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CONSTRUCTING AND RECONSTRUCTING OF ALGEBRAS

Dedicated to the memory of Professor Roman Sikorski

0. Introduction

In this paper we are concerned with a general method of building new algebras from a family of algebras of the same type indexed by a set with a semilattice structure and reconstructing algebras by means of the defined construction.

Let $(S,\cdot)$ be a (meet) semilattice and Ω a set of operation symbols having arity at least two. Then $(S,\cdot)$ may be considered as an Ω-algebra on setting

$$x_1 \dots x_n \omega := x_1 \cdot \dots \cdot x_n \tag{0.1}$$

for each n-ary ω in Ω. Such an algebra is called an Ω-semilattice. Conversely, given an Ω-semilattice (S,Ω) one may define a binary operation $\cdot$ on S by

$$x \cdot y := xy \dots y\,\omega \tag{0.2}$$

for each ω in Ω. The equation (0.1) will then hold. Thus the variety SL of all semilattices can be considered as the variety of Ω-algebras. Recall that an identity is regular if exactly the same variables appear on both sides of it. (See [17]). It is well known that SL satisfies exactly all regular identities between Ω-words. A variety V of Ω-algebras is called regular if each identity satisfied in $\underline{V}$ is regular. Otherwise, $\underline{V}$ is called irregular. Then $\underline{V}$ contains the variety $\underline{SL}$ of Ω-semilattices if and only if $\underline{V}$ is regular. Each algebra (A,Ω) in a regular variety $\underline{V}$ has a homomorphism, say h, onto an Ω-semilattice (S,Ω). In the case (A,Ω) is plural, i.e. (A,Ω) is idempotent and all operations of Ω have arity at least two, the fibres $A_s := h^{-1}(s)$ for each s in S are subalgebras and (A,Ω) is said to be a semilattice of algebras (A_s,Ω). In particular, (A,Ω) is an Ω-semilattice replica of its subalgebras. (See Malcev [14]). The natural question arises: Is there a general method of reconstructing (A,Ω) from its Ω-semilattice quotient and corresponding fibres? There is an example of such a construction for algebras in so called regularised varieties. The regularisation or regularised variety $\tilde{\underline{V}}$ of a variety $\underline{V}$ is defined to be the class of Ω-algebras satisfying the regular identities that are satisfied in $\underline{V}$. (See Płonka [17] and [19]). By Płonka's results [19] it is known that an algebra in the regularisation $\tilde{\underline{V}}$ of an irregular variety $\underline{V}$ of plural algebras is the Płonka sum of algebras in $\underline{V}$. However if $\underline{V}$ is an arbitrary regular variety then not every algebra in $\underline{V}$ can be reconstructed as a Płonka sum of its subalgebras.

Now consider more general situation. Given a set Ω of operation symbols having arity at least two, define a multisemilattice or Ω-multisemilattice to be an algebra (M,Ω) for which each reduct (M,ω) with ω in Ω is an $\{\omega\}$-semilattice. For each reduct (M,ω), (0.2) defines a binary semilattice operation. Denote this operation by $\cdot_\omega$. There is a corresponding semilattice order on M, denoted $\leq_\omega$. For a plural variety $\underline{V}$ of Ω-algebras let $\underline{MV}$ denote the variety of Ω-multisemilattices (M,Ω) lying in $\underline{V}$. The variety $\underline{MV}$ is non-trivial if and only if $\underline{V}$ is multiregular, i.e. each identity satisfied by each $\underline{V}$-algebra and involving only one operation ω from Ω is regular. Note that multiregular variety is not necessarily regular. (Take for example the variety $\underline{DL}$ of distributive lattices). Let h be a homomorphism of a plural algebra (A,Ω)

onto a multisemilattice (M,Ω). Then the fibres $A_m := h^{-1}(m)$ for each m in M are subalgebras of (A,Ω). In this case (A,Ω) is said to be the multisemilattice (M,Ω) of algebras (A_m,Ω). In particular, each plural algebra (A,Ω) is a multisemilattice replica of its subalgebras, and a semilattice of Ω-algebras is a special case of a multisemilattice of Ω-algebras. Now we can ask as before whether there is a general method of reconstructing (A,Ω) from its Ω-multisemilattice quotient and corresponding fibres. Consider the following construction.

Let (M,Ω) be a multisemilattice. For each m in M, let an algebra (A_m,Ω) be given, and for each k-ary ω in Ω and each pair (m,n) in M^2 with $n \leqq_\omega m$, a mapping $\varphi^\omega_{m,n}: A_m \longrightarrow A_n$ satisfying

(0.3) $\varphi^\omega_{m,m}$ is the identity mapping on A_m.

Define an operation ω on the disjoint union A of A_m, $m \in M$, by

(0.4) $$a_1 \ldots a_k\omega := a_1\varphi^\omega_{m_1,m} \ldots a_k\varphi^\omega_{m_k,m},$$

where $a_i \in A_{m_i}$, $m_i \in M$, $m = m_1 \ldots m_k\omega$. The algebra (A,Ω) defined in this way is well defined Ω-algebra having M as the multisemilattice quotient and the subalgebras A_m, $m \in M$, as corresponding fibres. However this construction of a new algebra from the family of Ω-algebras indexed by a set with an Ω-multisemilattice structure is too general to reconstruct any Ω-multisemilattice of Ω-algebras from its multisemilattice quotient and corresponding fibres. It is not always possible to define mappings $\varphi^\omega_{m,n}: A_m \longrightarrow A_n$ for $m \geqq_\omega n$ such that the condition (0.4) holds. As we will see in section 2 this difficulty can be avoided by taking a mapping from A_m in some extension of A_n instead of a mapping from A_m in A_n, for $m \geqq_\omega n$. There is also one more difficulty. Each non-trivial multisemilattice (M,Ω) of algebras (A_m,Ω) must satisfy multiregular identities satisfied in all (A_m,Ω). To use the described construction to reconstructing a multisemilattice of algebras from its multisemilattice quotient and corresponding fibres one needs such a definition that secures this property for a construced algebra. One gets this by putting some additional condition on mappings $\varphi^{\omega}_{m,n}$ that generalises functoriality condition in the definition of Płonka sum. In particular, Płonka sums are (very) special case of the construction described here.

The paper is devided into two parts. In the first section one defines and investigates extensions of Ω-algebras generalising some ideas known from semigroup theory. The notion of sink (or trunk) (see [20]) plays a crucial role in this section. Using results of the first part one defines the main construction of multisemilattice sum of algebras in section two. One investigates its properties and some special cases. The main theorem of this section reads that every multisemilattice of Ω-algebras is a multisemilattice sum. Examples of applications of the construction of the multisemilattice sum in theory of semirings, bisemilattices, modes and modals are then given.

As for basic facts concerning algebras the reader is referred to [1]. The notation and terminology is similar to that in [1]. An Ω-algebra is denoted by (A,Ω) or briefly A when there is no danger of confusion.

Finally note that the paper contains an expanded version of some material that will appear in the book [27]. Some problems of the paper were discussed with J.D.H. Smith and the final form of some definitions was proposed by him.

1. Sinks and sink extensions

A subset T of an Ω-algebra A is said to be a <u>sink</u> or <u>trunk</u> if for each n-ary ω in Ω, t in T and $a_1,\dots,a_{i-1},a_{i+1},\dots,a_n$ in A and $i = 1,\dots,n$, $a_1\dots a_{i-1}ta_{i+1}\dots a_n\omega \in T$. The notion of a sink was investigated by F.S. Poyatos [20]. For example, $\{0\}$ is a sink of the algebra $(R,\cdot)$ of real numbers under multiplication, and the set 2Z of even integers is a sink of the subalgebra $(Z,\cdot)$ of $(R,\cdot)$. The interior $I^0 = (0,1)$ of the unit interval $I = [0,1]$ is a sink of the algebra (I, I^0) with infinitely many operations $\underline{i}$, where $i \in I^0$, defined by $xy\underline{i} :=$ $:= (1-i)x + iy$. (See [27], [28], [29]). The empty set is a sink of an algebra (A,Ω) if and only if Ω does not contain any constant symbols. The set A is <u>improper</u> sink of an algebra (A,Ω), and all other sinks are called <u>proper</u>. An algebra (A,Ω) is called <u>impermeable</u> if it has no proper non-empty sinks. For example lattices, semigroups in irregular varieties and idempotent commutative medial quasigroups (see [9], [11]) are impermeable. In semigroup theory, as well as in less known theory of distributive groupoids sinks are called (two sided) ideals. (See [16], [8]). Sinks of (meet) semilattices are also called order ideals, initial segments, hereditary sets, down sets, lower ends and decreasing sets. Clearly every sink T of an algebra A is its subalgebra. The set $T(A)$ of all sinks of A is a lattice with respect to the set theoretical unions and intersections. Sometimes it is convenient to speak about sinks of an Ω'-reduct (A, Ω'), where $\Omega' \subset \Omega$, of an algebra (A,Ω). Such sinks are called Ω'-sinks and in the case $\Omega' = \{\omega\}$, simply ω-sinks.

In some cases the notion of a sink of an algebra A is closely related to the notion of a semilattice replica of A.

1.1. **Lemma.** Let $\underline{V}$ be a regular variety of plural Ω-algebras. Let A be an algebra in $\underline{V}$ and h a homomorphism of A onto its semilattice replica I. If for each i in I, the fibre $A_i = h^{-1}(i)$ satisfies an irregular identity, then a subset B of A is a sink of (A,Ω) if and only if $B = \cup (A_i \mid i \in J)$, where J is a sink of I.

Proof. Let J be a sink of I, and $B = \cup (A_i \mid i \in I)$. Let ω be n-ary operation in Ω and $a_{i_k} \in A_{i_k}$ for $k = 1,\dots,n$. Then $a_{i_1} \dots a_{i_n} \omega \in A_{i_1 \dots i_n \omega}$. Suppose $i_j \in J$ and $a_{i_j} \in A_{i_j}$. Then $a_{i_1} \dots a_{i_j} \dots a_{i_n} \omega \in A_{i_1 \dots i_j \dots i_n} \subseteq \cup (A_i \mid i \in J)$, since $i_1 \dots i_j \dots i_n \leq i_j$, whence $i_1 \dots i_j \dots i_n \in J$. Consequently, $B = \cup (A_i \mid i \in J)$ is a sink.

Now suppose B is a sink of A. If $a_i \in A_i \cap B$ then $A_i \subseteq B$. Indeed, since A_i satisfies an irregular identity, it follows that there is a binary word $w(x,y)$ with $w(x,y) = x$. Hence for any $b_i \in A_i$, $b_i = \bar{w}(b_i, a_i)$, which implies that $b_i \in B$. Consequently, $B = \bigcup (A_i \mid i \in J)$ for some $J \subseteq I$, and it is easy to show that J is a sink of I.

1.2. **Proposition.** Let $\underline{V}$ be a regular variety of plural Ω-algebras and A an algebra in $\underline{V}$ with semilattice replica I and fibres A_i, $i \in I$, satisfying an irregular identity. Then the lattice $T(A)$ of all sinks of A is isomorphic to the lattice $T(I)$ of all sinks of I.

For a in an Ω-algebra A, denote by T_a the least sink containing a. Then evidently the relation θ_T defined by $(a,b) \in \theta_T$ if and only if $T_a = T_b$ is an equivalence relation (See [20]). If the conditions of Proposition 1.2 are satisfied, then the relation θ_T coincides with the least congruence of A having as a quotient an Ω-semilattice. This holds in particular for idempotent commutative medial groupoids from [9] and [11], and meet-distributive bisemilattices form [22].

Let T be a sink of an Ω-algebra A. The relation τ_T defined on A by $(a,b) \in \tau_T$ if and only if $a,b \in T$ or $a = b$ is a congruence relation of A. It is called the <u>Rees congruence</u>, similarly as in semigroup theory. The quotient algebra A/τ_T is denoted by A/T, and called <u>Rees quotient</u>.

An element 0_ω of an Ω-algebra A is an <u>ω-zero</u> if $\{0_\omega\}$ is an ω-sink of (A,ω). An element 0 of A is a <u>zero</u> if the set $\{0\}$ is a sink of (A,Ω). Note that the quotient A/T is isomorphic to the Ω-algebra $((A-T) \cup \{0\}, \Omega)$, where each n-ary ω in Ω is defined by

$$a_1 \ldots a_n \omega := \begin{cases} a_1 \ldots a_n \omega & \text{if } \ a_i,\ a_1 \ldots a_n \omega \notin T, \\ 0 & \text{otherwise.} \end{cases}$$

Since T is a sink, the τ_T-class T acts as the zero of A/T. For example the set I^o is a sink of the algebra (I, I^o), and the quotient I/I^o is the free semilattice $\{0,1\}SL$ on two generators 0 and 1.

Let A be an Ω-algebra. If T is a sink of A then A is said to be a sink extension of T by A/T, or briefly an extension of T. In particular, each algebra A is an extension of itself. An extension A of T is proper if $A \neq T$. For example, the algebra (I, I^o) is an extension of (I^o, I^o) by the semilattice $\{0,1\}$ SL. The notion of a sink extension was firstly introduced in semigroup theory under the name of ideal extension. (See [16]). The aim of this section is to generalize some ideas concerning extensions of semigroups to the case of Ω-algebras.

First note that the set of all identities that are satisfied in a proper extension E of an Ω-algebra A by an Ω-algebra Q with zero is contained in the set of identities satisfied in both A and Q. The next lemma shows that identities satisfied in E must be regular in many cases.

1.3. Lemma. Let E be a proper extension of an Ω-algebra A by an Ω-algebra Q with zero. If Q is an idempotent algebra then none of irregular identities satisfied in A and Q is satisfied in E.

Proof. Let $w(x_1,\dots,x_n) = w'(y_1,\dots,y_k)$ be an irregular identity satisfied in A and Q. Assume that $x_i \notin \{y_1,\dots,y_k\}$. Let $a \in E-A$ and $b \in A$. Substitute a for all x_j, $j \neq i$, and for all y_l, $l = 1,\dots,k$. Substitute b for x_i. Then $a\dots aba\dots a\bar{w} \in A$ since A is a sink of E, and $a\dots a\bar{w}' = a$ since Q is idempotent, whence $a\dots aba\dots a\bar{w} \neq a\dots a\bar{w}'$.

The following Definition 1.4 and Theorem 1.8 below give a method of constructing new extensions of an Ω-algebra A by an Ω-algebra Q with zero, from an arbitrary extension of A. Since the algebra A is its own extension, the theorem gives a general method for constructing extensions of arbitrary algebras.

1.4. Definition. Let (D,Ω) be an extension of (A,Ω). Let (Q,Ω) be an algebra with zero. Denote $Q^* :=$ $:= Q - \{0\}$. Then a partial homomorphism of Q into D is a mapping $\varphi: Q^* \longrightarrow D$ such that

$$a_1 \ldots a_n \omega\varphi = a_1\varphi \ldots a_n\varphi\omega \text{ if } a_i, a_1 \ldots a_n\omega \in Q^*,$$

$$\text{and } a_1\varphi \ldots a_n\varphi\omega \in A \text{ if } a_1 \ldots a_n\omega = 0 \text{ in } Q.$$

Given a partial homomorphism φ of Q into D, the <u>extension</u> of A by Q <u>induced</u> <u>from</u> φ is the algebra (E,Ω) with underlying set $E = Q^* \cup A$ and with operations ω in Ω defined by setting

$$(1.5) \quad a_1 \ldots a_n\omega := \begin{cases} a_1 \ldots a_n\omega \text{ in } Q & \text{if } a_i, a_1 \ldots a_n\omega \in Q^*, \\ a_1\psi \ldots a_n\psi\omega \text{ in } D & \text{otherwise}, \end{cases}$$

where

$$a\psi := \begin{cases} a\varphi & \text{for } a \in Q^*, \\ a & \text{for } a \in A. \end{cases}$$

1.6. Lemma. For each Ω-word w of arity n and arbitrary $a_1, \ldots, a_n$ in E in Definition 1.4

$$(1.7) \quad a_1 \ldots a_n\overline{w} = \begin{cases} a_1 \ldots a_n\overline{w} & \text{in } Q \text{ if } a_1, \ldots, a_n \in Q^* \text{ and for each subword } t \text{ of } w \text{ acting on } a_{i_1}, \ldots, a_{i_k} \ (k \leq n),\ a_{i_1} \ldots a_{i_k}\overline{t} \in Q^*, \\ a_1\psi \ldots a_n\psi\overline{w} & \text{otherwise}. \end{cases}$$

In particular, $a_1 \ldots a_n\overline{w} = a_1\varphi \ldots a_n\varphi\overline{w}$ for $a_1, \ldots, a_n \in Q^*$ with $a_1 \ldots a_n\overline{w} = 0$ in Q.

Proof. The proof goes by induction on the number of Ω-operations constituting w, the result holding by the definition of Ω on E, if this number is 1. Otherwise suppose $a_1 \ldots a_n\overline{w} = a_1 \ldots a_i(a_{i+1} \ldots a_{i+m}\omega)a_{i+m+1} \ldots a_n\overline{w}$ for m-ary $(m \leq n)$ operation ω and n-m+1-ary derived operation $\overline{w}$. Recall that $a_{i+1} \ldots a_{i+m}\omega$ is defined by (1.5). Hence if there is $j \in \{1, \ldots, i\} \cup \{i+m+1, \ldots, n\}$ with $a_j \in A$, then evidently by induction hypothesis (1.7) holds. Now suppose, for each j,

$a_j \in Q^*$. If there is k such that $a_{i+k} \in A$ for $k = 1,\dots,m$, or if all $a_{i+k} \in Q^*$ and $a_{i+1}\dots a_{i+m}\omega = 0$ in Q for $k = 1,\dots,m$, then by (1.5) $a_{i+1}\dots a_{i+m}\omega = a_{i+1}\psi\dots a_{i+m}\psi\omega \in A$, and by induction hypothesis (1.7) holds. Also in the last case, all $a_i \in Q^*$ and $a_{i+1}\dots a_{i+m}\omega \in Q^*$, (1.7) follows easily from the induction hypothesis. Finally, note that if all $a_i \in Q^*$ and for every k-ary ($k \leq n$) subword t of w acting on $a_{i_1},\dots,a_{i_k}$, $a_{i_1}\dots a_{i_k}\bar{t} \neq 0$, then also $a_1\dots a_n\bar{w} \neq 0$. Hence if $a_1\dots a_n\bar{w} = 0$, then there is a subword t of w and elements $a_{i_1},\dots,a_{i_k}$ in Q such that $a_{i_1}\dots a_{i_k}\bar{t} = 0$ in Q, whence $a_{i_1}\dots a_{i_k}\bar{t} = a_{i_1}\varphi\dots a_{i_k}\varphi\bar{t} \in A$, what implies the last statement.

1.8. **P r o p o s i t i o n .** If (Q,Ω) is a non-trivial plural algebra with zero, then all identities satisfied in an extension (E,Ω) of an algebra (A,Ω) as in Definition 1.4 are just all regular identities satisfied in both (D,Ω) and (Q,Ω).

P r o o f . By Lemma 1.3, each identity holding in a proper extension of A must be regular. Let

$$w(x_1,\dots,x_n) = w'(y_1,\dots,y_n) \tag{1.9}$$

be a (regular) identity satisfied in D and in Q. Evidently $\{x_1,\dots,x_n\} = \{y_1,\dots,y_n\}$. Let $a_1,\dots,a_n$, $b_1,\dots,b_n \in E$, $\{a_1,\dots,a_n\} = \{b_1,\dots,b_n\}$. Substitute a_i for x_i and b_j for y_j in (1.9) and consider three cases.

(i) All $a_i \in Q^*$, and hence all $b_j \in Q^*$, and for each subword t of w acting on $a_{i_1},\dots,a_{i_k}$, $a_{i_1}\dots a_{i_k}\bar{t} \in Q^*$, and similarly for each subword s of w' acting on $b_{j_1},\dots,b_{j_l}$, $b_{j_1}\dots b_{j_l}\bar{s} \in Q^*$. Then evidently $a_1\dots a_n\bar{w}$ in E equals $a_1\dots a_n\bar{w}$ in Q and $a_1\dots a_n\bar{w} = b_1\dots b_n\bar{w}'$.

(ii) All $a_i \in Q^*$, and hence all $b_j \in Q^*$, but there is a subword t of say w acting on $a_{i_1},\dots,a_{i_k}$ with $a_{i_1}\dots a_{i_k}\bar{t} = 0$. Then in Q, $a_1\dots a_n\bar{w} = 0 = b_1\dots b_n\bar{w}'$, and by Lemma 1.6,

$a_1 \ldots a_m\bar{w}$ in E equals $a_1\varphi \ldots a_n\varphi\bar{w}$ and $b_1 \ldots b_n\bar{w}'$, equals $b_1\varphi \ldots b_n\varphi\bar{w}'$. Since all a_i, b_j are in D and (1.9) holds in D, it follows that $a_1 \ldots a_n\bar{w} = b_1 \ldots b_n\bar{w}'$ in E.

(iii) If for certain i, $a_i \in A$, then by Lemma 1.6, $a_1 \ldots a_n\bar{w}$ in E equals $a_1\psi \ldots a_n\psi\bar{w}$ and moreover each $a_i \in D$. Since $a_i \in \{b_1, \ldots, b_k\}$, also $b_1 \ldots b_n\bar{w}'$ in E equals $b_1\psi \ldots b_n\psi\bar{w}'$. Since (1.9) holds in D, it follows that also in this case $a_1 \ldots a_n\bar{w} = b_1 \ldots b_n\bar{w}'$ in E.

A congruence θ on an algebra (A,Ω) with sink (T,Ω) is said to preserve (T,Ω) if its restriction to T is the equality relation. An extension (E,Ω) of an algebra (A,Ω) is called an envelope of (A,Ω) if equality relation is the only congruence on (E,Ω) preserving (A,Ω). In semigroup theory envelopes are called dense extensions. It is easy to see that each algebra (A,Ω) is an envelope of itself and the extension in the example above is an envelope. For a convex polytop A in an Euclidean space, the algebra (A,I^o) is an envelope of $(\mathrm{Int}\ A, I^o)$. On the other hand, the semilattice $(\{1,2,3\},\vee)$ is not an envelope of $\{3\}$.

Now if (E,Ω) is an extension of an algebra (A,Ω) and is a congruence of (E,Ω) preserving (A,Ω) then the quotient E/θ is also an extension of (A,Ω).

1.10. L e m m a . Let (E,Ω) be an extension of an algebra (A,Ω) and θ a congruence of (E,Ω) preserving (A,Ω). Then the extension $(E/\theta,\Omega)$ of (A,Ω) is an envelope if and only if θ is a maximal congruence of (E,Ω) preserving (A,Ω).

For each extension (E,Ω) of an algebra (A,Ω) there is a homomorphism onto an envelope of (A,Ω). In this sense, envelopes can be considered as minimal proper extensions of (A,Ω). The next theorem shows that each extension of (A,Ω) can be constructed from an envelope of (A,Ω).

1.11. T h e o r e m . Each extension (E,Ω) of an algebra (A,Ω) is induced from a partial homomorphism φ of $Q = (E-A)\cup\{0\}$ into an envelope (D,Ω) of (A,Ω) with $Q^*\varphi = D-A$.

P r o o f . Let X be the set of congruences on (E,Ω) preserving (A,Ω). The set X is non-empty, since it contains the equality relation. By Kuratowski-Zorn's Lemma, X has a maximal element τ. Take $D = E/\tau$ and $Q = (E-A) \cup \{0\}$. (Note that (Q,Ω) is isomorphic to the Rees quotient $(E/A,\Omega)$). Let $n_\tau : E \longrightarrow E/\tau$ be the natural homomorphism under τ. Take φ to be the restriction of n_τ to $E-A = Q^*$. If $a_1,\dots,a_n \in Q^*$, ω is n-ary operation in Ω, and $a_1\dots a_n\omega \neq 0$ in Q, then $a_1\varphi\dots a_n\varphi\omega = a_1n_\tau\dots a_nn_\tau\omega = a_1\dots a_n\omega n_\tau = a_1\dots a_n\omega\varphi$. If $a_1\dots a_n\omega = 0$ in Q, then $a_1\varphi\dots a_n\varphi\,\omega = a_1n_\tau\dots a_nn_\tau\omega = a_1\dots a_n\omega n_\tau = a_1\dots a_n\omega \in A$, as required. Evidently $D = E/\tau = A \cup Q^*\varphi$, and D is an envelope of (A,Ω) by Lemma 1.10. Now one needs to show that the conditions (1.5) are satisfied. Let ω be n-ary operation in Ω. Let $\{1,\dots,n\} = \{i_1,\dots,i_k\} \cup \{j_{k+1},\dots,j_n\}$ both last sets being non-empty and $a_{i_r} \in A$, $a_{j_s} \in Q^*$. Then $a_1\dots a_n\omega = a_1\dots a_n\omega n_\tau = a_1n_\tau\dots a_nn_\tau\omega = a_1\psi\dots a_n\psi\omega$, where $a_{i_r}\psi = a_{i_r}$ and $a_{j_s}\psi = a_{j_s}\varphi$. If $a_i \in Q^*$ for $i = 1,\dots,n$, and $a_1\dots a_n\omega = 0$ in Q, then in a similar way, $a_1\dots a_n\omega = a_1n_\tau\dots a_nn_\tau\omega = a_1\varphi\dots a_n\varphi\omega$. The last case can be verified analogously. It follows that the envelope D has required properties.

Note that not each regular identity satisfied in an Ω-algebra A must be also satisfied in an envelope D of A. To show this consider the following example. Let $S_1 = \{1\}$ be one element normal band and $S_2 = \{a,b\}$ be a left-zero band. Let $D := S_1 \cup S_2$ and moreover $a1 = 1a = a$ and $b1 = 1b = b$. The algebra $(D,\cdot)$ defined in this way is evidently a semigroup and it is easy to check that D is an envelope of S_2. However the identity $xyz = xzy$ is not satisfied in D since $1ab = a \neq b = 1ba$.

1.12. P r o p o s i t i o n . Let (E,Ω) be an extension of an algebra (A,Ω) by an Ω-algebra Q with zero constructed in Definition 1.4 from an envelope D of A. Then E is a subdirect product of D and Q.

P r o o f . As in the proof of Theorem 1.11, $D = E/\tau$. Let $n_\tau : E \to E/\tau$ and $\mu : E \to E/A$ be the natural homomorphism. Define $\gamma : E \to D \times \mathbb{C}$ by $e \mapsto (en_\tau, e\mu)$. Then evidently γ is a homomorphism. Now let $a,b \in E$. If $a\gamma = b\gamma$, then $an_\tau = bn_\tau$ implies $a = b$; if $a,b \in E-A$, then $a\mu = b\mu$ implies $a = b$ and finally if one of elements a,b, say a, is in A, and the second one, say b, is in $E-A$, then $a\mu = 0$ and $b\mu \neq 0$, which is impossible. Thus γ is bijective, and since both n_τ and μ are surjective, $E\gamma$ is a subdirect product of D and $\mathbb{C}$.

2. Constructing algebras in multiregular varieties

2.1. D e f i n i t i o n . Let (M,Ω) be an Ω-multisemilattice. For each m in M, let an algebra (A_m,Ω) be given, and for each ω in Ω, an extension (E^ω_m,ω) of (A_m,ω). For each k-ary operation ω in Ω and each pair (m,n) in M^2 with $n \leqq_\omega m$, let $\psi^\omega_{m,n} : A_m \to E^\omega_n$ be a mapping satisfying

(a) $\psi^\omega_{m,m}$ is the embedding of A_m into E^ω_m;

(b) for $m_1,\dots,m_k \in M$ and $m_1 \dots m_k\omega = m$

$$(A_{m_1}\psi^\omega_{m_1,m})\dots(A_{m_k}\psi^\omega_{m_k,m})\omega \subseteq A_n;$$

(c) for any $n \leqq_\omega m_1 \dots m_k\omega = m$ and a_i in A_{m_i} for $i = 1,\dots,k$

$$a_1\psi^\omega_{m_1,m}\dots a_k\psi^\omega_{m_k,m}\omega\psi^\omega_{m,n} = a_1\psi^\omega_{m_1,n}\dots a_k\psi^\omega_{m_k,n}\omega.$$

Define an Ω-algebra structure on the disjoint union A of the underlying sets A_m, m in M, by

$$(2.2)\quad \omega : A_{m_1} \times \dots \times A_{m_k} \to A_m,\ a_1 \dots a_k\omega := a_1\psi^\omega_{m_1,m}\dots a_k\psi^\omega_{m_k,m}\omega,$$

where $m = m_1 \dots m_k\omega$. The algebra (A,Ω) is said to be the <u>sum of algebras A_m over the multisemilattice M by the mappings</u> $\psi^\omega_{m,n}$, or more briefly the <u>multisemilattice sum</u>.

Note that by 2.1(b), $a_1\psi^{\omega}_{m_1,m}\ldots a_k\psi^{\omega}_{m_k,m}\omega \in A_m$, whence the left hand side of the equality 2.1(c) is well-defined. Further, note that if each E^{ω}_m is just A_m then 2.1(c) is equivalent to the commuting of the following diagram

$$\begin{array}{ccccc} A_n^k & \xleftarrow{\psi^{\omega}_{m_1,n}\times\cdots\times\psi^{\omega}_{m_k,n}} & A_{m_1}\times\cdots\times A_{m_k} & \xrightarrow{\psi^{\omega}_{m_1,m}\times\cdots\times\psi^{\omega}_{m_k,m}} & A_m^k \\ \downarrow\omega & & & & \downarrow\omega \\ A_n & \xleftarrow{\qquad\qquad} & \psi^{\omega}_{m,n} & \xrightarrow{\qquad} & A_m \end{array}$$

Once the Ω-algebra structure on (A,Ω) has been defined, the condition 2.1(c) may be rewritten more simply as

(2.3) $$a_1\ldots a_k\omega\psi^{\omega}_{m,n} = a_1\psi^{\omega}_{m_1,n}\ldots a_k\psi^{\omega}_{m_k,n}\omega.$$

For $n \leq_{\omega} m$, the mappings $\psi^{\omega}_{m,n}$ are ω-homomorphisms.

To prove this, put $m_1 = \ldots = m_k = m$ in (2.3). By 2.1(a), the algebras (A_m,Ω) are subalgebras of (A,Ω). Further, (A,Ω) is a multisemilattice M of algebras (A_m,Ω). Note that Płonka sums are special cases of multisemilattice sums in which (M,Ω) is an Ω-semilattice and each of the E^{ω}_m for ω in Ω is just A_m. Now let Ω be a disjoint sum of Ω_1 and Ω_2, and let $x\cdot_1 y := x\cdot_{\omega_1} y = x\cdot_{\omega_1'} y$ for ω_1, ω_1' in Ω_1 and $x\cdot_2 y := x\cdot_{\omega_2} y = x\cdot_{\omega_2'} y$ for ω_2, ω_2' in Ω_2. If $(M,\cdot_1,\cdot_2)$ is a lattice, $E^{\omega}_m = A_m$ for each m in M, and (A,Ω_1) and (A,Ω_2) are Płonka sums of (A_m,Ω_1) and (A_m,Ω_2) respectively then one obtains another special case of multisemilattice sum introduced and investigated under the name of double sum by E. Graczyńska [4], [5].

Two of the most important cases of multisemilattice sums are given in the following definitions.

2.4. D e f i n i t i o n . A multisemilattice sum is said to be a Lallement sum if the following two conditions are satisfied for each m in M and ω in Ω:

(a) $$E_m^\omega = \{a\psi^\omega_{n,m} \mid m \leq_\omega n, \quad a \in A_n\};$$

(b) (E_m^ω, ω) is an envelope of (A_m, ω).

A Lallement sum is said to be strict if $E_m^\omega = A_m$ for each $\omega \in \Omega$ and m in M.

Lallement sums were first introduced in semigroup theory. (See [13] and [16]). See also related construction of bisemilattices in [22].

. The significanse of Lallement sums is explained in the following theorem.

2.5. Theorem. Every multisemilattice of algebras is a Lallement sum.

Proof. Let (A,Ω) be an Ω-multisemilattice (M,Ω) of algebras (A_m,Ω). For each m in M and ω in Ω, set $A_m^\omega = \bigcup_{m \leq_\omega n} A_n$. Then (A_m^ω,ω) is a subalgebra of (A,ω) and an extension of (A_m,ω) by the Rees quotient $(A_m^\omega/A_m,\omega)$ with singleton sink $(\{A_m\},\omega)$. By Theorem 1.11, this extension is induced from a partial homomorphism φ_m^ω of $(A_m^\omega/A_m,\omega)$ into an envelope (E_m^ω,ω) of (A_m,ω), with $((A_m^\omega/A_m) - \{A_m\})\varphi_m^\omega = E_m^\omega - A_m$. Identifying each (A_n,ω) for $m <_\omega n$ with its isomorphic image in $(A_m^\omega/A_m,\omega)$ under the Rees homomorphism, define $\psi^\omega_{n,m}: A_n \longrightarrow E_m^\omega$ to be the restriction to A_n of the partial homomorphism $\varphi_m^\omega: ((A_m^\omega/A_m)-\{A_m\}) \longrightarrow E_m^\omega$. Define $\psi^\omega_{m,m}: A_m \longrightarrow A_m$ to be the identity mapping on A_m. Then the conditions 2.4(a), 2.4(b), and 2.1(a) are certainly satisfied.

Now let $m = m_1 \ldots m_k\omega$ in M and $a_i \in A_{m_i}$ for $i = 1,\ldots,k$. If $m_i = m$ for some i in $\{1,\ldots,k\}$, then by 1.5
$$a_1\psi^\omega_{m_1,m}\ldots a_k\psi^\omega_{m_k,m}\omega = a_1\varphi_m^\omega\ldots a_{i-1}\varphi_m^\omega a_i a_{i+1}\varphi_m^\omega\ldots a_k\varphi_m^\omega\omega \in A_{m_i} = A_m,$$
since (A_m,ω) is an ω-sink of (E_m^ω,ω). Otherwise, all a_i for $i = 1,\ldots,k$ belong to $A_m^\omega - A_m$ and $a_1\ldots a_k\omega \in A_m$. Hence, since

$a_1 \ldots a_k\omega$ is the zero of A_m^{ω}/A_m it follows that $a_1\psi^{\omega}_{m_1,m} \ldots a_k\psi^{\omega}_{m_k,m}\,\omega = a_1\varphi^{\omega}_m \ldots a_k\varphi^{\omega}_m\,\omega \in A_m$ what implies 2.1(b), and moreover $a_1 \ldots a_k\omega = a_1\psi^{\omega}_{m_1,m} \ldots a_k\psi^{\omega}_{m_k,m}\,\omega$. To verify 2.1(c), it then suffices to verify the equivalent (2.3). But, with $n <_{\omega} m$, one has $a_1 \ldots a_k\omega\psi^{\omega}_{m,n} = a_1 \ldots a_k\omega\varphi^{\omega}_n = a_1\varphi^{\omega}_n \ldots a_k\varphi^{\omega}_n\,\omega = a_1\psi^{\omega}_{m_1,n} \ldots a_k\psi^{\omega}_{m_k,n}\,\omega$, as required. Also, (2.3) with $n = m$ is is immediate.

2.6. **Corollary.** Every algebra in a multiregular variety of plural algebras is a Lallement sum of its indecomposable subalgebras over its multisemilattice replica.

Theorem 2.5 and Corollary 2.6 concern in particular semigroups, semirings, bisemilattices, modes and modals. Semirings are well known algebras. (See [15] for examples of semirings represented as Płonka sums). Bisemilattices, algebras with two semilattice structures, raises still more interest among algebraists. (See for instance [3], [18], [21] - [28]). One of the major and best known classes of bisemilattices is that of so-called dissemilattices or meet-distributive bisemilattices, in which one of semilattice operations distributes over the second. (See [22], [24], [27], [28] for the role of Lallement sums in the theory of dissemilattices). Dissemilattices play a big rôle in the theory of modes and modals ([27], [28], [29]). A mode is an idempotent and entropic algebra. An algebra (A,Ω) is entropic if for each n-ary ω in Ω the mapping $\omega:(A^n,\Omega) \to (A,\Omega)$ is an Ω-homomorphism. (One uses also different names as for example medial or abelian for this property). Perhaps the most interesting modes are CIM-groupoids (i.e. idempotent commutative medial groupoids) of Ježek and Kepka ([9], [10], [11], [12]) and barycentric algebras. From the description of the variety of CIM-groupoids in [12], Thm. 6 in [2] saying that each irregular CIM-groupoid is a quasigroups, and results of Płonka ([17] and [19]) it follows that each groupoid in a non-trivial regular subvariety of the variety of all CIM-groupoids is the Płonka sum of quasigroups.

Other CIM-groupoids are Lallement sums of irregular subgroupoids. In particular this answers the question of Ježek, Kepka [10] about a method of reconstructing a CIM-groupoid from its semilattice replica and corresponding classes. Barycentric algebras are homomorphic images of convex subsets of real affine spaces considered as algebras with infinitely many binary operations $\underline{r}$, where $r \in I^{0} = (0,1)$, defined by $xy\underline{r} := (1-r)x + ry$. (See [6], [7], [27], [30]). A modal or an idempotent entropic operator semilattice (see [29], [27], [28]) is an algebra $(M,+,\Omega)$ such that $(M,+)$ is a semilattice, (M,Ω) is a mode, and the operations Ω distribute over +. The name "modal" is intended both to refer to the relationship with modes and to suggest the analogy with "modules", which are also algebras $(M,+,R)$ in which a set R of operations distributes over the structure of $(M,+)$. Typical examples are modals of non-empty submodes of modes. Other examples are provided by real numbers with the operation of maximum and operations I^{0}, distributive lattices and more general dissemilattices, then semilattice-normal semirings (see [15]). Each modal in a variety $\underline{V}$ is a Lallement sum of indecomposable submodals over its dissemilattice replica. Modals in an irregular variety $\underline{V}$ are Lallement sums of submodals over (distributive) lattice replica. Further each modal in a regular variety $\underline{V}$ is a Lallement sum of submodals over its semilattice replica. If $\tilde{\underline{V}}$ is a regularised variety of an irregular variety $\underline{V}$ of plural algebras then each algebra in $\tilde{\underline{V}}$ is the Płonka sum of its subalgebras in $\underline{V}$ over its semilattice replica. (See [17], [19]). Algebras in regular varieties not being regularisation can be Płonka sums only in individual cases.

Another important special case arises from a need to correlate the various different Ω-reducts in a multisemilattice sum. This leads to the following definition.

2.7. D e f i n i t i o n . A multisemilattice or Lallement sum is said to be coherent if the following three conditions are satisfied:

(a) the multisemilattice (M,Ω) is an Ω-semilattice;

(b) for each m in M, there is an extension (E_m,Ω) of (A_m,Ω) such that $(E_m,\omega) = (E_m^{\omega},\omega)$ for each ω in Ω;

(c) for each element (n,m) of the relation $\leq.$ on the Ω-semilattice (M,Ω), there is a mapping $\varphi_{m,n}: A_m \longrightarrow E_n$ such that $\varphi_{m,n} = \varphi^{\omega}_{m,n}$ for each ω in Ω.

Note, in particular, that Płonka sums are both strict and coherent. In a coherent sum, the description of the basic operations ω given in Definition 2.1 may be extended to derived operations.

2.8. Proposition. Let (A,Ω) be a coherent sum over the Ω-semilattice (M,Ω). Then for each Ω-word $w(x_1,\ldots,x_n)$ and for elements a_i of A_{m_i}, with $m = m_1\ldots m_n\overline{w}$ in (M,Ω),

$$a_1\ldots a_n\overline{w} = a_1\varphi_{m_1,m}\ldots a_n\varphi_{m_n,m}\overline{w}$$

Proof. The proof goes by induction on the number of occurences of elements of Ω in the word w, the result holding by Definition 2.2 if this number is 0 or 1. Otherwise, suppose $a_1\ldots a_n\overline{w} = a_1\ldots a_i a_{i+1}\ldots a_{i+p}\omega a_{i+p+1}\ldots a_n\overline{v}$ for an p-ary operation ω in Ω and an Ω-word $v(x_1,\ldots,x_{n-p+1})$. Let $m' = m_{i+1}\ldots m_{i+p}\omega$, so that $m = m_1\ldots m_i m' m_{i+p+1}\ldots m_n\overline{v} \leq. m'$. Then by induction

$$a_1\ldots a_n\overline{w} = a_1\ldots a_i(a_{i+1}\varphi_{m_{i+1},m'}\ldots a_{i+p}\varphi_{m_{i+p},m'}\omega)a_{i+1}\ldots a_n\overline{v} =$$

$$= a_1\varphi_{m_1,m}\ldots a_i\varphi_{m_i,m}(a_{i+1}\varphi_{m_{i+1},m'}\ldots$$

$$\ldots a_{i+p}\varphi_{m_{i+p},m'}\omega)\varphi_{m',m}a_{i+p+1}\varphi_{m_{i+p+1},m}\ldots a_n\varphi_{m_n,m}\overline{v} =$$

$$= a_1\varphi_{m_1,m}\ldots a_i\varphi_{m_i,m}(a_{i+1}\varphi_{m_{i+1},m}\ldots$$

$$\ldots a_{i+p}\varphi_{m_{i+p},m}\omega)a_{i+p+1}\varphi_{m_{i+p+1},m}\ldots a_n\varphi_{m_n,m}\overline{v} = a_1\varphi_{m_1,m}\ldots a_n\varphi_{m_n,m}\overline{w}$$

as required, the ultimate equality holding by 2.1(c).

2.9. Proposition. Let (A,Ω) be a non-trivial multisemilattice sum of algebras (A_m,Ω), m in M. Then (A,Ω) satisfies, for each $\omega\in\Omega$, all regular identities involving only the operation ω and satisfied by each of the extensions E_m^ω of (A_m,ω). Irregular identities of this type are not satisfied in (A,Ω).

Proof. Evidently all identities satisfied in (A,ω), for each ω in Ω, must be regular. It will be shown that all such identities are satisfied in A. Let w and w' be Ω-words of arity k involving only one operation ω in Ω. Let $w(x_1,\dots,x_k) = w'(y_1,\dots,y_k)$ be an identity satisfied in all extensions E_m^ω of (A_m,ω). Substitute $a_i\in A_{m_i}$ for x_i, $i=1,\dots,k$ and $b_j\in A_{n_j}$ for y_j, $j = 1,\dots,k$. Since $\{x_1,\dots,x_k\} = \{y_1,\dots,y_k\}$, one has $\{m_1,\dots,m_k\} = \{n_1,\dots,n_k\}$, whence $m_1\dots m_k\bar{w} = m_1\cdot\dots\cdot m_k = n_1\cdot\dots\cdot n_k = n_1\dots n_k\bar{w} = m$, say. Then by Proposition 2.8 $a_1\dots a_k\bar{w} = a_1\psi^\omega_{m_1,m}\dots a_k\psi_{m_k,m}\bar{w}$ and $b_1\dots b_k\bar{w}' = b_1\psi^\omega_{n_1,m}\dots b_k\psi^\omega_{n_k,m}\bar{w}'$. Since $a_i\psi^\omega_{m_i,m}$, $b_j\psi^\omega_{n_j,m}\in E_m^\omega$, where the identity $w(x_1,\dots,x_k) = w'(y_1,\dots,y_k)$ holds, $a_1\psi^\omega_{m_1,m}\dots a_k\psi^\omega_{m_k,m}\bar{w} = b_1\psi^\omega_{n_1,m}\dots b_k\psi^\omega_{n_k,m}\bar{w}'$, whence $a_1\dots a_k\bar{w} = b_1\dots b_j\bar{w}'$, as required.

Note that an identity satisfied in all A_m, m in M, and involving at least two operations from Ω need not to be satisfied in the sum A of A_m, m in M. To show this consider the following example. Let $(\{0,1\},+,\cdot)$ be a lattice $(0<1)$. Let $A_0 = \{b,c\}$ be a lattice with $b<c$ and $A_1 = \{a\}$ be one element lattice. Let $D_0^\cdot = D_0^+ = A_0$ and $D_1^\cdot = D_1^+ = A_1$. Further, let $a\psi^\cdot_{1,0} = b$, $b\psi^+_{0,1} = c\psi^+_{0,1} = a$. Then the sum A of A_0 and A_1 over $\{0,1\}$ is evidently a bisemilattice. Both summands of the sum as well as the lattice of indices are distributive lattices. However neither distributive law nor the absorption laws are preserved in the sum. Indeed, $c+ac = c \neq b = c(a+c)$ and $c(a+c) = b \neq c$.

2.10. Proposition. A coherent sum (A,Ω) satisfies all the regular identities satisfied by each of the extensions (E_m,Ω).

Proof. Using Proposition 2.8, the proof is exactly analogous to the proof of Płonka's theorem [17] that Płonka sums satisfy regular identities satisfied by each of the fibres.

The following corollary generalises Płonka's theorem.

2.11. Corollary. The identities satisfied by a strict coherent sum over a non-trivial (multi-) semilattice are precisely the regular identities satisfied by each of the fibres.

Coherent multisemilattice sums can be described also by means of some subdirect products. Let (A,Ω) be an algebra and 0 an element not contained in A. Then $(A \cup \{0\},\Omega)$, where 0 acts as the zero, is also well defined Ω-algebra and satisfies the same regular identities as (A,Ω).

2.12. Corollary. Let (A,Ω) be a coherent sum of algebras (A_m,Ω) over a (multi-)semilattice M. Then (A,Ω) is a subdirect product of algebras (B_m,Ω), m in M, where $B_m = E_m$, if m is the zero of M, and $B_m = E_m \cup \{0_m\}$, where 0_m acts as the zero, otherwise.

Proof. Let $\alpha_m : A \longrightarrow B_m$ be defined by

$$a\alpha_m = \begin{cases} a\varphi_{k,m} & \text{if } a \in A_k,\ k \geqq m, \\ 0_m & \text{otherwise.} \end{cases}$$

Let ω be an n-ary operation from Ω and let $a_{k_1},\dots,a_{k_n} \in A_{k_i}$. Then $a_{k_1}\alpha_m \dots a_{k_n}\alpha_m\omega = a_{k_1}\varphi_{k_1,m}\dots a_{k_n}\varphi_{k_n,m}\omega = a_{k_1}\varphi_{k_1,k_1\dots k_n}\dots$

$\dots a_{k_n}\varphi_{k_n,k_1\dots k_n}\omega\varphi_{k_1\dots k_n,m} = a_{k_1}\dots a_{k_n}\omega\varphi_{k_1\dots k_n,m} =$

$= a_{k_1}\dots a_{k_n}\omega\alpha_m$ in the case $k_1\dots k_n \geq m$, and $a_{k_1}\alpha_m\dots a_{k_n}\alpha_m\omega =$

$= 0_m = a_{k_1}\dots a_{k_n}\omega\alpha_m$ otherwise. Hence α_m is an Ω-homomorphism.

Now it is easy to see that the mapping $\beta: A \to \Pi\{B_m | m \in M\}$ defined by $a \mapsto (a\alpha_m)_{m\in M}$ is an Ω-homomorphism. Moreover β is injective. Indeed, suppose $a\alpha_m = b\alpha_m$ for all m in M, and let $a \in A_k$, $b \in A_l$. Then $k \geq m$ if and only if $l \geq m$ for all m in M what implies $k = l$. But then for $k = l = m$, one obtains $a = a\alpha_m = b\alpha_m = b$. Consequently, β is an isomorphism of A onto subdirect product of B_m, $m \in M$.

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