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ON PARTIALLY ORDERED SETS WITH SMALL INITIAL SEGMENTS

Dedicated to the memory of Professor Roman Sikorski

I. Introduction

We base our considerations on Zermelo-Fraenkel set theory with the axiom of choice. In particular: an ordinal coincides with the set of all its predecessors, a cardinal is an initial ordinal. Every well ordered set is isomorphic with an ordinal (ordered by $\in$) and every set X can be well ordered isomorphically to a certain cardinal (which is called the cardinality of X and denoted by $|X|$).

We denote cartesian product of the family $\{A_i : i \in I\}$ by $\prod_{i\in I} A_i$ and its cardinality by $\prod_{i\in I} |A_i|$. If $(\forall i \in I)\ |A_i| = m$ and $|I| = n$, then $\prod_{i\in I} |A_i|$ is denoted by m^n. It is easy to see that

$$(1) \qquad \text{if} \quad (\forall i \in I) m_i \leqslant m \quad \text{then} \quad \prod_{i\in I} m_i \leqslant m^{|I|}.$$

Let $\leqslant$ be a partial order of a set P and $x,y \in P$. We call x,y incomperable if $x \nleqslant y$ and $y \nleqslant x$. A set of mutually incomparable elements is called antichain. An m-antichain is an antichain of cardinality m. Every antichain is included in a maximal antichain.

An initial segment of P is a set of the form $\{y \in P : y \leq x\}$ and it is denoted by x). If $A \subseteq P$ then the set $\{y \in P : (Ex \in A) y \leq x\}$ is denoted by A).

A subset P_1 of P is called cofinal with P if

$$(\forall x \in P)(\exists y \in P_1)\ x \leq y$$

i.e. if $P = P_1$). Let m be a cardinal. The least cardinal n such that the set m (ordered by $\in$) contains a cofinal subset of cardinality n is denoted by cf(m). If cf(m) = m then m is called a regular cardinal. If $|I| < cf(m)$ and $(\forall i \in I) \cdot |X_i| < m$ then $|\bigcup_{i \in I} X_i| < m$. The reader who needs more basic or advanced informations on set theory and cardinals can find them in [1] or [2].

Let n(P) be the least cardinal n such that

$$(\forall x \in P) |x)| < n.$$

We call P a set with small initial segments (s.i.s.) if $n(P) < |P|$.

One can prove without difficulty that a linearly ordered set is not a set with s.i.s. It is easy to find examples of partially ordered set with s.i.s., e.g. antichains. The family P of all finite subsets of a certain uncountable set is another such example (P is ordered by inclusion, obviously $n(P) = \aleph_0$).

The main purpose of this paper is to prove that in many cases (and under some assumptions of set-theoretic nature - always) partially ordered set P with s.i.s. is "thick", i.e. it contains a $|P|$-antichain.

II. Introduction of the results and problems

T h e o r e m 1. If P is an infinite partially ordered set with s.i.s. and m is an arbitrary cardinal less than $|P|$ then P contains an m-antichain.

Theorem 2. If P is an infinite partially ordered set with s.i.s. and $cf(|P|) > n(P)$ then P contains a $|P|$-antichain.

Theorem 3. Let P be a partially ordered set with s.i.s., $cf(|P|) < |P|$ and P satisfies the following condition (+): $m^k < |P|$ for each cardinal $k < cf(|P|)$ and each cardinal $m < |P|$. Then P contains a $|P|$-antichain.

Corollary 1. If P is a partially ordered set with s.i.s. such that $cf(|P|) = \aleph_0$ then P contains a $|P|$-antichain.

Proof. If P is a countable set then the proposition is a consequence of Theorem 2. If P is uncountable then it satisfies trivially the assumptions of Theorem 3 (k is finite).

For some cardinals $|P|$ fulfilment of the condition (+) depends on certain additional axioms of the set theory. The following corollary is an easy example of such situation when each cardinal satisfies the desired condition.

Corollary 2. If the generalized continuum hypothesis holds then every infinite partially ordered set P with s.i.s. contains a $|P|$-antichain.

Proof. If P is a regular cardinal we can apply Theorem 2. Otherwise $cf(|P|) < |P|$ and $P = \aleph_\lambda$ for a limit ordinal λ. Thus for ordinals $\alpha, \beta < \lambda$

$$\aleph_\alpha^{\aleph_\beta} \leq (2^{\aleph_\alpha})^{\aleph_\beta} = 2^{\aleph_\alpha \cdot \aleph_\beta} = 2^{\aleph_\gamma} = \aleph_{\gamma+1} < \aleph_\lambda$$

where $\gamma = \max(\alpha, \beta)$. The condition (+) is satisfied and we can apply Theorem 3.

The natural problem on antichains arises in the sets which are beyond the reach of Theorems 2 and 3 i.e. such sets P with s.i.s. that there exist cardinals $m < |P|$ and $k < cf(|P|)$ and

$$\aleph_0 < cf(|P|) \leq n(P) < |P| \leq m^k.$$

This problem remains unsolved (at least by the author of this paper). In particular the following problem is open (it seems to be the simplest case of the above situation). Assume that $|P| = \aleph_{\omega_1} = 2^{\aleph_0}$ and that all initial segments of P are at most countable (i.e. $n(P) = \aleph_1$). Does P contain an $\aleph_{\omega_1}$-antichain? (by Theorem 1 it contains an antichain of an arbitrary cardinality less than $\aleph_{\omega_1}$).

III. Proofs

P r o o f of Theorems 1 and 2. Let us define two transfinite sequences of sets $(P_\alpha : \alpha < \lambda)$ and $(A_\alpha : \alpha < \lambda)$ where $P_\alpha \subseteq P$ and A_α is a maximal antichain in P_α

$P_0 = P$

$P_{\alpha+1} = P_\alpha - A_\alpha)$

$P_\beta = \bigcap_{\alpha<\beta} P_\alpha$ for a limit ordinal β.

Let λ be the first ordinal α such that $P_\alpha = \emptyset$. If one of the antichains A_α has cardinality $|P|$ then the conclusions of both the theorems are satisfied. Thus we can assume that $|A_\alpha| < |P|$ for all $\alpha < \lambda$.

Let us observe that the set $\bigcup_{\alpha<\lambda} A_\alpha$ is cofinal with P since if $a \in P$ then $a \in P_\alpha - P_{\alpha+1}$ for some $\alpha < \lambda$. Therefore $a \in A_\alpha)$ hence $a \leqslant b$ for some $b \in A_\alpha$. The above property can be expressed by equality

(2) $$P = \bigcup_{\alpha<\lambda} A_\alpha).$$

We shall now prove now that

(3) $$\lambda \leqslant n(P).$$

Let us assume that $n(P) < \lambda$. Thus $P_{n(P)} \neq \emptyset$. Let a be an arbitrary element of $P_{n(P)}$. $a \in P_\alpha$ for each $\alpha < n(P)$ and therefore it is comparable with some $a_\alpha \in A_\alpha$ (as A_α is a maximal antichain in P_α). But if $a \leqslant a_\alpha$ then $a \in A_\alpha)$ what implies $a \notin P_{\alpha+1}$. Thus $a_\alpha \leqslant a$ for each $\alpha \leqslant n(P)$. It is easy to see that

$a_\alpha \neq a_\beta$ for $\alpha, \beta < n(P)$, $\alpha \neq \beta$. Hence $|a)| \geq n(P)$ which contradicts the assumption.

Assume now (in spite of the conclusion of Theorem 1) that there exists a cardinal $m < |P|$ such that for every ordinal $\alpha < \lambda$

$$(4) \qquad |A_\alpha| < m.$$

Inequalities (3) and (4) imply

$$\Big|\bigcup_{\alpha<\lambda} A_\alpha\Big| \leq n(P)\cdot m < |P|$$

hence

$$\Big|\bigcup_{\alpha<\lambda} A_\alpha)\Big| < |P|$$

in spite of the equality (2).

For proving Theorem 2 it is sufficient to write equality (2) in the following form

$$P = \bigcup_{\alpha<\lambda} \bigcup_{x\in A_\alpha} x).$$

In $cf(|P|) > n(P)$, then, by (3), $\Big|\bigcup_{x\in A_{\alpha_0}} x)\Big| = |P|$ for a certain $\alpha_0 < \lambda$. It implies immediately that $|A_{\alpha_0}| = |P|$.

L e m m a 1. If the cardinal $|P|$ satisfies condition (+) then there exists a transfinite sequence $(n_\alpha : \alpha < cf(|P|))$ of cardinals having the following properties:

1. n_α is a regular cardinal
2. $n_\alpha < n_\beta$ if $\alpha < \beta < cf(|P|)$
3. $\sup\{n_\alpha : \alpha < cf(|P|)\} = |P|$
4. $\prod_{\xi<\alpha} n_\xi < n_\alpha$.

P r o o f . The required sequence is defined by transfinite induction in a routine way, with the help of the inequality

$$\prod_{\xi<\alpha} n_\xi \leqslant \left[\sup\{n_\xi : \xi < \alpha\}\right]^{|\alpha|}$$

which is obvious consequence of (1).

L e m m a 2. Let P satisfy the assumptions of theorem 3. There exists a transfinite sequence $(B_\alpha : \alpha < cf(|P|))$ of antichains which have the following properties:

1. $|B_\alpha|$ is a regular cardinal, $|B_\alpha| > n(P)$, $|B_\alpha| > cf(|P|)$
2. $|B_\alpha| < |B_\beta|$ if $\alpha < \beta < cf(|P|)$
3. $\sup\left\{|B_\alpha| : \alpha < cf(|P|)\right\} = |P|$
4. if $\alpha < \beta < cf(|P|)$, $a \in B_\alpha$ and $b \in B_\beta$ then $b \nleq a$
5. if $\alpha < cf(|P|)$ then $\left|\prod_{\xi<\alpha} B_\xi\right| < |B_\alpha|$.

P r o o f . Let $(n_\alpha : \alpha < cf(|P|))$ be a sequence of cardinals which satisfies the conditions 1 - 4 of lemma 1. We can assume that $n_\alpha > cf(|P|)$ and $n_\alpha > n(P)$ for each $\alpha < cf(|P|)$. Let $\left\{a_\xi : \xi < |P|\right\}$ be an enumeration of all elements of the set P (under some well ordering of P), let for $\alpha < cf(|P|)$

$$P_\alpha = \left\{a_\xi : \xi < n_\alpha\right\}$$

and

$$Q_\alpha = P_{\alpha+1}) - P_\alpha).$$

It is easy to see that $|P_\alpha)| = n_\alpha \cdot n(P)$ hence $|Q_\alpha| = n_{\alpha+1}$. Moreover Q_α is a set with s.i.s. therefore it satisfies the assumptions of Theorem 2. Thus Q_α contains an $n_{\alpha+1}$-antichain B_α. It is easy to verify that this sequence of antichains $(B_\alpha : \alpha < cf(|P|))$ has the required properties 1-5.

P r o o f of Theorem 3. Let $(B_\alpha : \alpha < cf(|P|))$ be a sequence of antichains which satisfies the conditions 1-5 of Lemma 2. We shall construct a double sequence $(C_{\alpha,\beta} : \alpha \leqslant \beta < cf(|P|))$ of antichains such that the following conditions are satisfied: let $\alpha, \beta, \gamma, \lambda$, be ordinals such that $\alpha \leqslant \beta < cf(|P|)$, $\alpha \leqslant \gamma < cf(|P|)$, $\alpha \leqslant \lambda < cf(|P|)$, λ is a limit ordinal. Then

1. $C_{\alpha,\beta} \subseteq B$,
2. if $\beta<\gamma$, then $C_{\alpha,\gamma} \subseteq C_{\alpha,\beta}$,
3. $|C_{\alpha,\alpha}| = |B_\alpha|$,
4. $|C_{\alpha,\beta} - C_{\alpha,\beta+1}| < |B_\alpha|$,
5. $\left|\left(\bigcap_{\alpha\leq\beta<\lambda} C_{\alpha,\beta}\right) - C_{\alpha,\lambda}\right| < |B_\alpha|$
6. $\bigcup_{\xi\leq\beta} C_{\xi,\beta}$ is an antichain.

Observe that the above conditions 1-5 and the condition 1 of lemma 2 imply for each $\beta>\alpha$ the equality

(5) $$|C_{\alpha,\beta}| = |B_\alpha|.$$

The construction will be based on transfinite induction on the ordinal variable β.

We put $C_{0,0} = B_0$. Let us assume now that the antichains $C_{\alpha,\gamma}$ with $\alpha\leq\gamma<\beta$ have already been defined and they satisfy conditions 1-5. One must define $C_{\alpha,\beta}$ for $\alpha\leq\beta$. Let $D_{\alpha,\beta} = \bigcap_{\alpha\leq\gamma<\beta} C_{\alpha,\gamma}$ (if $\beta = \gamma+1$ then obviously $D_{\alpha,\beta} = C_{\alpha,\gamma}$). For a fixed well ordering $\prec_\alpha$ of the set $D_{\alpha,\beta}$ and for $x \in D_{\alpha,\beta}$ let $\overline{x}$ denotes the set $\{y \in D_{\alpha,\beta} : y \prec_\alpha x\}$. Given a function f with β as the domain and such that $f(\alpha) \in D_{\alpha,\beta}$ (i.e. $f \in \prod_{\alpha<\beta} D_{\alpha,\beta}$) let E_f be the set defined by the following equality

(6) $$E_f = \left\{b \in B_\beta : (\forall\alpha <\beta)\ b) \cap D_{\alpha,\beta} \subseteq \overline{f(\alpha)}\right\}.$$

It follows by the inequalities

$$|b) \cap D_{\alpha,\beta}| \leq |b)| < n(P) < |B_\beta|$$

and by the regularity of the cardinal $|B_\beta|$) that

$$(\forall\ b \in B_\beta)(\exists f \in \prod_{\alpha<\beta} D_{\alpha,\beta})\ b \in E_f$$

thus

$$B_\beta = \bigcup\left\{E_f : f \in \prod_{\alpha<\beta} D_{\alpha,\beta}\right\}.$$

But $\left|\prod_{\alpha<\beta} D_{\alpha,\beta}\right| \le \left|\prod_{\alpha<\beta} B_\alpha\right| < |B_\beta|$. Thus, again by the regularity of $|B_\beta|$, there exists $f_0 \in \prod_{\alpha<\beta} D_{\alpha,\beta}$ that $|E_{f_0}| = |B_\beta|$. We put now

$$C_{\beta,\beta} = E_{f_0} \tag{7}$$

$$C_{\alpha,\beta} = D_{\alpha,\beta} - \overline{f_0(\alpha)} \quad \text{for } \alpha<\beta. \tag{8}$$

The conditions 1-5 are obviously satisfied. It remains to prove the same for condition 6. Let $a,b \in \bigcup_{\alpha\le\beta} C_{\alpha,\beta}$. If both these elements are in $C_{\beta,\beta}$ or in $C_{\alpha,\beta}$ for some $\alpha<\beta$ then they are incomparable by condition 1. If $a \in C_{\alpha,\beta}$, $b \in C_{\gamma,\beta}$ for some $\alpha, \gamma<\beta$ then $a \in C_{\alpha,\beta_1}$, $b \in C_{\gamma,\beta_1}$ for some ordinal β_1 such that $\alpha, \gamma \le \beta_1 < \beta$. Therefore a and b belong to $\bigcup_{\xi\le\beta_1} C_{\xi,\beta_1}$ which is an antichain by the inductive assumption. Let us assume now that $a \in C_{\beta,\beta}$ and $b \in C_{\alpha,\beta}$ for some $\alpha<\beta$. Thus $a \nleq b$ by condition 1 and point 4 of Lemma 2. Let $b \le a$. It implies that $b \in a) \cap D_{\alpha,\beta}$ hence, by (7) and (6), $b \in \overline{f_0(\alpha)}$ in spite of (8). Thus $b \nleq a$.

We define now a set C by the following two equalities

$$F_\alpha = \bigcap_{\alpha\le\beta<\mathrm{cf}(|P|)} C_{\alpha,\beta},$$

$$C = \bigcup_{\alpha<\mathrm{cf}(|P|)} F_\alpha,$$

C is an antichain. Indeed, if $a \in F_{\alpha_1}$ and $b \in F_{\alpha_2}$ for some $\alpha_1 \le \alpha_2 < \mathrm{cf}(|P|)$, then for an arbitrary $\beta_1 \ge \alpha_2$ we obtain that $a \in C_{\alpha_1,\beta_1}$ and $b \in C_{\alpha_2,\beta_1}$. Thus a, b belong to $\bigcup_{\xi\le\beta_1} C_{\xi,\beta_1}$ and they are incomparable by condition 6.

The antichains are pairwise disjoint by condition 1. $|F_\alpha| = |B_\alpha|$ by conditions 3, 4, 5 and by the inequality $|B_\alpha| < cf(|P|)$. Therefore

$$|C| = \sup \{ |F_\alpha| : \alpha < cf(P) \} = |P|$$

by point 3 of Lemma 2. This completes the proof.

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