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ON VARIATIONS OF THE WIENER TYPE

Dedicated to the memory of Professor Roman Sikorski

1. Introduction

As in [2], we shall denote by M, N and M_k, N_k $(k=0,1,2)$ the pairs of non-negative continuous convex functions complementary in the sense of W.H. Young. For the inverse functions the symbols M^{-1}, N^{-1} etc. will be used.

Let f be an arbitrary complex-valued function defined in a closed interval $\langle a,b\rangle$. Consider partitions

$$\Pi = \{a = t_0 < t_1 < \ldots < t_r = b\} \tag{1}$$

and finite sequences

$$U = (u_0, u_1, \ldots, u_{r-1}) \tag{2}$$

of non-negative numbers u_j such that

$$\sum_{j=0}^{r-1} N(u_j) \leq 1. \tag{3}$$

Introduce the quantities

$$V_M(f;a,b) = \sup_{\Pi} \sum_{j=0}^{r-1} M(|f(t_{j+1}) - f(t_j)|),$$

$$V_M^*(f;a,b) = \sup_{\Pi} \left\{ \sup_{U} \sum_{j=0}^{r-1} |f(t_{j+1}) - f(t_j)|\, u_j \right\}$$

called the first and the second M-variation of f on $\langle a,b\rangle$, respectively.

By the inequality of W.H. Young,

$$(4) \qquad V_M^*(f;a,b) \leqslant V_M(f;a,b) + 1.$$

If $\gamma = V_M^*(f;a,b)$ is a positive number, then

$$(5) \qquad V_M(f/\gamma;a,b) \leqslant 1.$$

Moreover,

$$(6) \qquad V_M^*(\lambda f;a,b) = |\lambda|\, V_M^*(f;a,b)$$

for each complex number λ. In the special case

$$M(u) = u^{\alpha}/\alpha \quad (\alpha > 1) \text{ for all } u \geqslant 0,$$

$$V_M^*(f;a,b) = \beta^{1/\beta} V_{\alpha}^*(f;a,b),$$

where $\beta = \alpha/(\alpha-1)$ and

$$V_{\alpha}^*(f;a,b) = \sup_{\Pi} \left\{ \sum_{j=0}^{r-1} |f(t_{j+1}) - f(t_j)|^{\alpha} \right\}^{1/\alpha}$$

(see [2], p.24, 89-91, 86-87 or [8], p.16, 25, 171, 175).

Denote by $BV_M(a,b)$ $[BV_M^*(a,b)]$ the class of all complex-valued functions f such that

$$V_M(f;a,b)<\infty \quad [V_M^*(f;a,b)<\infty].$$

Both these classes consist merely of bounded functions f having at most enumerable sets of discontinuity points x at which the finite one-sided limits $f(x \pm 0)$ exist (see [5] and [7], p.582-583). In view of (4), $BV_M(a,b)\subset BV_M^*(a,b)$. For functions $f\in BV_M^*(a,b)$, the non-negative functional

$$V_M^*(f) = V_M^*(f;a,b)$$

can be treated as a certain seminorm.

Consider now a complex-valued function f defined in the interval $\langle a,b\rangle$, and a positive integer number n. Following Čanturija [1], let us introduce the modulus of variation of f

$$v(n,f;a,b) = \sup_{\Pi_n} \sum_{k=0}^{n-1} |f(x_{2k+1}) - f(x_{2k})|,$$

where the supremum is taken over all systems Π_n of non-overlapping intervals $(x_{2k},x_{2k+1})\subset (a,b)$, $k = 0,1,\dots,n-1$. It can easily be observed,

$$(7) \qquad v(n,f;a,b)\leq n\, M^{-1}\left(\frac{1}{n}\right) V_M^*(f;a,b)$$

(see [5], p.454). Consequently,

$$\lim_{n\to\infty} \frac{1}{n}\, v(n,f;a,b) = 0 \quad \text{when} \quad f\in BV_M^*(a,b).$$

In this note some simple facts connected with the second M-variations will be presented.

2. Special properties

Let us begin with the following

Theorem 1. Suppose that

$$M(u) \leq M_0(u) \quad \text{for all} \quad u \in \langle 0, \delta \rangle, \tag{8}$$

δ being a positive fixed number. Then, there is a positive number c (depending only on M, M_0 and δ) such that

$$V_M^*(f; a,b) \leq c\, V_{M_0}^*(f; a,b) \tag{9}$$

for each $f \in BV_{M_0}^*(a,b)$ $(-\infty < a < b < \infty)$.

Proof. Write

$$M_0(u) = \int_0^u p_0(t)dt, \quad N_0(u) = \int_0^u q_0(s)ds \quad \text{for} \quad u \geq 0.$$

As is known ([2], p.24), for all $s \geq 0$,

$$q_0(s)s = M_0(q_0(s)) + N_0(s) \quad \text{and} \quad q_0(s)s \leq M(q_0(s)) + N(s).$$

Consequently,

$$M_0(q_0(s)) + N_0(s) \leq M(q_0(s)) + N(s) \quad \text{when} \quad s \geq 0.$$

By the assumption,

$$M_0(q_0(s)) \geq M(q_0(s)) \quad \text{if} \quad 0 \leq q_0(s) \leq \delta .$$

Therefore, the previous estimate ensures that

$$N_0(s) \leq N(s) \quad \text{if} \quad 0 \leq q_0(s) \leq \delta .$$

Denoting by s_0 a positive number such that $q_0(s_0) \leq \delta$, we then have

$$N_0(s) \leq N(s) \quad \text{for every} \quad s \in \langle 0, s_0 \rangle. \tag{10}$$

Let c be a number satisfying the conditions

$$c \geq 1 \quad \text{and} \quad c\, s_0 \geq N^{-1}(1).$$

Introduce the pair of functions

$$\widetilde{M}_0(u) = c\, M_0(u),\ \widetilde{N}_0(u) = c\, N_0\left(\frac{u}{c}\right) \quad (u \geq 0)$$

complementary in the sense of Young ([2], p.23). It is easily seen, for any f bounded on $\langle a,b\rangle$,

$$(11) \qquad V^*_{\widetilde{M}_0}(f;a,b) \leq c\, V^*_{M_0}(f;a,b).$$

Consider an arbitrary partition (1) and a finite sequence (2) of non-negative numbers u_j satisfying the condition (3). By the inequality (1.14) of [2] (p.17) and (10),

$$N(u_j) \geq c\, N\left(\frac{u_j}{c}\right), \quad N\left(\frac{u_j}{c}\right) \geq N_0\left(\frac{u_j}{c}\right),$$

because $c \geq \max\,(1, N^{-1}(1)/s_0)$. Hence

$$\sum_{j=0}^{r-1} \widetilde{N}_0(u_j) \leq 1$$

and, consequently,

$$(12) \qquad V^*_{\widetilde{M}_0}(f;a,b) \geq V^*_M(f;a,b).$$

From (11) - (12) the desired assertion follows immediately.

R e m a r k 1. If the inequality (8) holds for all $u \in \langle 0, \infty)$, then the number c occuring in (9) is independent of δ.

For example, let

$$(13) \qquad M(u) = e^{-u^{-\alpha}} \quad (\alpha > 0) \quad \text{for} \quad u \in (0, \delta\rangle,$$

where $\delta = \{\alpha/(\alpha+1)\}^{1/\alpha}$. Choose a positive integer $k = k(\alpha)$ such that $\alpha k > 1$, and introduce the convex function

$$M_0(u) = k!\, u^{\alpha k} \quad \text{for all} \quad u \geq 0.$$

It can easily be observed that for every $u \in \langle 0, \delta \rangle$ the inequality (8) is true. Thus, for arbitrary finite a, b and $f \in BV^*_{M_0}(a,b)$, the estimate (9) holds with c depending only on M. Consequently,

$$\lim_{h\to 0+} V^*_M(f;x,x+h) = 0 \left[\lim_{h\to 0+} V^*_M(f;x-h,x) = 0 \right],$$

whenever f is right- [left-] sidely continuous at $x \in \langle a,b)$ [$x \in (a,b\rangle$] (see [3], Sects. 4, 5).

T h e o r e m 2. Suppose that M possesses the complementary function N satisfying the condition

$$\lim_{s\to 0+} \frac{N(s\,w)}{s\,N(w)} = 0 \tag{14}$$

uniformly in w on some finite interval $(0,\rho\rangle$. Then, under the assumption $f \in BV^*_M(a,b)$,

$$\lim_{h\to 0+} V^*_M(f;x,x+h) = 0 \tag{15}$$

for every $x \in \langle a,b)$ at which the function f is right-sidely continuous.

P r o o f . In view of (5), there is a positive number θ such that

$$V_M(\theta f;a,b) \leq 1.$$

Supposing f right-sidely continuous at x ($a \leq x < b$), and putting $g = \theta f$, we have

$$\lim_{\sigma\to 0+} V_M(g;x,x+\sigma) = 0 \tag{16}$$

(see [3], Sects. 4, 5).

By (14), for any given $\varepsilon > 0$ and sufficiently large $\varrho = \varrho(\varepsilon) > 1$,

$$\varrho N\left(\frac{w}{\varrho}\right) < \varepsilon N(w) \tag{17}$$

uniformly in $w \in (0, \varrho\rangle$. Applying (16) we get

$$V_M(g; x, x+d) < \frac{\varepsilon}{\varrho} \tag{18}$$

if the positive number d is small enough.

Consider now a partition

$$\{x = x_0 < x_1 < \dots < x_r = x + d\}$$

and some non-negative numbers u_j satisfying the condition (3), such that

$$\sum_{j=0}^{r-1} |g(x_{j+1}) - g(x_j)| u_j > V_M^*(g; x, x+d) - \varepsilon .$$

This and the inequality of W.H. Young lead to

$$V_M^*(g; x, x+d) < \varepsilon + \sum_{j=0}^{r-1} \widetilde{M}(|g(x_{j+1}) - g(x_j)|) + \sum_{j=0}^{r-1} \widetilde{N}(u_j),$$

where

$$\widetilde{M}(u) = \varrho M(u), \ \widetilde{N}(u) = \varrho N\left(\frac{u}{\varrho}\right) \quad (u \geqslant 0).$$

Consequently,

$$V_M^*(g; x, x+d) < \varepsilon + \varrho \sum_{j=0}^{r-1} M(|g(x_{j+1}) - g(x_j)|) + \sum_{j=0}^{r-1} \varrho N\left(\frac{u_j}{\varrho}\right) \leqslant$$

$$\leqslant \varepsilon + \varrho V_M(g; x, x+d) + \varepsilon \sum_{j=0}^{r-1} N(u_j) < 3\varepsilon \ \text{ when } \varrho > N^{-1}(1).$$

by (17) and (18). Analogously, in the case $\varrho \leq N^{-1}(1)$,

$$V_M^*(g;x,x+d) < \varepsilon + \frac{N^{-1}(1)}{\varrho}\left\{\sum_{j=0}^{r-1} \tilde{M}(|g(x_{j+1})-g(x_j)|) + \sum_{j=0}^{r-1} \tilde{N}\left(\frac{u_j}{N^{-1}(1)}\varrho\right)\right\} \leq$$

$$\leq \varepsilon + \frac{N^{-1}(1)}{\varrho}\left\{\varrho V_M(g;x,x+d) + \sum_{j=0}^{r-1} \varrho N\left(\frac{1}{\varrho}\cdot\frac{u_j}{N^{-1}(1)}\varrho\right)\right\} <$$

$$< \varepsilon + \frac{N^{-1}(1)}{\varrho}\left\{\varepsilon + \varepsilon\sum_{j=0}^{r-1} N\left(\frac{u_j}{N^{-1}(1)}\varrho\right)\right\} \leq \left\{2 + \frac{N^{-1}(1)}{\varrho}\right\}\varepsilon .$$

Observing that

$$V_M^*(g;x,x+d) = \theta\, V_M^*(f;x,x+d) \quad \text{and} \quad V_M^*(f;x,x+h) \leq V_M^*(f;x,x+d)$$

for every $h \in \langle 0,d\rangle$, we obtain the relation (15).

For example, the functions

1^0 $N(u) = u^{\alpha}/|\log u|^{\beta}$ $(\alpha > 1,\ \beta \geq 0)$,

2^0 $N(u) = (1+u)\log(1+u) - u$

satisfy the condition (14) uniformly in w, respectively, on the interval $(0,1/2\rangle$ and $(0,\varrho\rangle$ for every finite $\varrho > 0$.

For the function N complementary to M defined by (13) the condition (14) does not hold uniformly on $(0,\varrho\rangle$ with any positive ϱ. Indeed, it can easily be verified that

$$M^{-1}(y) = 1/|\log y|^{1/\alpha} \quad \text{for small} \quad y > 0.$$

Since

$$y < M^{-1}(y)\, N^{-1}(y) \leq 2y \quad (y > 0)$$

(see [2], p.25), we have

$$y\,|\log y|^{1/\alpha} < N^{-1}(y) \leq 2y\,|\log y|^{1/\alpha}$$

for sufficiently small $y \in (0,1/e)$. Consequently, there are positive numbers u_α and k_α such that, for all $u \in (0,u_\alpha)$,

$$\frac{k_\alpha u}{|\log u|^{1/\alpha}} \leq N(u) < \frac{u}{|\log u|^{1/\alpha}} \quad (\alpha > 0).$$

By this estimate, for all $w \in (0, u_\alpha)$ and $s \in (0,1)$,

$$\frac{N(sw)}{sN(w)} \geq \frac{k_\alpha sw}{|\log sw|^{1/\alpha}} \cdot \frac{|\log w|^{1/\alpha}}{sw} = k_\alpha \left|\frac{\log w}{\log sw}\right|^{1/\alpha}.$$

Observing that

$$\lim_{w \to 0+} \frac{\log w}{\log sw} = 1 \quad \text{for} \quad s \in (0,1),$$

we get the desired result.

3. Two applications

A. Given an arbitrary complex-valued function f of period 2π, Lebesgue-integrable over $\langle -\pi, \pi \rangle$, let us consider the n-th partial sum

$$S_n[f](x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

of its Fourier series. As is well-known, if at a fixed x some finite one-sided limits $f(x \pm 0)$ exist, then

$$(19) \quad S_n[f](x) - \frac{1}{2}\left\{f(x+0) + f(x-0)\right\} = \frac{1}{\pi}\int_0^\pi \varphi_x(t)\, D_n(t)\,dt,$$

where φ_x is the 2π-periodic function defined by

$$\varphi_x(t) = \begin{cases} f(x+t) + f(x-t) - f(x+0) - f(x-0) & \text{when } 0 < t < 2\pi, \\ 0 & \text{when } t = 0 \end{cases}$$

and

$$D_n(t) = \frac{1}{2} \sum_{k=1}^{n} \cos kt = \frac{\sin\left(n + \frac{1}{2}\right)t}{2\sin\frac{1}{2}t} \qquad (n \geq 0).$$

In Sect. 2 of [4] the following basic estimate is proved.

L e m m a . Suppose that φ_x is bounded in the interval $\langle 0,\pi\rangle$; write

$$t_j = \frac{2\pi j}{2n+1} \quad (j = 0,1,\dots,n).$$

Then, for every positive integer n,

$$(20)\qquad \left|\frac{1}{\pi}\int_0^{\pi}\varphi_x(t)D_n(t)dt\right| \leq \frac{5}{3n}\, v(n,\varphi_x;0,\pi) +$$

$$+ \frac{7}{6}\, v(1,\varphi_x;0,t_1) + \frac{4}{3}\sum_{k=1}^{n-1}\frac{v(k,\varphi_x;0,t_{k+1})}{k(k+1)} .$$

Applying this lemma we shall present our first application of M-variations to Fourier series.

T h e o r e m 3. If $f \in BV_M^*(0,2\pi)$, then

$$\left|S_n[f](x) - \frac{1}{2}\left\{f(x+0) + f(x-0)\right\}\right| \leq$$

$$\leq 19\sum_{k=1}^{n}\frac{1}{k}\, M^{-1}\left(\frac{k}{n}\right) V_M^*(\varphi_x;0,\frac{\pi}{k})$$

for every positive integer n and all real x.

P r o o f The inequality (7) gives

$$v(k+1,\varphi_x;0,t_{k+1}) \leq (k+1)M^{-1}\left(\frac{1}{k+1}\right)V_M^*(\varphi_x;0,t_{k+1}) \qquad (k=0,1,\dots).$$

Hence, in view of (19) and (20),

$$\left|S_n[f](x) - \frac{1}{2}\left\{f(x+0) + f(x-0)\right\}\right| \leq$$

$$\leq \frac{3}{n}\, v(n,\varphi_x;0,\pi) + \frac{7}{6}\, v(1,\varphi_x;0,t_1) +$$

$$+ \frac{4}{3}\sum_{k=1}^{n-2}\frac{v(k,\varphi_x;0,t_{k+1})}{k(k+1)} \leq \frac{3}{n}\, v(n,\varphi_x;0,\pi) +$$

$$+ \frac{7}{6} M^{-1}(1)V_M^*(\varphi_x;0,t_1) + \frac{4}{3} \sum_{j=2}^{n-1} \frac{2}{j} M^{-1}\left(\frac{1}{j}\right) V_M^*(\varphi_x;0,t_j) \leq$$

$$\leq \frac{3}{n} v(n,\varphi_x;0,\pi) + 2M^{-1}(1)V_M^*(\varphi_x;0,t_1) +$$

$$+ \sum_{j=2}^{n-1} \frac{4}{j+1} M^{-1}\left(\frac{2}{j+1}\right) V_M^*(\varphi_x;0,t_j).$$

Further,

$$\sum_{j=1}^{n-1} \frac{1}{j+1} M^{-1}\left(\frac{2}{j+1}\right) V_M^*(\varphi_x;0,t_j) \leq$$

$$\leq \int_{\pi/n}^{\pi} \frac{1}{t} M^{-1}\left(\frac{2\pi}{nt}\right) V_M^*(\varphi_x;0,t)\, dt =$$

$$= \int_1^n \frac{1}{s} M^{-1}\left(\frac{2s}{n}\right) V_M^*(\varphi_x;0, \frac{\pi}{s})\, ds \leq$$

$$\leq \sum_{k=1}^{n-1} \frac{1}{k} M^{-1}\left(\frac{2(k+1)}{n}\right) V_M^*(\varphi_x;0, \frac{\pi}{k}) .$$

Consequently,

$$\left|S_n[f](x) - \frac{1}{2}\left\{f(x+0) + f(x-0)\right\}\right| \leq 3M^{-1}\left(\frac{1}{n}\right) V_M^*(\varphi_x;0,\pi) +$$

$$+ 16 \sum_{k=1}^{n} \frac{1}{k} M^{-1}\left(\frac{k}{n}\right) V_M^*(\varphi_x;0, \frac{\pi}{k}),$$

and, the desired thesis is now evident (cf. [4], Th. 2(i)).

C o r o l l a r y . Suppose that

$$\sum_{k=1}^{\infty} \frac{1}{k} M^{-1}\left(\frac{1}{k}\right) < \infty, \tag{21}$$

and that

$$\lim_{t\to 0+} V_M^*(\varphi_x;0,t) = 0. \tag{22}$$

Then

$$\lim_{n\to\infty} S_n[f](x) = \frac{1}{2}\left\{f(x+0) + f(x-0)\right\}. \tag{23}$$

If f is continuous at every point lying in $\langle a,b\rangle$ and if the relation (22) holds uniformly in x on $\langle a,b\rangle$, then

$$\lim_{n\to\infty} S_n[f](x) = f(x) \quad \text{uniformly in } x\in\langle a,b\rangle.$$

In fact, under the assumption (22), for every $\varepsilon > 0$ there is an integer $k_0 = k_0(\varepsilon,x) \geq 1$ such that

$$V_M^*(\varphi_x;0,\tfrac{\pi}{k}) < \varepsilon \quad \text{if} \quad k \geq k_0.$$

Hence

$$\sum_{k=k_0}^{n} \frac{1}{k} M^{-1}\left(\frac{k}{n}\right) V_M^*(\varphi_x;0,\tfrac{\pi}{k}) < \varepsilon \sum_{k=k_0}^{n} \frac{1}{k} M^{-1}\left(\frac{k}{n}\right) \quad \text{if} \quad n\geq k_0.$$

The condition (21) ensures that

$$\sum_{k=1}^{n} \frac{1}{k} M^{-1}\left(\frac{k}{n}\right) \leq 2\int_1^{n+1} \frac{1}{t} M^{-1}\left(\frac{t}{n}\right) dt \leq 2\int_{1/2}^{\infty} \frac{1}{s} M^{-1}\left(\frac{1}{s}\right) ds < \infty$$

uniformly in $n \geq 1$. Thus,

$$\sum_{k=1}^{n} \frac{1}{k} M^{-1}\left(\frac{k}{n}\right) V_M^{*}(\varphi_x; 0, \frac{\pi}{k}) < \varepsilon + 2\varepsilon \int_{1/2}^{\infty} \frac{1}{s} M^{-1}\left(\frac{1}{s}\right) ds$$

for sufficiently large n, and, by Theorem 3 the relation (23) follows.

Remark 2. If the function N complementary to M satisfies the assumption of Theorem 2, then, for any $f \in BV_M^{*}(0,2\pi)$ the relation (22) holds at every real x.

B. As the second application we shall give an estimate for some Stieltjes integrals.

Theorem 4. Suppose that $f \in BV_{M_1}^{*}(a,b)$, $g \in BV_{M_2}^{*}(a,b)$ $(-\infty < a < b < \infty)$ and that

$$S = \sum_{k=1}^{\infty} M_1^{-1}\left(\frac{1}{k}\right) M_2^{-1}\left(\frac{1}{k}\right) < \infty .$$

Then the Riemann-Stieltjes integral of f and g having no common discontinuities exists and

$$(24) \quad \left| \int_a^b \left\{f(t) - f(b)\right\} dg(t) \right| \leqslant C \, V_{M_1}^{*}(f; a,b) \, V_{M_2}^{*}(g; a,b),$$

where

$$C = S + 1/\left\{N_1^{-1}(1) N_2^{-1}(1)\right\}.$$

Proof. By (5), there are positive numbers θ_1, θ_2 such that

$$\theta_1 f \in BV_{M_1}(a,b), \quad \theta_2 g \in BV_{M_2}(a,b).$$

Hence, in view of Theorem (5.5) of [7], the Riemann-Stieltjes integral

$$I = \int_a^b \left\{\theta_1 f(t) - \theta_1 f(b)\right\} d\, \theta_2 g(t)$$

exists.

Applying the inequality (2) of [6], we obtain

$$|I| \leq C\, V^{*}_{M_1}(\theta_1 f; a,b)\, V^{*}_{M_2}(\theta_2 g; a,b);$$

whence the thesis follows immediately (see (6)).

In particular, taking

$$g(t) = \int_a^t K(s)ds \quad (a < t < b),$$

with the integrand K measurable and bounded in $\langle a,b\rangle$, we can estimate (in absolute value) the Lebesgue integral

$$\int_a^b \{f(t) - f(b)\} K(t)dt$$

by the right-hand side of (24), too.

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