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ON THE DIVISIBILITY BY 3 OF $\# K_2 O_F$ FOR REAL QUADRATIC FIELDS F

Dedicated to the memory of Professor Roman Sikorski

1. Introduction

Let F be a real quadratic field with the discriminant d, and let O_F be its ring of integers. It is known that the group K_2O_F is finite, where K_2 is the functor of Milnor. It seems that there are some relations between this group and the class group of F. In the present paper we obtain some such relations.

We prove (Theorem 2) that $\# K_2O_F$ is divisible by 3 if and only if the class number of the field $Q(\sqrt{-3d})$ is divisible by 3 or $d \equiv 6 \mod 9$. It follows that if 3 divides the class number h(d) of F, then $3 \mid \# K_2O_F$.

By a result of B. Mazur and A. Wiles the odd parts of the integers $\# K_2O_F$ and $w_F \zeta_F(-1)$ are equal, where ζ_F is the Dedekind zeta function of F, and $w_F = 24$ for $d \neq 5, 8$, $w_F = 120$ resp. 48 for d = 5 resp. 8. Moreover there exist formulas expressing $\zeta_F(-1)$ and the class number of $Q(\sqrt{-3d})$ as sums of Kronecker symbols (see [Bö], [Le], [Mo]). Therefore our proof is based on transformations of sums of Kronecker symbols. Some particular cases of Theorem 2 have been proved earlier by C. Queen [Qu] and Lu Hong-Wen [Lu]. Recently J. Urbanowicz [Ur] has also given a simple proof of Theorem 2.

2. Formulas containing Kronecker symbols

We collect here some formulas containing Kronecker symbols that will be used later. We consider Kronecker symbols $\left(\frac{d}{l}\right)$, where l is an integer, and d is a fundamental discriminant, i.e. $d \equiv 1 \bmod 4$, $d \neq 1$, squarefree, or $d = 4d'$, $d' \equiv 2, 3 \bmod 4$, d' squarefree. Let τ be the number of units of $Q(\sqrt{d})$ for d negative.

The following formulas hold:

(1) $$\left(\frac{d}{l}\right) = \left(\frac{d}{l'}\right) \quad \text{for} \quad d>0,\ l \equiv \pm l \bmod d,$$

(2) $$\left(\frac{d}{l}\right) = \left(\frac{d}{l'}\right) \operatorname{sgn}(ll') \quad \text{for} \quad d<0,\ l \equiv l' \bmod d,$$

(3) $$\left(\frac{d}{l}\right) = -\left(\frac{d}{|d|-l}\right) \quad \text{for} \quad d<0,\quad 0<l<|d|,$$

(4) $$\sum_{l=1}^{|d|} \left(\frac{d}{l}\right) = 0 \quad \text{for every} \quad d,$$

(5) $$\sum_{l=1}^{d} \left(\frac{d}{l}\right) l = 0 \quad \text{for} \quad d>0,$$

(6) $$h(d) = \frac{\tau}{2d} \sum_{l=1}^{|d|} \left(\frac{d}{l}\right) l \quad \text{for} \quad d<0,$$

(7) $$h(d) = \frac{\tau}{3-\left(\frac{d}{3}\right)} \sum_{l=1}^{[|d|/3]} \left(\frac{d}{l}\right) \quad \text{for} \quad d<-3,$$

(8) $$h(-3d) = 2 \sum_{l=1}^{[d/3]} \left(\frac{d}{l}\right) \quad \text{for} \quad d>4,\ 3 \nmid d.$$

All these formulas are given in [Le], (1)-(5) on pp.337--338, (6) on p.341, formula (3), (7) on p.402, formula (30*), (8) on p.408, formula (44), see also [Mo].

L e m m a 1. For $F = Q(\sqrt{d})$, $d>0$ we have

$$w_F \zeta_F(-1) = \frac{1}{d}\sum_{l=1}^{d} \left(\frac{d}{l}\right) l^2.$$

P r o o f . In the well known formula

$$w_F \zeta_F(-1) = d \sum_{l=1}^{d} \left(\frac{d}{l}\right) B_2\left(\frac{l}{d}\right), \quad \text{where } B_2(x) = x^2 - x + \frac{1}{6},$$

the terms corresponding to $-x + \frac{1}{6}$ vanish in view of (4) and (5).

L e m m a 2. If $d<0$, then

$$\sum_{\substack{l=1\\ l\equiv 0(3)}}^{|d|} \left(\frac{d}{l}\right) = - \sum_{\substack{l=1\\ 2|d|+l\equiv 0(3)}}^{|d|} \left(\frac{d}{l}\right) \quad \text{and} \quad \sum_{\substack{l=1\\ |d|+l\equiv 0(3)}}^{|d|} \left(\frac{d}{l}\right) = 0.$$

P r o o f . It is sufficient to substitute $|d| - l$ for l and to apply (3).

3. Main results

Unfortunately, we cannot avoid some tedious computations with Kronecker symbols, but we collect them in the proof of Theorem 1.

T h e o r e m 1. If d is a negative fundamental discriminant, $3\nmid d$, then

$$\sum_{l=1}^{3|d|} \left(\frac{-3d}{l}\right) l^2 = 6d\left(\frac{d}{3}\right)\left\{ 3 \sum_{\substack{l=1\\ |d|+l\equiv 0(3)}}^{|d|} \left(\frac{d}{l}\right) \frac{l+|d|}{3} + \frac{|d|}{\tau} h(d)\left(1-\left(\frac{d}{3}\right)\right)\right\}.$$

P r o o f . From $\left(\frac{-3}{l}\right) = \pm 1$ for $l \equiv \pm 1 \bmod 3$ it follows that

$$\sum_{l=1}^{3|d|} \left(\frac{-3d}{l}\right) l^2 = -\left(\frac{d}{3}\right)\left(\sum_{\substack{l=1 \\ l\equiv |d|\,(3)}}^{3|d|} \left(\frac{d}{l}\right) l^2 - \sum_{\substack{l=1 \\ l\equiv 2|d|\,(3)}}^{3|d|} \left(\frac{d}{l}\right) l^2 \right) =$$

$$= -\left(\frac{d}{3}\right)\left(\sum_{\substack{l=1 \\ l\equiv |d|\,(3)}}^{|d|} \left(\frac{d}{l}\right) l^2 + \sum_{\substack{l=1 \\ |d|+l\equiv |d|\,(3)}}^{|d|} \left(\frac{d}{l}\right) (l+|d|)^2 + \right.$$

$$+ \sum_{\substack{l=1 \\ 2|d|+l\equiv |d|\,(3)}}^{|d|} \left(\frac{d}{l}\right) (l+2|d|)^2 - \sum_{\substack{l=1 \\ l\equiv 2|d|\,(3)}}^{|d|} \left(\frac{d}{l}\right) l^2 -$$

$$\left. - \sum_{\substack{l=1 \\ |d|+l\equiv 2|d|\,(3)}}^{|d|} \left(\frac{d}{l}\right) (l+|d|)^2 - \sum_{\substack{l=1 \\ 2|d|+l\equiv 2|d|\,(3)}}^{|d|} \left(\frac{d}{l}\right) (l+2|d|)^2 \right).$$

Let us observe that all six sums containing $\left(\frac{d}{l}\right) l^2$ cancel, and the sum of remaining terms is equal to

$$-\left(\frac{d}{d}\right) \left(2|d| \sum\nolimits_1 + |d|^2 \sum\nolimits_2\right),$$

where

$$\sum\nolimits_1 = \sum_{\substack{l=1 \\ l\equiv 0(3)}}^{|d|} \left(\frac{d}{l}\right) l + 2 \sum_{\substack{l=1 \\ |d|+l\equiv 0(3)}}^{|d|} \left(\frac{d}{l}\right) l - \sum_{\substack{l=1 \\ 2|d|+l\equiv 0(3)}}^{|d|} \left(\frac{d}{l}\right) l -$$

$$- 2 \sum_{\substack{l=1 \\ l\equiv 0(3)}}^{|d|} \left(\frac{d}{l}\right) l = 3 \sum_{\substack{l=1 \\ |d|+l\equiv 0(3)}}^{|d|} \left(\frac{d}{l}\right) l - \sum_{l=1}^{|d|} \left(\frac{d}{l}\right) l =$$

$$= 3 \sum_{\substack{l=1 \\ |d|+l\equiv 0(3)}}^{|d|} \left(\frac{d}{l}\right) (l+|d|) + \frac{2|d|}{\tau} h(d)$$

in view of (4) and (6), and

$$\sum\nolimits_2 = \sum_{\substack{l=1\\ l\equiv 0(3)}}^{|d|} \left(\frac{d}{l}\right) + 4 \sum_{\substack{l=1\\ |d|+l\equiv 0(3)}}^{|d|} \left(\frac{d}{l}\right) - \sum_{\substack{l=1\\ 2|d|+l\equiv 0(3)}}^{|d|} \left(\frac{d}{l}\right) -$$

$$- 4 \sum_{\substack{l=1\\ l\equiv 0(3)}}^{|d|} \left(\frac{d}{l}\right) = -2 \sum_{\substack{l=1\\ l\equiv 0(3)}}^{|d|} \left(\frac{d}{l}\right),$$

in view of Lemma 2. Moreover from (7) it follows that

$$\sum_{\substack{l=1\\ l\equiv 0(3)}}^{|d|} \left(\frac{d}{l}\right) = \left(\frac{d}{3}\right) \sum_{t=1}^{[|d|/3]} \left(\frac{d}{t}\right) = \frac{1}{\tau}\left(\frac{d}{3}\right)\left(3 - \left(\frac{d}{3}\right)\right) h(d) =$$

$$= \frac{1}{\tau}\left(3\left(\frac{d}{3}\right) - 1\right) h(d).$$

Collecting all above results we obtain the theorem.

T h e o r e m 2 (see [Qu] and [Lu]). If $d > 0$ is the discriminant of the field $F = Q(\sqrt{d})$, then $3 \mid \# K_2 O_F$ if and only if the class number of the field $Q(\sqrt{-3d})$ is divisible by 3 or $d \equiv 6 \bmod 9$.

P r o o f . Since the odd parts of numbers $\# K_2 O_F$ and $w_F \zeta_F(-1)$ are equal, it is sufficient to investigate the divisibility by 3 of $w_F \zeta_F(-1)$.

(1) Suppose that $3 \nmid d$. Then we have mod 3:

$$w_F \zeta_F(-1) = \frac{1}{d} \sum_{l=1}^{d} \left(\frac{d}{l}\right) l^2 \equiv \frac{1}{d} \sum_{\substack{l=1\\ 3\nmid l}}^{d} \left(\frac{d}{l}\right) = -\frac{1}{d} \sum_{\substack{l=1\\ 3\mid l}}^{d} \left(\frac{d}{l}\right) =$$

$$= -\frac{1}{2d}\left(\frac{d}{3}\right) h(-3d) \equiv h(-3d) \bmod 3,$$

in view of (8), and the theorem follows in this case.

(2) Suppose that $3|d$. Put $d = -3d'$. Then $3 \nmid d'$, and d' is a negative fundamental discriminant. We apply to it Theorem 1. Then we obtain mod 3:

$$w_F \zeta_F(-1) = \frac{-1}{3d'} \sum_{l=1}^{3|d'|} \left(\frac{d'}{l}\right) l^2 \equiv -2\left(\frac{d'}{3}\right) \frac{|d'|}{\tau} h(d') \left(1 - \left(\frac{d'}{3}\right)\right).$$

Thus $w_F \zeta_F(-1) \equiv 0 \bmod 3$ if and only if $h(d')\left(1-\left(\frac{d'}{3}\right)\right) \equiv 0 \bmod 3$, i.e. $h(d') \equiv 0 \bmod 3$ or $d' \equiv 1 \bmod 3$.

Since $d = -3d'$, we have $h(d')$ = the class number of $Q(\sqrt{-3d})$, and $d' \equiv 1 \bmod 3$ is equivalent to $d \equiv 6 \bmod 9$.

Corollary. Let $d > 0$ be the discriminant of the field $F = Q(\sqrt{d})$. Then $3|h(d)$ implies $3|\#K_2O_F$.

Proof. By a theorem of A. Scholz [Sch] if $3|h(d)$, then 3 divides the class number of $Q(\sqrt{-3d})$. Therefore the result follows from Theorem 2.

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