DEMONSTRATIO MATHEMATICA
Vol. XVIII No 1 1985

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# COMMUTATORS IN ORTHOMODULAR LATTICES

*Dedicated to the memory of Professor Roman Sikorski*

## 1. Introduction

In this paper, the notion of a commutator of a subset of an orthomodular lattice (abbreviated: OML) is introduced. It is a generalization of the commutator introduced by Marsden [9] for a two-element subset, and by Beran [1] for a finite subset of an OML. Some properties of that commutator, were used by Bruns-Kalmbach [2], Beran [1] and Poguntke [11] to describe finitely generated OML-s. The notion of a commutator has been also used in [12] to formulate the necessary and sufficient condition for the existence of joint probability distributions of observables on a quantum logic, i.e. the event structure of a quantum mechanical system, which is supposed to be an orthomodular $\sigma$-lattice.

We shall study, in Section 3, the properties of commutators in L that is an OML L. We show that, for any subset M of L, the set $J(M) = \{a \in L(M) : a \leq \overline{\mathrm{com}}\, F,\ F$ is a finite subset of $M\}$ is a p-ideal in the sub-OML L(M) of L, generated by the set M. Moreover, J(M) is identical with the Marsden [9] p-ideal $[L(M), L(M)]^{o}$ in L(M), generated by all the commutators $\overline{\mathrm{com}}\, \{m,n\}$, $m,n \in L(M)$. The commutator $\overline{\mathrm{com}}\, M$ of the set M is the join of elements of J(M) provided it exists.

In Section 4, commutators in a $\sigma$-OML L are studied. It is shown that the set $J(M) = \{a \in L(M) : a \leq \bigvee_{i=1}^{\infty} \overline{\text{com}}\, F_i,\ F_i$ is a finite subset of M for any $i=1,2,\dots\}$ is a $\sigma$-p-ideal in L(M), which is identical with the $\sigma$-p-ideal $[L(M), L(M)]^{o}$ in L(M), generated by all the commutators $\overline{\text{com}}\{m,n\}$, $m,n \in L(M)$.

In Section 5, the properties of commutators with respect to the notion of partial compatibility, introduced in [12], are studied. It is shown that com M, if it exists, is the largest element in L, with respect to which the set M is partially compatible. It holds that $\overline{\text{com}}\, M = \overline{\text{com}}\, L(M)$, provided $\overline{\text{com}}\, M$ exists. It has been proved that the maximal subset $Q \subset L$, which is p.c. to some element $a \in L$, is a sub-OML (sub-$\sigma$-OML, complete sub-OML) of L, if L is an OML (a $\sigma$-OML, complete OML). It is also shown that if M is p.c.a ($a \in L$), then also $M^{cc}$ is p.c.a.

In Section 6, the results obtained in the preceding paragraphs, are applied to the commutators of observables. The commutator of the set $\{x_\alpha : \alpha \in A\}$ of observables on a $\sigma$-OML L is defined as com$(\bigcup\{R(x_\alpha) : \alpha \in A\})$, where R(x) is the range of the observable x. Some expressions for the commutator of observables are derived. Finally, it is shown that the joint distribution of a set $\{x_\alpha : \alpha \in A\}$ of observables exists in a state m if and only if $m(a) = 0$ for all $a \in J(M)$, where $M = \bigcup\{R(x_\alpha) : \alpha \in A\}$. This result is a generalization of the result obtained in [12].

## 2. Preliminaries

Let $(L, \leq, 0, 1, \perp)$ be an orthomodular lattice (see [8] for all the details), i.e. a lattice with the orthocomplementation $\perp : L \to L$ such that $(a^{\perp})^{\perp} = a$, $a \vee a^{\perp} = 1$, $a \leq b \Longrightarrow a^{\perp} \geq b^{\perp}$ for all $a,b \in L$. The orthomodularity property claims that $a \leq b \Longrightarrow b = a \vee (a^{\perp} \wedge b)$, $a,b \in L$.

The elements $a,b \in L$ are orthogonal ($a \perp b$) if $a \leq b^{\perp}$.

The elements $a,b \in L$ are compatible ($a \leftrightarrow b$) if there is a Boolean subalgebra of L containing them. The compatibility relation has the following properties: $a \leftrightarrow b$ iff $a = (a \wedge b) \vee (a \wedge b^{\perp})$, $a \leq b \Rightarrow a \leftrightarrow b$, $a \leftrightarrow b \Rightarrow a^{\perp} \leftrightarrow b$, $a \leftrightarrow b$ and $a \leftrightarrow c \Rightarrow \{a, b, c\}$ is a distributive triple, $a \leftrightarrow b_i$, $i = 1,2,\dots,n \Rightarrow a \leftrightarrow \bigvee_{i=1}^{n} b_i$ and $a \wedge \bigvee_{i=1}^{n} b_i = \bigvee_{i=1}^{n} (a \wedge b_i)$.

The centre $L^{c}$ of $L$ is the set $L^{c} = \{a \in L : a \leftrightarrow b$ for any $b \in L\}$. $L^{c}$ is a Boolean subalgebra of $L$.

If $M$ is any subset of $L$, the sub-OML $L(M)$ of $L$, generated by $M$ (i.e. the smallest sub-OML of $L$ containing $M$) is the set of all $p(m_1, m_2, \dots, m_n)$, where $p$ is a lattice polynomial, $m_1, m_2, \dots, m_n \in M \cup M^{\perp}$, and $M^{\perp} = \{m^{\perp} : m \in M\}$.

We shall write $a \leftrightarrow A$, $A \subset L$, if $a \leftrightarrow b$ for all $b \in A$. A subset $A$ of $L$ is compatible if $a \leftrightarrow b$ for any $a, b \in A$. A sub-OML $B$ of $L$ is a Boolean subalgebra iff $B$ is compatible. Any OML can be written as a set-union of (maximal) Boolean subalgebras.

The direct sum $L_1 \oplus L_2$ of OML-s $L_1$ and $L_2$ is the set of all $(a,b)$, $a \in L_1$, $b \in L_2$, with lattice operations and orthocomplementation defined "coordinatewise", i.e. $(a_1,b_1) \vee (a_2,b_2) =$ $(a,b)^{\perp} = (a^{\perp}, b^{\perp})$. If $c$ is an element in the centre $L^{c}$ of $L$, then $L$ is isomorphic to the direct sum of $\langle 0,c \rangle$ and $\langle 0,c^{\perp} \rangle$, i.e. $L \simeq \langle 0,c \rangle \oplus \langle 0,c^{\perp} \rangle$.

A subset $\mathcal{P}$ of $L$ is a p-ideal in L if

(i) $b_i \in \mathcal{P}$, $i=1,2,\dots n \Rightarrow \bigvee_{i=1}^{n} b_i \in \mathcal{P}$, $n \in N$.

(ii) $b \in \mathcal{P}$, $a \in L \Rightarrow (b \vee a^{\perp}) \wedge a \in \mathcal{P}$.

The map $\mu : L_1 \to L_2$, where $L_1$ and $L_2$ are OML-s, is a p-morphism if

(i) $\mu\left(\bigvee_{i=1}^{n} b_i\right) = \bigvee_{i=1}^{n} \mu(b_i)$

(ii) $b \perp c \Rightarrow \mu(b) \perp \mu(c)$

P r o p o s i t i o n 1. (See [10]). The image of an OML under a p-morphism $\mu$ is an OML and

(1) $$\mu(0) = 0$$

(2) $$\mu(b^{\perp}) = \mu(b)^{\perp} \wedge \mu(1)$$

(3) $$\mu\left(\bigwedge_{i=1}^{n} b_i\right) = \bigwedge_{i=1}^{n} \mu(b_i).$$

If $A \subset L$, the intersection of all p-ideals containing A is a p-ideal, it is the p-ideal generated by A.

P r o p o s i t i o n 2. (See [10]). If z is the maximal element of a p-ideal $\mathcal{P}$, then z belongs to the centre $L^c$ of L. Conversely, if z belongs to $L^c$, then the segment $\langle 0, z \rangle$ is a p-ideal in L.

We recall that, for any $a \in L$, the segment $\langle 0, a \rangle = \{b \in L : b \leq a\}$ is an OML with the relative orthocomplementation $b \mapsto b^{\perp} \wedge a$.

A congruence relation on L is an equivalence relation $\theta$ on L such that $x_1 \theta y_1$ and $x_2 \theta y_2$ imply $x_1 \vee x_2 \theta y_1 \vee y_2$, $x \theta y$ implies $x^{\perp} \theta y^{\perp}$ [8].

P r o p o s i t i o n 3 (See [5]). A reflexive binary relation $\theta$ on L is a congruence relation iff for all $x, y, z \in L$ we have:

(i) $$x \theta y \iff x \wedge y \,\theta\, x \vee y$$

(ii) $$x \leq y \leq z,\ x \theta y,\ y \theta z \Longrightarrow x \theta z$$

(iii) $$x \leq y,\ x \theta y \Longrightarrow x \wedge z \,\theta\, y \wedge z \quad (x \vee z \,\theta\, y \vee z)$$

(iv) $$x \theta y \Longrightarrow x^{\perp} \theta y^{\perp}.$$

The following statements can easily be checked using Proposition 3 (see [1], [8], [9]).

If $\theta$ is a congruence of L and D is a class of $\theta$, then D is a convex sublattice of L and $D^{\perp} = \{d^{\perp} : d \in D\}$ is also a class of $\theta$. Further, $J = [0]\theta$ is a p-ideal in L.

If $\theta$ is a congruence and $J = [0]\theta$, then $x\theta y$ iff $(x\vee y)\wedge(x^{\perp}\vee y^{\perp})\in J$.

If J is a p-ideal, the binary relation defined by $x\theta y$ iff $(x\vee y)\wedge(x^{\perp}\vee y^{\perp})\in J$ is a congruence in L and $[0]\theta = J$.

If $\theta$ is a congruence of L and $[0]\theta = J$, let L/J denote the set of all classes of $\theta$. L/J is an OML and the map $\phi : L \to L/J$, $\phi(a) = [a]\theta$, is a p-morphism. It is called the p-morphism induced by $\theta$.

The kernel $\mu^{-1}(0) = \{b\in L_1 : \mu(b) = 0\}$ of a p-morphism $\mu : L_1 \to L_2$ is a p-ideal in $L_1$.

P r o p o s i t i o n 4 [8]. There is a one-to-one correspondence between the p-ideals in L and the congruences of L.

3. Commutators in an OML

For the elements a, b of an OML L put

$$(1)\qquad [a,b] = (a\vee b)\wedge(a^{\perp}\vee b)\wedge(a\vee b^{\perp})\wedge(a^{\perp}\vee b^{\perp}).$$

The element $[a,b]$ has been introduced by Marsden [9]. It is called the commutator of $\{a,b\}$. It is easily seen that $[a,b] = 0$ iff $a \leftrightarrow b$.

For a finite subset $F = \{a_1,a_2,\dots,a_n\}$ of L let us set, following to Beran [1]:

$$(2)\qquad \overline{\mathrm{com}}\,F \equiv \overline{\mathrm{com}}\{a_1,a_2,\dots,a_n\} = \bigwedge_{d\in D^n}\left(a_1^{d_1}\vee\dots\vee a_n^{d_n}\right),$$

$$(3)\qquad \underline{\mathrm{com}}\,F \equiv \underline{\mathrm{com}}\{a_1,a_2,\dots,a_n\} = \bigvee_{d\in D^n}\left(a_1^{d_1}\wedge\dots\wedge a_n^{d_n}\right),$$

where $D = \{0,1\}$, $d = (d_1,d_2,\dots,d_n)$, $a^0 = a^{\perp}$, $a^1 = a$. We shall call $\overline{\mathrm{com}}\,F$ ($\underline{\mathrm{com}}\,F$) the upper (lower) commutator of the set F. Evidently, $[a,b] = \overline{\mathrm{com}}\{a,b\}$ and $\overline{\mathrm{com}}\,F = (\underline{\mathrm{com}}\,F)^{\perp}$. If $F_1$ and $F_2$ are finite subsets of L such that $F_1\subset F_2$, then it is easy to see that $\overline{\mathrm{com}}\,F_1\leqslant\overline{\mathrm{com}}\,F_2$, $\underline{\mathrm{com}}\,F_1\geqslant\underline{\mathrm{com}}\,F_2$. It is also easy to see that the following statements are equivalent: (i) F is a compatible set, (ii) $\underline{\text{com}}\ F = 0$, (iii) $\underline{\text{com}}\ F = 1$, (iv) $\overline{\text{com}}\{a,b\}=0$ ($\underline{\text{com}}\ \{a,b\} = 1$) for any $a,b \in F$.

Let M and N be any subsets of L. Let us denote by $[M,N]$ the p-ideal generated by all commutators $[m,n]$, $m \in M$, $n \in N$. It has been shown by Marsden [9] that

$$(4)\quad [L,L] = \{x \in L:\ x \leq \bigvee_{i=1}^{n} [a_i,b_i]:\ a_i,b_i \in L,\ \ i=1,2,\dots,n\}$$

T h e o r e m 1 [9]. Let L be an OML and let $J = [L,L]$. Then L/J is Boolean. Moreover, if I is a p-ideal for which L/I is Boolean, then $I \supset J$.

T h e o r e m 2. For any subset G of L, $[G,G] = [(L(G), L(G)]$.

P r o o f . From $G \subset L$ we obtain that $[G,G] \subset [L(G),L(G)]$. Put $J = [G,G]$. Let $\phi$ be the p-morphism induced by J. For any $x,y \in L$, $\phi[x,y] = [\phi(x), \phi(y)]$. As G generates L(G), $\phi(G)$ generates $\phi(L(G))$. We have $[\phi(g), \phi(h)] = 0$ for all $g,h \in G$, i.e. $\phi(h) \leftrightarrow \phi(g)$. This implies that $\phi(x) \leftrightarrow \phi(y)$ for all $x,y \in L(G)$, i.e. $[x,y] \in J$. Hence $J = [L(G),L(G)]$.

T h e o r e m 3. Let L be an OML generated by $G = \{g_1,g_2,\dots,g_n\}$. Let $c = \overline{\text{com}}\ G$. Then $[L,L] = \langle 0,c\rangle$.

P r o o f . For any $g,h \in G$, $[g,h] \leq c$, i.e. $[G,G] = [L,L] \subset \langle 0,c\rangle$. Put $J = [L,L]$ and let $\phi$ be the p-morphism induced by J. Then $\phi[g,h] = [\phi(g), \phi(h)] = 0$ for any $g,h \in G$ implies $\phi(c) = 0$, i.e. $c \in J = [L,L]$. Hence $[L,L] = \langle 0,c\rangle$.

As a consequence of Theorem 1 and Theorem 3 we obtain, once again, the following well-known statement, which has been proved in [2] and reproved in [1] and [11].

T h e o r e m 4. Let L, G, and c be as in Theorem 3. Then c is in the centre of L, the OML $\langle 0,c^\perp\rangle$ is Boolean, and the OML $\langle 0,c\rangle$ is tightly generated by the set $G \wedge c = \{g \wedge c : g \in G\}$ (i.e. $\underline{\text{com}}\ \{g_1 \wedge c,\dots,g_n \wedge c\} = 0$), and $L = \langle 0,c\rangle \oplus \langle 0,c^\perp\rangle$.

Now let $M \subset L$ be any subset of L. Let $M^c = \{b \in L : b \leftrightarrow a$ for any $a \in M\}$. It is known that $M^c$ is a sub-OML of L, and the map $N \mapsto N^c$ ($N \subset L$) has the following properties:

$$N_1 \subset N_2 \Longrightarrow N_2^c \subset N_1^c, \qquad N \subset N^{cc}, \qquad N^c = (N^{cc})^c = (N^c)^{cc}.$$

If $M$ is compatible, i.e. $M \subset M^c$, then $M^{cc} \subset M^c = (M^{cc})^c$, i.e. $M^{cc}$ is a Boolean subalgebra of $L$.

The following statement gives us a relation between $[M,M]$ and $[M^{cc}, M^{cc}]$.

**Theorem 5.** For any subset $M$ of $L$, if $[M,M] \subset \langle 0,c \rangle$ and $c \in L^c$, then $[M^{cc}, M^{cc}] \subset \langle 0,c \rangle$.

**Proof.** Put $[M \wedge c^{\perp}] = \{ b \in \langle 0, c^{\perp} \rangle : b \leftrightarrow M \wedge c^{\perp} \}$. We show that $M^c \wedge c^{\perp} = [M \wedge c^{\perp}]'$. The proof is exactly the same as the proof of Theorem 13 (see § 5). From this we obtain that $M^{cc} \wedge c^{\perp} = [M \wedge c^{\perp}]''$. As $[m,n] \in \langle 0,c \rangle$ for all $m,n \in M$, $[m \wedge c^{\perp}, n \wedge c^{\perp}] = [m,n] \wedge c^{\perp} = 0$, so that $M \wedge c^{\perp}$, hence $[M \wedge c^{\perp}]''$ is compatible and this in turn implies that $M^{cc} \wedge c^{\perp}$ is compatible. From this it follows that $[m \wedge c^{\perp}, n \wedge c^{\perp}] = [m,n] \wedge c^{\perp} = 0$ for all $m,n \in M^{cc}$, i.e. $[M^{cc}, M^{cc}] \subset \langle 0,c \rangle$.

**Lemma 1.** Let $M \subset L$ be any subset, let $y_i = p_i(m_1^{\,i}, m_2^{\,i}, \dots, m_{n_i}^{\,i})$, $i = 1,2,\dots,k$, where $p_i$ are lattice polynomials and $m_1^{\,i}, m_2^{\,i}, \dots, m_{n_i}^{\,i} \in M \cup M^{\perp}$, $i = 1,2,\dots,k$. Then

$$\overline{\mathrm{com}} \{ y_1, y_2, \dots, y_k \} \leq \overline{\mathrm{com}} \left( \bigcup_{i=1}^{k} \{ m_1^{\,i}, m_2^{\,i}, \dots, m_{n_i}^{\,i} \} \right).$$

**Proof.** Put $\mathcal{P}_1 = M \cup M^{\perp}$, and for $n > 1$, $\mathcal{P}_n = \{ a \vee b, a \wedge b : a,b \in \mathcal{P}_{n-1} \}$. Then $L(M) = \bigcup_{i=1}^{\infty} \mathcal{P}_i$.

If $p_1, \dots, p_k \in \mathcal{P}_1$, the statement of Lemma 1 holds.

Now for any $x_i, y_i \in L$, $i = 1,2,\dots,k$,

$$\overline{\mathrm{com}} \{ (x_1 \vee y_1), (x_2 \vee y_2), \dots, (x_k \vee y_k) \} =$$

$$= \bigwedge_{d \in D^k} \left( (x_1 \vee y_1)^{d_1} \vee (x_2 \ \ y_2)^{d_2} \vee \dots \vee (x_k \vee y_k)^{d_k} \right)$$

and for any $d = (d_1, d_2, \dots, d_k) \in D^k$, $D = \{0,1\}$, we have that

$$(x_1 \vee y_1)^{d_1} \vee (x_2 \vee y_2)^{d_2} \vee \dots \vee (x_k \vee y_k)^{d_k} \leq$$

$$\leq \left(x_1^{d_1} \vee y_1^{\varepsilon_1} \vee x_2^{d_2} \vee y_2^{\varepsilon_2} \vee \dots \vee x_k^{d_k} \vee y_k^{\varepsilon_k}\right) \wedge$$

$$\wedge \left(x_1^{\varepsilon_1} \vee y_1^{d_1} \vee x_2^{\varepsilon_2} \vee y_2^{d_2} \vee \dots \vee x_k^{\varepsilon_k} \vee y_k^{d_k}\right),$$

where $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k \in \{0,1\}$ and $\varepsilon_i = d_i$ if $d_i = 1$, $\varepsilon_i$ is arbitrary if $d_i = 0$.

Now it is only a technical routine to prove that

$$\overline{\text{com}}\{(x_1 \vee y_1), (x_2 \vee y_2), \dots, (x_k \vee y_k)\} \leq \overline{\text{com}}\{x_1, y_1, \dots, x_k, y_k\}.$$

The remain of the proof follows by the induction.

T h e o r e m 6. For a subset $M$ of an OML $L$ put $J(M) = \{a \in L(M) : a \leq \overline{\text{com}}\, F,\ F \text{ is a finite subset of } M\}$. Then $J(M)$ is a p-ideal in $L(M)$.

P r o o f . From the properties of the commutator it follows that $J(M) = J(M \cup M^{\perp})$. Let $x_i \in J(M)$, $i = 1, 2, \dots, n$. Then

$$\bigvee_{i=1}^{n} x_i \leq \bigvee_{i=1}^{n} \overline{\text{com}}\, F_i \leq \overline{\text{com}}\left(\bigcup_{i=1}^{n} F_i\right),$$

so that $\bigvee_{i=1}^{n} x_i \in J(M)$. Now let $x \in J(M)$, $y \in L(M)$. Then $y = p(m_1, m_2, \dots, m_n)$ with $m_1, m_2, \dots, m_n \in M \cup M^{\perp}$. We have

$$(\overline{\text{com}}\, F \vee y^{\perp}) \wedge y \leq \overline{\text{com}}\, F \vee [\overline{\text{com}}\, F, y] \quad (\text{by } [9]) \leq$$

$$\leq \overline{\text{com}}\, F \vee \overline{\text{com}}\, (F \cup \{y\}) \quad (\text{by Lemma 1}) =$$

$$= \text{com}(F \cup \{y\}) \leq \text{com}(F \cup \{m_1, m_2, \dots, m_n\}) \quad (\text{by Lemma 1}),$$

so that $(x \vee y^{\perp}) \wedge y \in J(M)$.

For a subset M of L, let us denote by $[M,M]^o$ the p-ideal generated by all the commutators $[a,b]$ $(a,b \in M)$ in the sub-OML L(M) of L generated by M. By Theorem 2, $[M,M]^o = [L(M),L(M)]^o$.

T h e o r e m 7. The p-ideal J(M) of Theorem 6 is equal to $[L(M),L(M)]^o$.

P r o o f . It is clear that $[a,b] \in J(M)$ for any $a,b \in M$. On the other hand, if $\Phi$ is the p-morphism induced by $[M,M]^o$, then $\Phi(\overline{com}\ F) = 0$, hence $\overline{com}\ F \in [M,M]^o$ for any finite subset F of M, i.e. $[M,M]^o = J(M)$.

D e f i n i t i o n 1. Let us put, for any $M \subset L$,

(5) $\overline{com}\ M = \bigvee \{\overline{com}\ F : F \text{ is a finite subset of } M\}$

(6) $\underline{com}\ M = \bigwedge \{\underline{com}\ F : F \text{ is a finite subset of } M\}$

if the elements on the right exist, and we shall call $\overline{com}\ M$ the upper commutator and $\underline{com}\ M$ the lower commutator of the set M.

The following definition has been introduced in [12].

D e f i n i t i o n 2. We say that a subset M of L is partially compatible with respect to an element a of L (abbreviated: M is p.c. a) if (i) $M \leftrightarrow a$, (ii) $M \wedge a = \{m \wedge a : m \in M\}$ is a compatible set.

C o r o l l a r y 1. If $\underline{com}\ M$ exists for a subset M of an OML L, then L(M) (and hence M) is p.c. $\underline{com}\ M$.

P r o o f . Let $b \in M$. We can write $\overline{com}\ M = \bigvee \{\overline{com}(F \cup \{b\}) : F \text{ is a finite subset of } M\}$. Then $b \leftrightarrow \overline{com}(F \cup \{b\})$ for all finite $F \subset M$ implies $b \leftrightarrow \overline{com}\ M$ ([8], p.24). Now $M \leftrightarrow \overline{com}\ M$ implies $L(M) \leftrightarrow \overline{com}\ M$. Let $a,b \in L(M)$, then $[a,b] \leq \overline{com}\ M$ (Theorem 7). Hence $[a,b] \wedge \underline{com}\ M = [a \wedge \underline{com}\ M,\ b \wedge \underline{com}\ M] = 0$, i.e. $a \wedge \underline{com}\ M \leftrightarrow b \wedge \underline{com}\ M$. This implies that L(M) is p.c. $\underline{com}\ M$.

R e m a r k 1. The element $\overline{com}\ M$ may not exist. If $\overline{com}\ M$ exists in L(M), then it belongs to J(M), and $\langle 0, \underline{com}\ M \rangle$ is the maximal Boolean factor of L(M). In [11] the following example is given. Consider the sub-OML M of the OML $D_2^{\omega} \times MO_2^{\omega}$

where $D_2$ is the two-element lattice and $MO_2$ is the OML on figure 1, containing the elements $((a_n)_{n<\omega},(b_m)_{m<\omega})$ for which $\{n: a_n \neq 0\} \cup \{m : b_m \neq 0\}$ or $\{n: a_n \neq 1\} \cup \{m : b_m \neq 1\}$ is finite. Then M does not have the largest Boolean factor, i.e. $\overline{\text{com}}$ M does not exist in M.

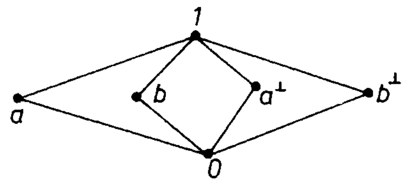


Fig.1

4. Commutators in $\sigma$-OML

The notions of a p-ideal and p-morphism can be generalized to the $\sigma$-p-ideal, $\sigma$-p-morphism and also to the complete c-p-ideal, c-p-morphism in an obvious way (see [10]). If L is an orthomodular $\sigma$-lattice (complete lattice), its centre $L^C$ is a Boolean $\sigma$-algebra (complete algebra), the set $M^C$ is a sub-$\sigma$-OML (complete sub-OML) for any subset M of L. Moreover, $b \leftrightarrow a_\alpha,\ \alpha \in A \Rightarrow b \leftrightarrow \bigvee\{a_\alpha : \alpha \in A\}$ and $b \wedge \left(\bigvee a_\alpha\right) = \bigvee (b \wedge a_\alpha)$, where A is countable (any) set of indexes, if $\{a_\alpha : \alpha \in A\}$ and b are elements of a $\sigma$-OML (a complete OML). The generalization of Theorem 1, in which the notion of a p-ideal is substituted by a $\sigma$-p-ideal (c-p-ideal) in a $\sigma$-OML (complete OML) also holds. In this section, we shall treat the case of a $\sigma$-OML.

L e m m a 2. Let A be a subset of L. Put $A_0 = M \cup M^{\perp}$, let $\Omega$ be the least uncountable ordinal, and $A_\alpha = \left\{ \bigvee_{i=1}^{n} a_i,\ \bigwedge_{i=1}^{n} a_i : (a_i)_{i=1}^{n} \subset \bigcup_{\beta<\alpha} A_\beta,\ n < \omega, \right\}$ where $\alpha < \Omega$. Then $\bigcup\{A_\alpha : \alpha < \Omega\}$ is the sub-$\sigma$-OML L(M) of L generated by M.

P r o o f . Clearly, $\bigcup\{A_\alpha : \alpha < \Omega\}$ is a sub-$\sigma$-OML of L. On the other hand, $\bigcup\{A_\alpha : \alpha < \Omega\} \subset L(M)$.

For a countable set $S = \{s_1, s_2, \dots\} \subset L$ with at most countable number of elements, the element com S exists, since the set of all finite subsets of an at most countable set is at most countable.

L e m m a 3. Let $M \subset L$, and let $\{y_i\}_{i=1}^{\infty}$ be the polynomials over $M \cup M^{\perp}$, $y_i = p_i(m_1^{\,i}, m_2^{\,i}, \dots)$, $i = 1, 2, \dots$ . Then

$$\overline{\mathrm{com}}\{y_1, y_2, \dots\} \leqslant \overline{\mathrm{com}}\Big(\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} m_j^{\,i}\Big). \tag{7}$$

P r o o f . As we shall see below (Corollary 5 in § 5), $\overline{\mathrm{com}}\, M = \overline{\mathrm{com}}\, L(M)$, provided $\overline{\mathrm{com}}\, M$ exists. Now as the $\sigma$-sub-OML of L generated by the set $\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} m_j^{\,i}$ contains all the polynomials $y_1, y_2, \dots$, we see that (7) holds.

T h e o r e m 8. Let M be a subset of $\sigma$-OML L. Put

$$J(M) = \Big\{ a \in L(M) : a \leqslant \bigvee_{i=1}^{\infty} \overline{\mathrm{com}}\, F_i, \text{ where } F_i \text{ is a finite subset of } M \text{ for any } i = 1, 2, \dots \Big\}. \tag{8}$$

Then J(M) is a $\sigma$-p-ideal in L(M).

P r o o f . It is enough to prove that for $y \in L(M)$, $(x \vee y^{\perp}) \wedge y \in J(M)$ for $x \in J(M)$. Let $x \leqslant \bigvee_{i=1}^{\infty} \overline{\mathrm{com}}\, F_i$, and $y = p(m_1, m_2, \dots)$ with $m_1, m_2, \dots \in M \cup M^{\perp}$. Then

$$\Big(\bigvee_{i=1}^{\infty} \overline{\mathrm{com}}\, F_i \vee y^{\perp}\Big) \wedge y = \bigvee_{i=1}^{\infty} (\overline{\mathrm{com}}\, F_i \vee y^{\perp}) \wedge y \leqslant \bigvee_{i=1}^{\infty} (\overline{\mathrm{com}}\, F_i \vee [\overline{\mathrm{com}}\, F_i, y])$$

$$(\text{by } [9]) \leqslant \bigvee_{i=1}^{\infty} \overline{\mathrm{com}}(F_i \cup (m_1, m_2, \dots\ )\ (\text{Lemma } 3) = \bigvee_{i=1}^{\infty} \bigvee_{j=1}^{\infty} \overline{\mathrm{com}}\, G_j^{\,i},$$

where $G_j^{\,i}$ is a finite subset of $F_i \cup \{m_1, m_2, \dots\} \subset M \cup M^{\perp}$.

Let us denote, as before, by $[M, M]^{\circ}$ the $\sigma$-p-ideal generated by all commutators $[m, n]$, $m, n \in M$, in L(M).

T h e o r e m 9. For any subset M of a $\sigma$-OML L, $J(M) = [L(M), L(M)]^{\circ}$.

P r o o f . As $[g,h] \in J(M)$ for all $g,h \in M$, we get $J(M) \supset [L(M), L(M)]^o$. Put $J = [L(M), L(M)]^o$. As $L(M)/J$ is Boolean, we have that $\phi(\overline{\text{com}}\ F) = \overline{\text{com}}(\phi(F)) = 0$ for any finite $F \subset M$, where $\phi$ is the p-morphism induced by J. Hence $\overline{\text{com}}\ F \in J$ for any $F \subset M$, i.e. $J(M) \subset J$.

## 5. Commutators and partial compatibility

By Corollary 1, if the element com M for a subset M of an OML exists, then M is p.c. com M. The following theorem shows that com M is the maximal element with respect to which the set M is partially compatible.

T h e o r e m 10. Let M be a subset of an OML L such that com M exists and let M be p.c. a for some $a \in L$. Then $a \leq \text{com}\ M$.

P r o o f . We recall that the subset $M \wedge a = \{m \wedge a : m \in M\}$ is compatible in L iff it is compatible in $\langle 0,a \rangle$. We can suppose that $a \neq 0$. Let $F = \{a_1, a_2, \dots, a_n\}$ be any finite subset of M. As F is p.c. a, the set $\{a_1 \wedge a, a_2 \wedge a, \dots, a_n \wedge n\}$ is compatible in $\langle 0,a \rangle$, so that

$$\bigvee_{d \in D^n} (a_1 \wedge a)^{d'_1} \wedge (a_2 \wedge a)^{d'_2} \wedge \dots \wedge (a_n \wedge a)^{d_n} = a,$$

where $(a_i \wedge a)^{d'_i} = \begin{cases} a_i \wedge a & \text{if } d'_i = 1 \\ a_i^{\perp} \wedge a & \text{if } d'_i = 0 \end{cases}$.

Hence $\bigvee_{d \in D^n} a_1^{d_1} \wedge \dots \wedge a_n^{d_n} \wedge a = a$, and because $a \leftrightarrow a_1^{d_1} \wedge \dots \wedge a_n^{d_n}$ for all $d \in D^n$, we have $\left( \bigvee_{d \in D^n} a_1^{d_1} \wedge \dots \wedge a_n^{d_n} \right) \wedge a = \text{com}\ F \wedge a = a$. Thus $a \leq \text{com}\ F$ for all finite $F \subset M$, hence $a \leq \text{com}\ M$.

R e m a r k 2. Takeuti [13] introduced the following element: $\perp\!\!\!\perp(M) = \bigvee \{x \in L : \forall m,n \in M,\ x \leftrightarrow m \text{ and } x \wedge m \leftrightarrow x \wedge n\}$. It can be shown that the elements $\perp\!\!\!\perp(M)$ and com M exist simultaneously and if they exist, they are identical [4].

The following theorem gives us a characterization of partial compatibility (see also [3]).

T h e o r e m 11. Let M be a subset of an OML L. Let com M exist. Then M is p.c. a iff $M \multimap a$ and $a \leq \underline{com}\, M$.

P r o o f . If M is p.c. a then $a \leq \underline{com}\, M$ by Theorem 10. Let $M \multimap a$ and $a \multimap \underline{com}\, M$. Let $b, c \in M$. Then $b \wedge \underline{com}\, M \multimap c \wedge \underline{com}\, M$, $b \wedge \underline{com}\, M \multimap a$, $c \wedge \underline{com}\, M \multimap a$ imply that $b \wedge a \wedge \underline{com}\, M \multimap c \wedge a \wedge \underline{com}\, M$. If $a \leq \underline{com}\, M$, we obtain that $b \wedge a \multimap c \wedge a$, i.e. M is p.c. a.

C o r o l l a r y 2. Let M be a subset of an ($\sigma$-complete, complete) OML L. Then if M is p.c. $a_\alpha$, $\alpha \in A$, then M is p.c. $\vee a_\alpha$ ($\wedge a_\alpha$), where A is a finite (countable, any) index set.

We shall say that the subset M of an OML L such that M is p.c. a, is maximal, if for $b \in L$, $M \cup \{b\}$ p.c. a implies $b \in M$. Similarly as in [12], we obtain the following result.

T h e o r e m 12. Let M be a maximal subset of a (complete, $\sigma$-complete) OML L, which is p.c. a for some $a \in L$. Then M is a (complete, $\sigma$-complete) sub-OML of L.

C o r o l l a r y 3. Let Q be a maximal subset of an OML L which is p.c. a ($a \neq 0$). Then $Q \wedge a$ is a maximal Boolean subalgebra of $\langle 0, a \rangle$.

T h e o r e m 13. Let M be a subset of an OML L and let $a \in M^c$, where $M^c = \{b \in L : b \multimap M\}$. Put $[M \wedge a]' = \{b \in \langle 0, a \rangle : b \multimap M \wedge a\}$. Then $M^c \wedge a = [M \wedge a]'$.

P r o o f . If $b \in M^c$ then also $b \wedge a \in M^c$. This implies that $b \wedge a \multimap M \wedge a$, i.e. $b \wedge a \in [M \wedge a]'$. Hence $M^c \wedge a \subset [M \wedge a]'$. To show the converse, let $b \in [M \wedge a]'$. If $m \in M$, then $m = (m \wedge a) \vee (m \wedge a^\perp)$, and $b = b \wedge a \multimap m \wedge a$. Moreover, $b \leq a \leq a \wedge m^\perp = (a^\perp \wedge m)^\perp$. This implies that $b \multimap a^\perp \wedge m$, and hence $b \multimap m$, i.e. $b \in M^c \wedge a$.

The following statement is a generalization of Theorem 3.7 in [12].

C o r o l l a r y 4. Let $M \subset L$ be p.c. a. Then $M^{cc}$ is also p.c. a.

P r o o f . By Theorem 13, $[M \wedge a]'' \supseteq M^{cc} \wedge a$. As $M \wedge a$ is a compatible subset of $\langle 0,a \rangle$, $[M \wedge a]''$ is a Boolean subalgebra of $\langle 0,a \rangle$. From this it follows that $M^{cc} \wedge a$ is a compatible subset of $\langle 0,a \rangle$, i.e. $M^{cc}$ is p.c. a.

C o r o l l a r y 5. For any subset $M \subset L$, com M = = com L(M), where L(M) is the sub-OML (sub-σ-OML, complete sub-OML) of an OML (σ-OML, complete OML) L, generated by M, provided com M exists.

P r o o f . As $L(M) \subset M^{cc}$, by Corollary 4 L(M) is p.c. com M. This implies by Theorem 10, that com $M \leq$ com F for any finite $F \subset L(M)$. Let $b \in L$ be such that $b \leq$ com F for all finite $F \subset L(M)$. Then, especially, $b \leq$ com G for all finite $G \subset M$, i.e. $b \leq$ com M.

R e m a r k 3. It would be interesting to study the properties of the family of all subsets M of L which are p.c. a for a fixed element $a \in L$. We know the following

$$N \subset M,\ M \text{ p.c. } a \Longrightarrow N \text{ p.c. } a,$$

M p.c. a, N p.c. a, and $M \wedge a \longleftrightarrow N \wedge a \Longrightarrow M \cup N$ p.c. a,

M p.c. a $\Longrightarrow M^{cc}$ p.c. a,

M p.c. a $\Longrightarrow$ there is a maximal subset

Q(M) of L which is p.c. a and contains M (by the Zorn lemma). Moreover, Q(M) is a sub-OML of L. The investigation of further properties of this family of sets exceedes the framework of this paper.

## 6. Commutators of observables

An observable on σ-complete OML L is a σ-homomorphism from the Borel subsets B(R) of the real line R to L. That is, an observable x is the map $x : B(R) \longrightarrow L$ such that: (i) $x(R) = 1$, (ii) $E \cap F = \emptyset$ implies $x(E) \perp x(F)$, (iii) $x(\bigcup E_i) = \bigvee x(E_i)$ for any sequence $\{E_i\} \subset B(R)$.

The spectrum $\sigma(x)$ of an observable x is the smallest closed subset C of R such that $x(C) = 1$. An observable x is said to have a pure point spectrum if $\sigma(x) = \{t_1, t_2, \dots\}$. Let $R(x) = \{x(E) : E \in B(R)\}$ be the range of the observable x. The observables $\{x_\alpha, \alpha \in A\}$ are compatible if $\cup\{R(x_\alpha) : \alpha \in A\}$ is a compatible subset of L.

We put

$$(9) \qquad \text{com}\{x_\alpha : \alpha \in A\} = \text{com}(\cup\{R(x_\alpha) : \alpha \in A\})$$

and we shall call com $\{x_\alpha : \alpha \in A\}$ the commutator of the observables $\{x_\alpha : \alpha \in A\}$, if it exists. It has been shown in [14], that the commutator exists for any at most countable set of observables.

A state m on a $\sigma$-OML L is the map $m : L \to \langle 0,1\rangle$ such that: (i) $m(1) = 1$, (ii) $m\left(\bigvee_{i=1}^{\infty} a_i\right) = \sum_{i=1}^{\infty} m(a_i)$ for any sequence $\{a_i\}$ of mutually orthogonal elements of L.

We say that the observables $x_1, x_2, \dots, x_n$ have a joint distribution in the state m if there is a measure $\mu$ on $B(R^n)$ such that

$$(10) \quad \mu(E_1 \times E_2 \times \dots \times E_n) = m(x_1(E_1) \wedge x_2(E_2) \wedge \dots \wedge x_n(E_n))$$

for any rectangle $E_1 \times E_2 \times \dots \times E_n \in B(R^n)$.

The notion of a joint distribution can be generalized as follows: we say that a set $\{x_\alpha : \alpha \in A\}$ of observables have a joint distribution in the state m if any finite subset of $\{x_\alpha : \alpha \in A\}$ has it.

It x is an observable and $a \in L$ is such that $a \leftrightarrow R(x)$, we denote by $x \wedge a$ the map $E \mapsto x(E) \wedge a$, $E \in B(R)$. It is easily seen that $x \wedge a$ is an observable on $\langle 0, a\rangle$.

We begin with the following observation.

T h e o r e m 14. Let $M_i$ be a finite subset of mutually orthogonal elements of an OML L for any $i \in \{1, \dots, n\}$. Then

$$(11) \quad \underline{\text{com}}\{M_1, M_2, \dots, M_n, Q\} = \wedge\{\underline{\text{com}}\{a_1, a_2, \dots, a_n, Q\} : (a_1, a_2, \dots, a_n) \in M_1 \times M_2 \times \dots \times M_n\},$$

where Q is any finite subset of L.

P r o o f . We shall proceed by induction on the cardinality of the sets $M_i$. First prove that

(12) $$\underline{\text{com}}\{M_1, b_2, \ldots, b_n\} = \bigwedge_{a \in M_1} \underline{\text{com}}\{a, b_2, \ldots, b_n\}$$

Let card $M_1 = 2$. Then

$$\underline{\text{com}}\{M_1, b_2, \ldots, b_n\} = \underline{\text{com}}\{a_1, a_2, b_2, \ldots, b_n\}.$$

Let us consider the set

$$A := \{a_1 \wedge b^d,\ a_2 \wedge b^d,\ a_1^{\perp} \wedge b^d,\ a_2^{\perp} \wedge b^d : d \in D^{n-1}\},$$

where we put $b^d = b_2^{d_2} \wedge \ldots \wedge b_n^{d_n}$. The set $A$ is composed of four-element classes for different $d \in D^{n-1}$. Any two elements of classes with different $d$-s are orthogonal. The elements in the same class are compatible, too, with the exception of $a_1^{\perp} \wedge b^d$ and $a_2^{\perp} \wedge b^d$. This implies that among any three elements of the set $A$ there is always one which is compatible with the other two, i.e. $A$ is a Foulis-Holland set. By [6], $A$ is distributive. Using the distributivity, we prove that

$$\underline{\text{com}}\{a_1, a_2, b_2, \ldots, b_n\} =$$
$$= \underline{\text{com}}\{a_1, b_2, \ldots, b_n\} \wedge \text{com}\{a_2, b_2, \ldots, b_n\}.$$

Now let us suppose that (12) holds for card $M_1 \leq k$. We prove it for $k+1$. We have

$$\underline{\text{com}}\{M_1, b_2, \ldots, b_n\} = \underline{\text{com}}\{a_1, a_2, \ldots, a_k, a_{k+1}, b_2, \ldots, b_n\} =$$
$$= \bigwedge_{i=1}^{k} \underline{\text{com}}\{a_i, a_{k+1}, b_2, \ldots, b_n\} =$$

$$= \bigwedge_{i=1}^{k} \underline{\mathrm{com}}\,\{a_i, b_2, \dots, b_n\} \wedge \underline{\mathrm{com}}\,\{a_{k+1}, b_2, \dots, b_n\} =$$

$$= \bigwedge_{i=1}^{k+1} \underline{\mathrm{com}}\,\{a_i, b_2, \dots, b_n\}$$

Further, let

$$\underline{\mathrm{com}}\,\{M_1, M_2, \dots, M_s, b_{s+1}, \dots, b_n\} =$$

$$= \bigwedge\{\underline{\mathrm{com}}\,\{a_1, \dots, a_s, b_{s+1}, \dots, b_n\} : (a_1, \dots, a_s) \in M_1 \times M_2 \times \dots \times M_s\}$$

hold for $s \leq j$. We shall prove it for $j+1$. If card $M_{j+1} = 1$, it is obvious. Let us suppose that it holds for card $M_{j+1} \leq k$. Let card $M_{j+1} = k+1$. Then

$$\underline{\mathrm{com}}\,\{M_1, \dots, M_j, M_{j+1}, Q\} = \underline{\mathrm{com}}\,\{M_1, M_2, \dots, M_j, c_1, c_2, \dots, c_k, c_{k+1}, Q\} =$$

$$= \bigwedge\{\underline{\mathrm{com}}\,\{a_1, a_2, \dots, a_j, c_i, c_{k+1}, Q\} : a_s \in M_s,\ s \in \{1, \dots, j\},\ i \in \{1, \dots, k\}\}.$$

By the first part of the proof we have

$$\underline{\mathrm{com}}\,\{a_1, a_2, \dots, a_j, c_i, c_{k+1}, Q\} =$$

$$= \underline{\mathrm{com}}\,\{a_1, \dots, a_j, c_i, Q\} \wedge \underline{\mathrm{com}}\,\{a_1, \dots, a_j, c_{k+1}, Q\},$$

which leads to

$$\underline{\mathrm{com}}\,\{M_1, M_2, \dots, M_j, M_{j+1}, Q\} =$$

$$= \bigwedge\{\underline{\mathrm{com}}\,\{a_1, \dots, a_j, a_{j+1}, Q\} : a_s \in M_s\}.$$

C o r o l l a r y 6. Let $x_1, \dots, x_n$ be observables on a complete OML L. Then

(13) $$\underline{\text{com}}\{x_1,\dots,x_n\} = \bigwedge_{(E_1,\dots,E_n)} \underline{\text{com}}\{x_1(E_1),\dots,x_n(E_n)\}$$

where the infimum is taken over all $E_1,\dots,E_n \in B(R)$.

P r o o f . Let us denote by b the right-hand side of (13). Observe that for mutually compatible elements $a_1,a_2,\dots,a_n$ we have

$$\underline{\text{com}}\{a_1,a_2,\dots,Q\} = \underline{\text{com}}\{\{a_1^{d_1}\wedge\dots\wedge a_n^{d_n} : d\in D^n\}, Q\} =$$

$$= \bigwedge_{d\in D^n} \text{com}\{a_1^{d_1}\wedge\dots\wedge a_n^{d_n}, Q\}$$

This implies that for any finite subset $F\subset \bigcup_{i=1}^n R(x_i)$, $b\le \underline{\text{com}}\, F$, which gives the desired result.

T h e o r e m 15. Let $x_1,\dots,x_n$ be observables on a $\sigma$-OML L with pure point spectra $\sigma(x_i) = \{t_1^i, t_2^i,\dots\}$, $i\in\{1,\dots,n\}$. Then

(14) $$\underline{\text{com}}\{x_1,\dots,x_n\} = \bigvee_{(i_1,\dots,i_n)\in N^n} x_1(\{t_{i_1}^1\})\wedge\dots\wedge x_n(\{t_{i_n}^n\})$$

where N is the set of all natural numbers.

P r o o f . Let us denote the right-hand side of (14) by b. Put $A := \{x_i(\{t_{j_i}^i\}) : j_i\in N,\ i\in\{1,\dots,n\}\}$. It is clear that the sub-$\sigma$-OML of L generated by A contains $\bigcup_{i=1}^n R(x_i)$. Corollary 5 then implies that $a := \underline{\text{com}}\, A = \underline{\text{com}}\{x_1,\dots,x_n\}$. It is easy to check that $b \leftrightarrow \bigcup_{i=1}^n R(x_i)$ and

$$b\wedge x_j(E) =$$

$$= \bigvee\{x_1(\{t_{i_1}^1\})\wedge\dots\wedge x_n(\{t_{i_n}^n\}) : (i_1,\dots,i_n)\in N^n,\ t_{i_j}^j\in E\}.$$

This means that $\bigcup_{i=1}^{n} R(x_i)$ is p.c. b. By Theorem 10 then $b \leq a$.
On the other hand,

$$a \wedge b = \bigvee_{(i_1,\dots,i_n)\in N^n} x_1\left(\left\{t_{i_1}^{1}\right\}\right) \wedge \dots \wedge x_n\left(\left\{t_{i_n}^{n}\right\}\right) \wedge a = a.$$

The last equality follows from the fact that $x_i \wedge a$, $i \in \{1,\dots,n\}$ are compatible observables on $\langle 0,a \rangle$ and $\sigma(x_i \wedge a) \subset \sigma(x_i)$. Hence $a = b$.

If the observables $x_1, x_2, \dots, x_n$ are p.c. a and a state m on L is such that $m(a) = 1$, then the observables $x_i \wedge a$, $i \in \{1,\dots,n\}$ are compatible on $\langle 0,a \rangle$. Moreover, for any $E \in B(R)$, $m(x_i(E)) = m[(x_i \wedge a)(E)]$. This means that, from the probabilistic point in view, the observables $x_1,\dots,x_n$ can be treated as compatible observables. It has been shown, that the joint probability distribution for $x_1,\dots,x_n$ exists exactly in such a case [12], [14]. If f is any Borel function, then $R(f(x)) \subset R(x)$ for any observable x, where $f(x) = x \circ f^{-1}$. This means that the set of all observables which are p.c. a is closed to the formations of functions. Now let $x_1,\dots,x_n$ be observables on the Hilbert space logic L(H) and let $A_1,\dots,A_n$ be the corresponding self-adjoint operators. Then $x_1,\dots,x_n$ p.c. P for some $P \in L(H)$ means that the subspace P reduces the operators $A_1,\dots,A_n$ and that the reduced operators $A_i/P$, $i \in \{1,\dots,n\}$ mutually commute. If the sum $A_1+A_2+\dots+A_n$ exists (i.e. is a self-adjoint operator), then also $A_1/P+\dots+A_n/P$ is a self-adjoint operator on P and $(A_1+\dots+A_n)/P = A_1/P+\dots+A_n/P$. From this it follows that the set of all observables which are p.c. P is closed also under the formations of sums. From Theorem 15 we see that the commutator of operators $A_1,\dots,A_n$ with pure point spectra is the closed subspace of H generated by all common eigenvectors (see also [7]).

Finally, we shall introduce a more general condition of the existence of joint distributions of observables. It can be applied also if the commutator does not exist and shows, that even in this case the observables can be considered, in the state $m$, as compatible observables.

T h e o r e m 16. The observables $\{x_\alpha : \alpha \in A\}$ have a joint distribution in the state $m$ iff $m(a) = 0$ for any $a \in J(M)$, where $M = \cup \{R(x_\alpha) : \alpha \in A\}$, and $J(M)$ is the $\sigma$-p-ideal of Theorem 8. In this case, the observables $\{[x_\alpha] : \alpha \in A\}$ on $L(M)/J(M)$ are compatible and $\tilde{m}([a]) = m(a)$, $a \in L(M)$, is a state on $L(M)/J(M)$.

P r o o f . I. Let the joint distribution in the state $m$ exist. From this it follows that for any countable subset $\{\alpha_1, \alpha_2, \dots\} \subset A$ we have $m(\overline{\mathrm{com}}\,\{x_{\alpha_1}, x_{\alpha_2}, \dots\}) = 0$. This implies that $m(\overline{\mathrm{com}}\, S) = 0$ for any countable subset $S \subset \cup \{R(x_\alpha) : \alpha \in A\}$. As $\bigvee_{i=1}^{\infty} \overline{\mathrm{com}}\, F_i \leq \overline{\mathrm{com}}\left(\bigcup_{i=1}^{\infty} F_i\right)$, we obtain that $m(a) = 0$ for any $a \in J(M)$.

II. Let $m(a) = 0$ for all $a \in J(M)$. Let $\Phi$ be the $\sigma$-p-morphism induced by $J(M)$ in $L(M)$. If $x$ is an observable on $L(M)$, then $[x](E) = \Phi(x(E))$, $E \in B(R)$, is an observable on $L(M)/J(M)$. As $L(M)/J(M)$ is Boolean, the observables $\{[x_\alpha], \alpha \in A\}$, are mutually compatible. The restriction of the state $m$ on $L(M)$ is a state on $L(M)$. Put $\tilde{m}(\Phi(a)) = m(a)$, $a \in L(M)$. We shall show that $\tilde{m}$ is a state on $L(M)/J(M)$. If $\Phi(a) = \Phi(b)$, then $(a \vee b) \wedge (a^\perp \vee b^\perp) \in J(M)$ (see § 2). This implies that $m((a \vee b) \wedge (a^\perp \vee b^\perp)) = m((a \vee b) \wedge (a \wedge b)^\perp) = 0$. From the orthomodularity we have that $a \vee b = (a \wedge b) \vee (a \vee b) \wedge (a \wedge b)^\perp$, which implies that $m(a \vee b) = m(a \wedge b)$, and hence $m(a) = m(b)$. Now $\tilde{m}(\Phi(1)) = m(1) = 1$, and if $\{\Phi(a_i)\}_i$ is an orthogonal sequence in $L(M)/J(M)$, put $b_1 = a_1$, $b_2 = a_2 \wedge a_1^\perp, \dots, b_n = a_n \wedge \left(\bigvee_{i=1}^{n-1} a_i\right)^\perp$. Then $\{b_i\}_i$ are mutually orthogonal, and $\Phi(b_i) = \Phi(a_i)$, $i = 1, 2, \dots$ . Then

$$\tilde{m}\left(\bigvee_{i=1}^{\infty} \Phi(a_i)\right) = \tilde{m}\left(\Phi\left(\bigvee_{i=1}^{\infty} b_i\right)\right) = \sum_{i=1}^{\infty} m(b_i) =$$

$$= \sum_{i=1}^{\infty} \tilde{m}(\phi(b_i)) = \sum_{i=1}^{\infty} \tilde{m}(\phi(a_i)).$$

As the observables $\{[x_\alpha], \alpha \in A\}$ are compatible, they have a joint distribution $\mu$ in the state $\tilde{m}$. Since

$$\tilde{m}([x_{\alpha_1}](E_1) \wedge \dots \wedge [x_{\alpha_n}](E_n)) =$$

$$= \tilde{m}\Big(\phi(x_{\alpha_1}(E_1) \wedge \dots \wedge x_{\alpha_n}(E_n))\Big) =$$

$$= m\Big(x_{\alpha_1}(E_1) \wedge \dots \wedge x_{\alpha_n}(E_n)\Big),$$

$\alpha_1, \dots, \alpha_n \in A$, $E_1, \dots, E_n \in B(R)$, $\mu$ is also the joint distribution of the observables $\{x_\alpha : \alpha \in A\}$ in the state $m$.

The author wishes to thank J. Hedlikova for valuable comments.

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Received July 2, 1984.