text{ and } \langle t_i^o, t_i \rangle. \end{cases}$$

There holds the inequality

$$v_{p_n}\left(x_n, \frac{1}{2^n}, \frac{1}{2^{n-1}}\right) \geq 2\, \frac{k_n}{2^{(n-1)p_n}} \geq 2 \frac{1}{2}\, 2^{p_n}\, n = 2^{p_n}\, n.$$

Define in $\langle 0,1 \rangle$ a function x by taking:

$$x(t) = \begin{cases} 0 & t = 0, \\ 0 & t = 1, \\ x_n(t) & \frac{1}{2^n} \leq t \leq \frac{1}{2^{n-1}} \quad \text{for} \quad n = 1,2,\dots \end{cases}$$

For $p_n > p$ there is

$$v_p(x) \geq \frac{1}{2^{p_n}}\, v_{p_n}\left(x_n, \frac{1}{2^n}, \frac{1}{2^{n-1}}\right) \geq \frac{1}{2^{p_n}}\, 2^{p_n}\, n \quad \text{for} \quad n = 1,2,\dots$$

Thus, $v_p(x,0,1) = \infty$ for arbitrary $p > 0$. Since for $\frac{1}{2^n} \leq h \leq \frac{1}{2^{n-1}}$ there is $\frac{x(h)}{h^\gamma} \leq \frac{1}{2^{n-1}}\, 2^{n\gamma} \to 0$ as $n \to \infty$. So, $\lim_{h \to 0} \frac{|x(0+h) - x(0)|}{h^\gamma} = 0$. If $t > 0$ then $x(t)$ belongs to one of the intervals on which either x is linear or t is the endpoint of two continuous intervals of linearity of $x(t)$. For sufficiently small h there is $x(t+h) - x(t) = ch$, so $\lim_{h\to 0} \frac{|x(t+h) - x(t)|}{|h|^{\gamma}} = 0$ (at $t = 1$, as $h \to 0-$). So $L^1_{\gamma}(x,t) = 0$ in the whole interval $\langle 0,1\rangle$.

Let us further notice that if $\gamma = 1$ then $L^1_1(x,t) < \infty$ for every t, but, as we have seen, x is not of p-BV for any p.

L e m m a 3. Let A be a nonempty subset of $\langle a,b\rangle$, right closed, this is to say, having the property: if $t_n \in A$, $t_0 < t_n$, $t_n \to t_0$ then $t_0 \in A$. Under this assumption A is measurable.

P r o o f . Denote $A' = \langle a,b\rangle \setminus A$. We claim that if $t_0 \in A'$, $a < t_0$ then there exists an interval $I(t_0) = \langle\alpha,\beta\rangle$ such that $(\alpha,\beta) \subset A'$, having the following properties:
(1) $t_0 \in \langle\alpha,\beta)$, (2) $\beta \in A$ or $\beta = b$, (3) $\alpha \in A$ or (3') $\alpha \in A'$ where α is the leftsided point of accumulation of A. Checking this is left to the reader. Consider now two different intervals $\langle\alpha',\beta'\rangle$, $\langle\alpha,\beta\rangle$ satisfying the conditions given above. Then, it is easy to see that (*) $(\alpha',\beta') \cap (\alpha,\beta) = \emptyset$. Let us take the set of all different intervals (α,β). It is at most denumerable, so, of the form $\bigcup_1^{\infty} (\alpha_i,\beta_i)$. Let $t_0 \in A'$. Its corresponding interval (α_0,β_0) is one of the (α_i,β_i). So, $t_0 \in (\alpha_i,\beta_i)$ or $t_0 = \alpha_i$. Let B denote the at most denumerable set of such endpoints t_0. Thus, we have shown $A = \bigcup_1^{\infty} (\alpha_i,\beta_i) \cup B$ what amounts to proving A' and also A is measurable.

L e m m a 4. Let y be a finite function on $\langle 0,2)$. Define

$$l^{\delta}(y,t) = \sup_{0<h\leq\delta} \frac{\varphi(|y(t+h) - y(t)|)}{\psi(h)}$$

for $t \in \langle 0,1\rangle$, $0 < \delta \leq 1$. Assume
(a) y is right continuous in $\langle 0,2)$,
(b) y is non-decreasing in $\langle 0,2\rangle$.

Then $l^{\delta}(y,t)$ is a measurable function in $\langle 0,1\rangle$.

P r o o f . Ad. (a). Denote $A = \{t \in \langle 0,1): l^{\delta}(y,t) \leqslant k\}$, $A = \{t \in \langle 0,1): \varphi(|y(t+h) - y(t)|) \leqslant k\psi(h),\ 0<h\leqslant\delta\}$. Let $0 \leqslant t_0 < 1$, $1 \geqslant t_i > t_0$, $t_i \to t_0$. Since y is right continuous at t_0, t_0+h, so $\varphi(|y(t_i+h)-y(t_i)|) \to \varphi(|y(t_0+h)-y(t_0)|)$ for any $0 \leqslant h \leqslant \delta$. There follows that A is right closed and by Lemma 3 - measurable.

Ad. (b). We claim (1) $l^{\delta}(y,t) = l^{\delta}(y^*,t)$ off a denumerable set, where y^* is the function starred y introduced in 3.1. (y^* exists because $y \in v\langle 0,2\rangle$). For $0 \leqslant t < 1$ such that $y^*(t) = y(t)$ there holds (2) $y^*(t+h) = y(t+h)$ off a denumerable set of values $0 \leqslant h < \delta$. So, one can find a dense set of h in which (2) holds. For a given $h<\delta$ there is

$$l^{\delta}(y^*,t) \geqslant \frac{\varphi(y^*(t+h_i) - y^*(t))}{\psi(h_i)} = \frac{\varphi(y(t+h_i) - y(t))}{\psi(h_i)} \geqslant$$

$$\geqslant \frac{\varphi(y(t+h) - y(t))}{\psi(h)}$$

with some $0<h<h_i$, $h_i \to h$. Therefore $l^{\delta}(y^*,t) \geqslant l^{\delta}(y,t)$ off a denumerable set of t. Let us now take $h_i<h<\delta$, $h_i \to h>0$ to have $y^*(t+h_i) = y(t+h_i)$ for t such that $y(t) = y^*(t)$. We have

$$\frac{\varphi(y^*(t+h_i) - y^*(t))}{\psi(h_i)} = \frac{\varphi(y(t+h_i) - y(t))}{\psi(h_i)} \leqslant$$

$$\leqslant \frac{\varphi(y(t+h) - y(t))}{\psi(h)} .$$

Hence

$$\frac{\varphi(y^*(t+h) - y^*(t))}{\psi(h)} \leqslant \frac{\varphi(y(t+h) - y(t))}{\psi(h)} \leqslant l^{\delta}(y,t).$$

Consequently $l^{\delta}(y^*,t) \leqslant l^{\delta}(y,t)$ off a denumerable set of t. There is then $l^{\delta}(y,t) = l^{\delta}(y^*,t)$ off a denumerable set and by (a) $l^{\delta}(y,t)$ is measurable.

C o r o l l a r y . If x is periodic of period 1, then for its starred function x^*, $L_\psi^\varphi(x^*,t)$ is measurable by Lemma 4 and 3.1. (**). In particular if x is continuous then $L_\psi^\varphi(x,t)$ is measurable. The same holds for x nondecreasing in $\langle 0,2\rangle$.

T h e o r e m 5. Let x be periodic with period 1, $x \in \mathcal{V}_\varphi\langle 0,1\rangle$.

A. For arbitrary φ, there is $L_1^\varphi(x,t) < \infty$ almost everywhere in $0,1$.

B. If $x \in \mathcal{V}_\varphi\langle 0,1\rangle$, φ is strictly increasing, φ_{-1} satisfies condition Δ_2 for small u, then $L_{\varphi_{-1}}^1(x,t) < \infty$ almost everywhere.

P r o o f . Denote $g(t) = v_\varphi(x,0,t)$, $0 \leq t \leq 1$. The subadditivity of $v_\varphi(x,\alpha,\beta)$ over intervals implies

$$(1) \qquad \frac{\varphi(|x(t+h) - x(t)|)}{h} \leq \frac{|g(t+h) - g(t)|}{h}.$$

Since g is nondecreasing, it has a derivative almost everywhere $g'(t) \geq 0$ in $\langle 0,1\rangle$ so by (1) $L_1^\varphi(x,t) \leq g'(t) < \infty$ almost everywhere. Assume φ as in hypothesis B and let for some t, $g'(t) < \infty$. Then $\varphi(|x(t+h) - x(t)|) \leq g'(t) + \varepsilon)h$ for $0 < h \leq h_0(t)$. Hence

$$(2) \qquad |x(t+h) - x(t)| \leq \varphi_{-1}((g\ (t)+\varepsilon)h).$$

But condition Δ_2 implies $\varphi_{-1}((g'(t)+\varepsilon)h) \leq C_t\varphi_{-1}(h)$ for sufficiently small h and $0 < \varepsilon \leq 1$. So there is $L_{\varphi_{-1}}^1(x,t) \leq C_t < \infty$ almost everywhere.

This theorem is a generalization of Th.1 from [8] by J. Marcinkiewicz, who considered the case $\varphi(u) = u^p$, $p > 1$. His method of proof is different from ours and uses the Vitali covering with tacit assumption that $L_1^\varphi(x,t)$ is measurable. It is not settled whether this is really so. Anyway, Marcinkiewicz does not mention this question. An improved version of Marcinkiewicz's theorem using an analogous method w. outer measure can be found in Gehring [4].

T h e o r e m 6. (of Riazanov type [13]). Let φ be a convex φ-function satisfying condition (δ) from Th.1. Let x be a periodic function of period 1, $v_\varphi(x,0,1)<\infty$.

For every $\varepsilon>0$ there exists a closed set $A\subset\langle 0,1\rangle$, $\mu(\langle 0,1\rangle\setminus A)<\varepsilon$ with the following properties:

(a) if F is the function corresponding to x and the set A (according to the definition 2(**)) then $F\in H_1^\varphi(C)$ with some constant C,

(b) $v_\varphi(F,0,1)\leq v_\varphi(x,0,1)$.

P r o o f . First of all, let us notice that for starred x, x^* according to the definition 3.1 there is $v_\varphi(x^*,0,1)<\infty$. So, by Th.5, $L_1^\varphi(x,t)<\infty$ in the set $a\subset\langle 0,1\rangle$ such that $\mu(a)=1$. Let for $n:1,2,\ldots$, $a_n=\left\{t\in a:\ \varphi(|x^*(t+h)-x^*(t)|)\leq nh,\right.$ for $\left.0\leq h\leq\frac{1}{n}\right\}$. We have $a_n\subset a_{n+1}$, $a=\bigcup\limits_1^\infty a_n$. As $a_n=$ $=\left\{t\in a:\ l^{1/n}(x^*,t)\leq n\right\}$, so, by Lemma 4, a_n are measurable. We can choose n_0 so that $\mu(a\setminus a_{n_0})<\varepsilon/2$. Take two points $t_1,t_2\in a_{n_0}$ such that $|t_1-t_2|\leq\frac{1}{n_0}$. From the definition of a_{n_0} follows that $\varphi(|x^*(t_1)-x^*(t_2)|)\leq n_0|t_1-t_2|$. Let $\sup\limits_{\langle 0,1\rangle}|x^*(t)|=k$. If $t_1,t_2\in a_{n_0}$, $|t_1-t_2|>\frac{1}{n_0}$ then

$$\varphi(|x^*(t_1)-x^*(t_2)|)\leq\varphi(2k)n_0\frac{1}{n_0}\leq\varphi(2k)n_0(|t_1-t_2|).$$

Consequently, $x^*\in H_1^\varphi(a_{n_0},c)$ where $c=n_0\sup(1,\varphi(2k))$.

However, $x^*(t)=x(t)$ in $\langle 0,1\rangle$ off a denumerable set. So, we can choose a closed set $A\subset a_{n_0}$ so that $\mu(a_{n_0}\setminus A)<\varepsilon/2$ and for $t\in A$, $x^*(t)=x(t)$. Considering the fact that $a=\langle 0,1\rangle$ but for a null set, we get $\mu(\langle 0,1\rangle\setminus A)<\varepsilon$ and $x\in H_1^\varphi(A,c)$. It is sufficient now to use Th.1.

Point (b) is a corollary from Th.2 in 2. Riazanov gives in [13] an analogous result for the case $\varphi(u)=u^p$, $p\geq 1$. His method of proof is different from ours. Let us notice that in the course of proof he makes use of measurability of some sets, which are measurable indeed but this fact is by no means obvious. We mean here sets from Lemma 4, p.(b).

L e m m a 5. Let y be defined on $\langle 0,2)$. Set

$$l^{*\delta}(y,t) = \operatorname*{supess}_{0<h\leq\delta} \frac{\varphi(|y(t+h)-y(t)|)}{h}$$

and $A(k) = \{t: l^{*\delta}(y,t) \leq k\}$. If y is continuous almost everywhere then $A(k)$ is a measurable set.

P r o o f . Let A denote the set of points of continuity in $\langle 0,1\rangle$ so $\mu(A) = 1$. Denote for a given t the set $e(t) = \{h: 0<h<\delta,\ t+h$ is a point of continuity of $y\}$. By assumption $\mu(e(t)) = \delta$. Denote by $a(t)$ the set of such $0<h<\delta$, for which $l^{*\delta}(y,t) = \sup\limits_{h\in a(t)} \frac{\varphi(|y(t+h) - y(t)|}{h}$ and $l(y,t) = \sup\limits_{h\in e(t)} \frac{\varphi(|y(t+h) - y(t)|)}{h}$. It is plain $\mu(a(t)) = \delta$. The definition of $\sup^*$ implies $l(y,t) \geq l^{*\delta}(y,t)$, because $\mu(e(t)) = \delta$. Let $h \in e(t_0)$. Choose a sequence of h_i such that $t_0+h_i \in a(t_0)$, $h_i \to h$ and in consequence

$$\frac{\varphi(|y(t_0+h_i) - y(t_0)|)}{h_i} \to \frac{\varphi(|y(t_0+h) - y(t_0)|)}{h}$$

and since $\frac{\varphi(|y(t_0+h_i) - y(t_0)|)}{h_i} \leq l^{*\delta}(y,t_0)$ so

$\frac{\varphi(|y(t_0+h) - y(t_0)|)}{h} \leq l^{*\delta}(y,t_0)$ for $h \in e(t_0)$. Hence $l(y,t_0) \leq$ $\leq l^{*\delta}(y,t_0)$. So, we have $l(y,t) = l^{*\delta}(y,t)$, this is $A(k) = \{t: l(y,t) \leq k\}$. We are going to show that $A(k)$ is measurable. Let $t_0 \in A$, $0 \leq t_0 \leq 1$, $t_0+h_0 \in e(t_0)$, $t_i \to t_0$, $t_i \in A(k)$. For $h \in e(t_i)$ there is $\frac{\varphi(|y(t_i+h) - y(t_i)|)}{h} \leq k$. Since $e = \bigcap_{1}^{\infty} e(t_i)$ has measure equal to δ, so one can choose

$h_i \to h_0$, $h_i \in e$ and there is $\frac{\varphi(|y(t_i+h_i) - y(t_i)|)}{h_i} \leq k$, for $i = 1,2,\ldots$ But at the points t_0+h_0, t_0 the function y is continuous, so $\frac{\varphi(|y(t_0+h_0) - y(t_0)|)}{h_0} \leq k$, that is at the point t_0 $\sup \frac{\varphi(|y(t_0+h) - y(t_0)|)}{h} \leq k$ for $h \in e(t_0)$ and in consequence $t_0 \in A(k) \cap A$. We have shown that the accumulation points of the set $A(k)$ included in A belong to $A(k)$. Thus we have proved $\bar{A}(k) \cap A = A(k) \cap A$, where $\bar{A}(k)$ is the closure of $A(k)$. Let $A' = \langle 0,1\rangle \setminus A$. We have then $\bar{A}(k) = (\bar{A}(k) \cap A) \cup$ $\cup\, (\bar{A}(k) \cap A')$. But $\bar{A}(k)$ is closed, $\bar{A}(k) \cap A'$ is of measure 0, so $A(k) \cap A$ is a measurable set. At the same time $A(k) \cap A'$ is of measure 0, so $A(k)$ is measurable.

R e m a r k . A more general theorem can be proved analogously: if y is continuous almost everywhere then $A(k) = \left\{t : l_\psi^{*\varphi}(y,t) \leq k\right\}$ are measurable, where

$$l_\psi^{*\varphi}(y,t) = \operatorname*{supess}_{0<h\leq\gamma} \frac{\varphi(|y(t+h) - y(t)|)}{\psi(h)} .$$

T h e o r e m 7. Let x be periodic with period 1, $v_\varphi(x,0,1) < \infty$. The function $L_1^{*\varphi}(x,t)$ is finite for almost all $t \in \langle 0,1\rangle$, measurable and there holds the inequality

$$(*) \qquad \int_0^1 L_1^{*\varphi}(x,t)dt \leq v_\varphi(x,0,1).$$

P r o o f . By inequality (1) in the proof of Th.5 where $g(t) = v_\varphi(x,0,1)$ we get (1) $L_1^{*\varphi}(x,t) \leq L_1^{\varphi}(x,t) \leq g'(t) < \infty$ for almost all $t \in \langle 0,1\rangle$. The function x is continuous almost everywhere (off a denumerable set) in $\langle 0,1\rangle$. From the definition of $L_1^{*\varphi}(x,t)$ there follows

$$L_1^{*\varphi}(x,t) = \lim_{n\to\infty} l^{*\,1/n}(x,t)$$

(comp. def. in Lemma 5) with $\ell^{*1/n}(x,t)$ being a nonincreasing sequence. Denote $a = \left\{t : L_1^{*\varphi}(x,t) < k\right\}$, $a_n=\left\{t: \ell^{*1/n}(x,t)< k\right\}$. Hence, we have $a = \bigcup_{n=1}^{\infty} (a_n \cap a_{n+1} \cap \ldots)$. By Lemma 5 the sets a_n are measurable, so a is measurable for any $k \geq 0$. From (1) there follows

$$\int_0^1 L_1^{*\varphi}(x,t)dt \leq \int_0^1 g'(t)dt < \infty$$

and further, by a classical theorem

$$\int_0^1 g'(t)dt \leq v(g,0,1) = g(1) - g(0) \leq v_\varphi(x,0,1).$$

R e m a r k . J. Marcinkiewicz gives in [8] a theorem of this type for $\varphi(u) = u^p$, $p \geq 1$, with $L_1^{*\varphi}(x,t)$ replaced by $L_1^{\varphi}(x,t)$ and assuming tacitly its measurability. This question is not settled here. It seems it should be settled in the affirmative, the way it is for $\varphi(u) = u$ (see Th.3) and also for x continuous (right-continuous) since then $L_1^{*\varphi}(x,t) = L_1^{\varphi}(x,t)$ (comp. Lemma 4).

T h e o r e m 8. Let $x \in \mathcal{V}_\varphi\langle 0,1\rangle$. Let φ be a strictly increasing φ-function, φ_{-1} satisfy the condition of submultiplicity $\varphi_{-1}(uv) \leq c\, \varphi_{-1}(u)\, \varphi_{-1}(v)$, $u,v \geq 0$. Then $L_{\varphi-1}^{*1}(x,t)$ is finite almost everywhere, measurable and

$$\int_0^1 \varphi\left(\frac{1}{c}\, L_{\varphi-1}^{*1}(x,t)\right)dt \leq v_\varphi(x,0,1).$$

P r o o f . The measurability of $L_{\varphi-1}^{*1}$ follows from a remark to Lemma 5. We have, as in the proof of Th.6.

$$\frac{\varphi(|x(t+h) - x(t)|)}{h} \leq \ell(h), \quad \ell(h) = \frac{g(t+h) - g(t)}{h}$$

and hence $\varphi(|x(t+h)-x(t)|)\leq \ell(h)h$. $|x(t+h)-x(t)|\leq \varphi_{-1}(\ell(h)h)\leq$ $\leq c\varphi_{-1}(\ell(h))\varphi_{-1}(h)$, whence, as $\ell(h)\to g'(t)$, $\frac{1}{c}L^{1}_{\varphi-1}(x,t)\leq$ $\leq\varphi_{-1}(g'(t))$ almost everywhere. Simultaneously $L^{*1}_{\varphi-1}(x,t)\leq$ $\leq L^{1}_{\varphi-1}(x,t)$ and thus

$$\int_0^1 \varphi\left(\frac{1}{c} L^{*1}_{\varphi-1}(x,t)\right)dt\leq \int_0^1 g'(t)dt\leq v_\varphi(x,0,1).$$

R e m a r k . It can be seen that the operator $U(x,t) = L^{*1}_{\varphi-1}(x,t)$ is homogeneous and subadditive in the following sense

$$U(x_1 + x_2,t)\leq U(x_1,t) + U(x_2,t)$$

almost everywhere. The theorem above implies that this operator is continuous and acting $v^{*}_{\varphi}\to L^{*\varphi}$, where $L^{*\varphi}$ is an Orlicz space of φ-integrable functions.

By way of application we are going to prove

T h e o r e m 9. Let φ be a φ-function strictly increasing and satisfying condition Δ_2 for small u. In the space $C_{\langle a,b\rangle}$ of continuous functions on $\langle a,b\rangle$, the set of periodic functions with period b-a, for which $v_\varphi(x) = \infty$ is residual.

P r o o f . It is known that in $C_{\langle a,b\rangle}$ the set of periodic functions with period b-a, for which $L^{1}_{\varphi-1}(x,t) = \infty$ everywhere is residual [3] (comp. also [12], Th.8). In view of Th.7 the set of continuous functions from v_φ is of I-Baire category.

A particular case of Th.9, with periodicity dropped, is Luxemburg's

T h e o r e m [7]. In the space $C_{\langle a,b\rangle}$ the set of continuous functions from v (space of functions of bounded Jordan variation) is of first category of Baire.

Let us remark that Luxemburg's proof is unnecessarily complicated. It is sufficient to apply a theorem of Mazurkiewicz-Banach [3] stating that the set of functions from $C_{\langle a,b\rangle}$, for which $L_1^1(x,t) = \infty$ everywhere is residual and Th. 7 or simply the theorem on differentiability almost everywhere of a function of bounded variation.

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