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C-DERIVED POLYADIC GROUPS

Dedicated to the memory of Professor Roman Sikorski

1. Introduction

The idea of n-groups derived from m-groups appeared in the very first paper [2] on polyadic groups (called also n-groups). This notion was subsequently generalized in various ways by several authors (cf. e.g. [21], [10], [1], [22], [23], [4]-[8]). In this paper we introduce a general construction, which contains all cases considered before, and we treat the problem from a new point of view. Usually, one asks what properties of an n-group (G,f) derived from an m-group (G,g) inherit from (G,g). We are interested when an n-group (G,f) is derived from an m-group (G,g) and what information about this m-group one can obtain from information about (G,f). Various applications and extensions of results presented here are given in [20]. This paper is a continuation of our papers devoted to various constructions of polyadic groups (cf. [12]-[19], [4]-[8]).

2. Some notions and notation

The terminology and notation of the present paper are the same as in [5], [6] (and in great parts as in [4], [18], [19], [12], [16]). To avoid repetitions, we fix the following notation: n, k, s are positive integers such that $n = s\cdot k$; (G,f) is a nonempty $(n+1)$-group, (G,g) is a nonempty $(k+1)$-group.

Let us recall after Post [21] a certain equivalence relation on the set of all polyads of a given polyadic group (cf. also [5], [13], [18]). Let (G,f) be an $(n+1)$-group. The relation $\underset{f}{\equiv}$ is defined as follows:

$\langle a_1^m\rangle \underset{f}{\equiv} \langle b_1^{m+un}\rangle$ if and only if for some i and for some elements $c_1,\dots,c_r \in G$ the equality $f_{(\cdot)}(c_1^i,a_1^m,c_{i+1}^r) = f_{(\cdot)}(c_1^i,b_1^{m+un},c_{i+1}^r)$ holds.

Post has proved (cf. [21]) that $\langle a_1^m\rangle \underset{f}{\equiv} \langle b_1^{m+un}\rangle$ if and only if for all i and r (where $r \geqslant i$ and $r+m \equiv 1 \pmod n$) the equality $f_{(\cdot)}(x_1^i,a_1^m,x_{i+1}^r) = f_{(\cdot)}(x_1^i,b_1^{m+un},c_{i+1}^r)$ holds for every sequence $x_1,\dots,x_r \in G$. In this paper we often consider different polyadic group operations f, g on the same set G, and so we write $\langle a_1^m\rangle \underset{f}{\equiv} \langle b_1^t\rangle$ or $\langle a_1^m\rangle \underset{g}{\equiv} \langle b_1^t\rangle$ to indicate from which polyadic group the Post relation comes.

3. C-systems

Consider a $(k+1)$-group (G,g). Let $\delta_1,\dots,\delta_n$ be a sequence of mappings from G into itself and $c_1,\dots,c_k \in G$. We denote the system of mappings and elements (called in this paper an s-system over G or simply a system over G) by $\langle \delta_1^n;c_1^k\rangle$ or briefly $\langle \underline{\delta};\underline{c}\rangle$. Any such s-system $\langle \underline{\delta};\underline{c}\rangle$ enables us to define an $(n+1)$-ary operation f on the set G by

$$(1)\qquad f(x_1^{n+1}) = g_{(s+1)}(x_1,\delta_1(x_2),\dots,\delta_n(x_{n+1}),c_1^k).$$

We say that the resulting $(n+1)$-groupoid (G,f) is $\langle \underline{\delta};\underline{c}\rangle$-derived from the $(k+1)$-group (G,g) and we write $(G,f) = \mathrm{der}^s_{\underline{\delta};\underline{c}}(G,g)$ (cf. [4], [5]). In general (G,f) need not be an $(n+1)$-group.

D e f i n i t i o n 1. An s-system $\langle\delta_1^n;c_1^k\rangle$ over a (k+1)-group (G,g) is said to be an s-G-system if the (n+1)-groupoid $(G,f) = \mathrm{der}^s_{\underline{\delta};\underline{c}}(G,g)$ is an (n+1)-group.

A criterion which decides when a system $\langle\underline{\delta};\underline{c}\rangle$ is a G-system was found in [6]. In the present paper we restrict ourselves to G-systems. It is easy to verify (cf. [6]) that for a G-system $\langle\underline{\delta};\underline{c}\rangle$ all mappings δ_i must be bijective. We have also the following

P r o p o s i t i o n 1. Let $\langle\delta_1^n;a_1^k\rangle$ and $\langle\delta_1^n;b_1^k\rangle$ be s-G-systems over a (k+1)-group (G,g). Then $\mathrm{der}^s_{\underline{\delta};\underline{a}}(G,g) = \mathrm{der}^s_{\underline{\delta};\underline{b}}(G,g)$ if and only if $\langle a_1^k\rangle \underset{g}{=} \langle b_1^k\rangle$.

Some additional assumptions are often imposed on G-systems under consideration (cf. [2], [21], [10], [11], [1], [7], [4], [19] etc.), resulting in special properties of derived (n+1)-groups. Now we try to state what do we mean by a condition C imposed on systems under consideration.

Consider the category Gr_{k+1} of (k+1)-groups (cf. [12]) and fix $n = s\cdot k$. Suppose that for any (k+1)-group (G,g) a set C(G,g) (possibly empty) of s-systems over (G,g) is chosen invariantly with respect to isomorphisms of (k+1)-groups (i.e., for any isomorphism $h:(A,g)\longrightarrow(B,g)$ and an s-system $\langle\underline{\delta};\underline{c}\rangle$ of the set C(A,g) the system $\langle h\underline{\delta}h^{-1};h(\underline{c})\rangle$ belongs to the set C(B,g)). Denote by C the class of all systems belonging to C(G,g) for any (k+1)-group (G,g). We will often say that a system $\langle\underline{\delta};\underline{c}\rangle$ satisfies a condition C or simply that it is a C-system if $\langle\underline{\delta};\underline{c}\rangle$ belongs to the class C. As was mentioned above, in this paper we assume that all systems of the class C (i.e., all C-systems) are G-systems. Given two conditions C, C′, the condition C is said to be stronger than C′ if $C(G,g)\subset C'(G,g)$ for any (k+1)-group (G,g). In this case we write $C\geqslant C'$ (and $C>C'$ if C is essentially stronger, i.e., $C\neq C'$). By CC′ we denote the intersection of the conditions C and C′. A group $(G,f) = \mathrm{der}^s_{\underline{\delta};\underline{c}}(G,g)$ is said to be an s-C-derived (or briefly: C-derived, if s is fixed) from (G,g) if $\langle\underline{\delta};\underline{c}\rangle$ is an s-C-system over (G,g). The (k+1)-group (G,g) is called then an s-C-creating (or $\langle \underline{\delta};\underline{c}\rangle$-creating) (k+1)-group of (G,f), and the system $\langle \underline{\delta};\underline{c}\rangle$ itself is called an s-C-creating system of (G,f). An (n+1)-group (G,f) is said to be a $C_{(k)}$-primitive (n+1)-group if it is not s-C-derived from any (k+1)-group and $n>1$. Consequently, (G,f) is a C-primitive (n+1)-group if it is $C_{(k)}$-primitive for every $k<n$.

The above terminology coincides with that of Dörnte, who considered in [2] the case where $\delta_i = id_G$ $(i = 1,\ldots,n)$ and $\langle c_1^k\rangle$ was an identity polyad in (G,g).

Definition 2. An s-system $\langle \delta_1^n;c_1^k\rangle$ over a (k+1)-group (G,g) is said to be an s-PE-system if δ_i is the identity mapping for every $i = 1,\ldots,n$ and $\langle c_1^k\rangle$ is an identity polyad in (G,g).

The above-defined condition is, in fact, the intersection of two conditions: P and E, which are defined and investigated in [20]. But in this paper we consider only PE-systems.

Note that in previous papers we used the symbol $\psi_s(G,g)$ (also $\psi_s(G)$) to denote the (n+1)-group PE-derived from (G,g). According to the terminology used here, in this paper we prefer the symbol $\operatorname{der}^s_{\underline{e}}(G,g)$ where $\underline{e} = \langle e_1^k\rangle$ denotes an identity polyad in (G,g).

The case of $n = 1$ should be treated separately. Since any (k+1)-group is 1-PE-derived from itself, for any condition C weaker than PE every (k+1)-group is 1-C-derived from itself. Therefore it is natural to consider the notion of C-primitive (n+1)-group only for $n>1$ (and $k<n$).

The first criterion for an (n+1)-group to be PE-derived from a (binary) group has been given by Dörnte in [2]. It was generalized by Post to the case of (n+1)-groups PE-derived from (k+1)-groups (cf. [21], and also [9] for a certain special case). This problem for conditions different from PE was considered in [5], [7], [22] and other papers. In [20] we give such criteria for various conditions C.

There are known some conditions C (e.g. the Hosszú condition of [4], which is described in section 5° of this paper) when C-primitive (n+1)-groups do not exist. Such conditions will be called nonrestrictive conditions, namely

D e f i n i t i o n 3. A condition C is said to be (s,k)-nonrestrictive if any (sk+1)-group is s-C-derived from a (k+1)-group.

D e f i n i t i o n 4. A condition C is said to be (s,k)-restrictive if there exists an (sk+1)-group which is not s-C-derived from any (k+1)-group.

It is evident that conditions which are weaker than a certain nonrestrictive condition are nonrestrictive. Similarly conditions which are stronger than a restrictive one are restrictive.

As was mentioned above, the Hosszú Condition (denoted in the sequel by H) defined in [4] is nonrestrictive, whereas the condition PE and also those studied in [5], [6], [19] are restrictive. For restrictive conditions C the problem arises of deciding when a given (n+1)-group is C-primitive. There is also problem of the reconstruction of C-creating (k+1)-groups, which makes sense for nonrestrictive conditions as well. This question was treated in [21]-[23], [11], [6]. In the present paper and in [20] we resolve the above problem in several new cases.

The notion of a nonrestrictive condition is closely related to a generalization of Hosszu theorem (cf. [4], Corollary 4). Namely, this generalized theorem states that an (n+1)-groupoid (G,f) is an (n+1)-group if and only if (G,f) is s-H-derived from a (k+1)-group. It is clear that this theorem remains true when we substitute the condition G for H. Moreover, we may substitute any nonrestrictive condition $C \geqslant G$ (and only such a condition). A natural question to ask at this point: Must such a condition be weaker than H? In other words

P r o b l e m 1. Does there exist a nonrestrictive condition essentially stronger than H?

P r o b l e m 2. Does there exist a nonrestrictive condition stronger than every nonrestrictive condition?

If the answer to Problem 2 is negative, then one may pose

P r o b l e m 3. Find a nonrestrictive condition such that any essentially stronger condition is restrictive.

Analogous question may be asked for restrictive conditions.

P r o b l e m 4. Does there exist a restrictive condition weaker than every restrictive condition?

If the answer to Problem 4 is negative, then we put

P r o b l e m 5. Find a restrictive condition such that any essentially weaker condition is nonrestrictive.

4. s-C-identity polyads

Now we formulate some notion which simplifies considerably the investigation of C-derived polyadic groups.

D e f i n i t i o n 5. Given an (n+1)-group (G,f), let $\langle e_1^k \rangle$ be a k-ad in G. The k-ad $\langle e_1^k \rangle$ is said to be an s-C-identity polyad in (G,f) if $\langle e_1^k \rangle$ is an identity polyad in some s-C-creating (k+1)-group of (G,f).

The introduction of this notion was inspired by [15], when we studied s-skew elements in polyadic groups with respect to the condition PE. One can also define s-C-inverse polyads and s-C-skew elements, which we will investigate in a separate paper. Note that a 1-PE-identity n-ad $\langle e_1^n \rangle$ in an (n+1)-group (G,f) is simply an identity n-ad in (G,f).

Consider an (n+1)-group (G,f). It is evident that any s-C-creating (k+1)-group of (G,f) determines some s-C-identity k-ads in (G,f) and conversely, any s-C-identity k-ad in (G,f) determines some s-C-creating (k+1)-groups of (G,f). Unfortunately, the correspondence between the set of all s-C-creating (k+1)-groups of (G,f) and the set of all s-C-identity k-ads in (G,f) is not necessarily bijective. Nevertheless, for certain conditions this is so.

D e f i n i t i o n 6. A condition C is said to be (s,k)-regular if for any (n+1)-group (G,f) the above correspondence is bijective.

It is easy to check that a condition stronger than a regular one is also regular, and a condition weaker than an irregular one is irregular.

By arguments of [21] or [15] the condition PE is (s-k)-regular for any s and k (cf. also [20]). In the following sections we will show that the condition H is (s,k)-irregular for $k \geqslant 2$, whereas for $k = 1$ it is (n,1)-regular (cf. Proposition 5 and Corollaries 8,9). We list now several problems about regularity of conditions between H and PE.

P r o b l e m 6. Does there exist an irregular condition stronger than every irregular condition?

If the answer to Problem 6 is negative, then one may pose

P r o b l e m 7. Find an irregular condition C such that any condition essentially stronger than C is regular.

P r o b l e m 8. Does there exist a regular condition weaker than every regular condition?

If the answer is no, we state

P r o b l e m 9. Find a regular condition C such that any condition essentially weaker than C is irregular.

As we mentioned above, for a regular condition C any s-C-creating (k+1)-group (G,g) of a given (n+1)-group (G,f) is determined by a unique s-C-identity k-ad $\langle e_1^k \rangle$ in (G,f). This may be false for an irregular condition. But for an arbitrary (regular or irregular) condition C any s-C-creating (k+1)-group (G,g) can be reconstructed from some s-C-identity k-ad $\langle e_1^k \rangle$ in (G,f) and some s-C-creating system $\langle \underline{\delta} ; \underline{c} \rangle$ of (G,f). Namely, we have the following

P r o p o s i t i o n 2. If an (n+1)-group (G,f) is $\langle \underline{\delta} ; \underline{c} \rangle$-derived from a (k+1)-group (G,g) with $\langle e_1^k \rangle$ as an identity k-ad, then the operation g is given by

$$g(x_1^{k+1}) = f_{(2)}(x_1, \gamma_1(x_2), \ldots, \gamma_k(x_{k+1}), d_1^{2n-k}) \tag{2}$$

where the mappings $\gamma_1, \ldots, \gamma_k$ are inverse of $\delta_1, \ldots, \delta_k$ and the (2n-k)-ad $\langle d_1^{2n-k} \rangle$ is an inverse of the k-ad $\langle \gamma_1(e_1), \ldots, \ldots, \gamma_k(e_k) \rangle$ in (G,f).

P r o o f . Let $(G,f) = \mathrm{der}^{s}_{\underline{\delta};\underline{c}}(G,g)$ (i.e. let f be given by (1) and let $\langle e_1^k \rangle$ be an identity k-ad in (G,g). First we consider the case $s > 1$.

Take elements $\tilde{d}_1,\ldots,\tilde{d}_{n-k} \in G$ such that the n-ad $\langle \delta_{k+1}(\tilde{d}_1),\delta_{k+2}(\tilde{d}_2),\ldots,\delta_n(\tilde{d}_{n-k}),c_1^k\rangle$ is an identity polyad in (G,g) and substitute $\tilde{d}_1,\ldots,\tilde{d}_{n-k}$ for $x_{k+2},\ldots,x_{n+1}$ in (1). Thus we get

$$(3)\qquad f(x_1^{k+1},\tilde{d}_1^{n-k}) = g(x_1,\delta_1(x_2),\ldots,\delta_k(x_{k+1})).$$

Take a (2n-k)-ad $\langle d_1^{2n-k}\rangle$ such that $\langle d_1^{2n-k}\rangle \underset{f}{=} \langle \tilde{d}_1^{n-k}\rangle$. Let γ_i denote the mapping inverse to δ_i for $i = 1,\ldots,k$. Then (3) becomes (2). Substituting the elements $e_1,\ldots,e_k$ for $x_2,\ldots,x_{k+1}$ in (2) we obtain

$$(4)\qquad x_1 = f_{(2)}(x_1,\gamma_1(e_1),\ldots,\gamma_k(e_k)d_1^{2n-k}).$$

It follows from (4) that $\langle d_1^{2n-k}\rangle$ is an inverse of the k-ad $\langle\gamma_1(e_1),\ldots,\gamma_k(e_k)\rangle$ in (G,f).

Next, let $s = 1$ (i.e., $n = k$). Note that

$$(5)\qquad f_{(2)}(x_1^{2k+1}) =$$
$$= g_{(4)}(x_1,\delta_1(x_2),\ldots,\delta_k(x_{k+1}),c_1^k,\delta_1(x_{k+2}),\ldots,\delta_k(x_{2k+1}),c_1^k).$$

Take elements $d_1,\ldots,d_k \in G$ such that the 3k-ad $\langle c_1^k,\delta_1(d_1),\ldots$ $\ldots,\delta_k(d_k),c_1^k\rangle$ is an identity polyad in (G,g) and put $d_1,\ldots,d_k$ in place of $x_{k+2},\ldots,x_{2k+1}$ in (5). As in the first part of the proof one can verify that g satisfies the assumptions of our theorem. This completes the proof of Proposition 2.

C o r o l l a r y 1. Let $\langle\underline{\delta};\underline{c}\rangle$ and $\langle\underline{\delta}';\underline{c}'\rangle$ be s-G-systems over (k+1)-groups (G,g) and (G,g'), resp., and let $\langle e_1^k\rangle$ be an identity k-ad in (G,g) as well as in (G,g'). If $\mathrm{der}^s_{\underline{\delta};\underline{c}}(G,g) = \mathrm{der}^s_{\underline{\delta}';\underline{c}'}(G,g')$ and $\delta_i = \delta_i'$ for $i = 1,\ldots,k$, then $g = g'$.

Fix an (n+1)-group (G,f) and a k-ad $\langle e_1^k\rangle$ in (G,f). It may be interesting to consider the question under what condition on $\gamma_1,\dots,\gamma_k$ and $d_1,\dots,d_{2n-k}$ formula (2) gives the operation g such that (G,g) is an s-C-creating (k+1)-group of (G,f) with $\langle e_1^k\rangle$ as an identity k-ad. The solution of this problem for certain conditions will be given in the following sections and in [20].

5. A-systems

Now we give a characterization of s-C-identity polyads for a certain condition C (this will be continued in [20]). First we define this condition, which is related to the condition H of [4].

D e f i n i t i o n 7. An s-system $\langle \delta_1^n;c_1^k\rangle$ over a (k+1)-group (G,g) is said to be an s-A-system if

1° δ_1 is an automorphism of (G,g);

2° $\delta_i = (\delta_1)^i$ for every $i = 1,\dots,n$;

3° for any $x\in G$ we have

$$g(\delta_n(x),c_1^k) = g(c_1^k,x); \tag{6}$$

4°

$$\langle \delta_1(c_1),\delta_1(c_2),\dots,\delta_1(c_k)\rangle \underset{g}{=} \langle c_1^k\rangle . \tag{7}$$

An (n+1)-group (G,f) A-derived from a (k+1)-group (G,g) will be denoted by $\mathrm{der}^s_{\delta;c_1^k}(G,g)$ (instead of $\mathrm{der}^s_{\delta_1^n;c_1^k}(G,g)$, since all mappings $\delta_1,\dots,\delta_n$ are powers of the same mapping δ; de facto $\delta = \delta_1$) or briefly $\mathrm{der}^s_{\delta;\underline{c}}(G,g)$. The A-creating system itself will be denoted by $\langle\delta;c_1^k\rangle$ or $\langle\delta;\underline{c}\rangle$.

From the definition of an A-system and by Proposition 1 we obtain

C o r o l l a r y 2. If $\langle\delta;a_1^k\rangle$ is an s-A-system over (G,g) and $\langle a_1^k\rangle \underset{g}{=} \langle b_1^k\rangle$, then $\langle\delta;b_1^k\rangle$ is also an s-A-system over (G,g) and $\mathrm{der}^s_{\delta;\underline{a}}(G,g) = \mathrm{der}^s_{\delta;\underline{b}}(G,g)$.

The condition A is a generalization of the well-known Hosszú condition for (binary) groups (cf. [21], [11], [10], [23], [1]). It differs from the condition H of [4] only in the formulation of 4^o, where equality (7) was of the form

(8) $$\delta_1(c_i) = c_i \quad \text{for every} \quad i = 1,\dots,k.$$

Any k-ad satisfying (8) also satisfies (7); so the condition H is stronger than A. The condition H is (s,k)-nonrestrictive for $k \geq 1$ (cf. [4]). which implies the (s,k)-nonrestrictivity of A. For $k \geq 2$ the condition A is essentially weaker than H. Indeed, this follows from

P r o p o s i t i o n 3. Let $\langle c_1^k \rangle$ and $\langle a_1^k \rangle$ be central polyads in a (k+1)-group (G,g). Define mappings $\delta_i : G \longrightarrow G$ by $\delta_i(x) = g_{(i)}(x, \underset{i}{\underbrace{a_1^k}})$ $(i = 1,\dots,n)$. Then $\langle \delta_1^n ; c_1^k \rangle$ is

1^o an A-system over (G,g) if and only if $\langle \underset{k}{\underbrace{a_1^k}} \rangle$ is an identity polyad in (G,g);

2^o an H-system over (G,g) if and only if $\langle a_1^k \rangle$ is an identity polyad in (G,g).

Recall that k-ads in a (k+1)-group (G,g) (to be exact: $\underset{g}{=}$ - equivalence classes of k-ads) may be treated as elements of the free covering group $(G^*, \cdot)$ of (G,g) and also as elements of the associated group $(G_0, \cdot)$ (which is actually a normal subgroup of $(G^*, \cdot)$; cf. [21], [18] and also [4], [5], [12], [13], [16]). For this reason we may interpret conditions 1^o and 2^o in terms of $(G^*, \cdot)$.

C o r o l l a r y 3. Let $k > 1$. If the associated group $(G_0, \circ)$ of a (k+1)-group (G,g) contains an element of order k from the center of the free covering group $(G^*, \cdot)$, then there exists an A-system over (G,g) which is not an H-system over (G,g).

We give the example of such a (k+1)-group. Let $k > 1$. Consider the (k+1)-group $(G,g) = \mathrm{der}_0^k(G, \cdot)$ where $(G, \cdot) = (Z_{k^2}, +)$ is the cyclic group of order k^2 (i.e., $g(x_1^{k+1}) \equiv x_1 + \dots + x_{k+1}$ (mod k^2)). Let $a_i = 1$ for $i = 1,\dots,k$ and let $\langle c_1^k \rangle$ be an arbitrary k-ad in G. Thus, by Proposition 2 $\langle \delta_1^n; c_1^k \rangle$ is an A-system and it is not an H-system over (G,g).

C o r o l l a r y 4. For $k>1$ the condition H is essentially stronger than A.

However, it is evident that for $k = 1$ this corollary is false. Namely,

P r o p o s i t i o n 4. For $k = 1$ the condition H is equal to A.

The main purpose of this section is to give a criterion for a k-ad to be an s-A-identity one.

L e m m a 1. If $\langle \delta; c_1^k \rangle$ is an A-system over (G,g) and $(G,f) = \mathrm{der}^s_{\delta;\underline{c}}(G,g)$, then the mapping δ is an automorphism of the (n+1)-group (G,f).

P r o o f . Indeed,

$$\delta(f(x_1^{n+1})) = \delta(g_{(s+1)}(x_1,\delta(x_2),\dots,\delta^n(x_{n+1}),c_1^k)) =$$
$$= g_{(s+1)}(\delta(x_1),\dots,\delta^{n+1}(x_{n+1}),c_1^k) = f(\delta(x_1),\dots,\delta(x_{n+1})).$$

T h e o r e m 1. A k-ad $\langle e_1^k \rangle$ is an s-A-identity polyad in an (n+1)-group (G,f) if and only if there exists an automorphism γ of (G,f) such that

$$(9)\quad \langle e_1,\gamma(e_2),\gamma^2(e_3),\dots,\gamma^{k-1}(e_k)\rangle \underset{f}{=} \langle \gamma(e_1),\gamma^2(e_2),\dots,\gamma^k(e_k)\rangle$$

and

$$(10)\qquad \gamma^k(x) = f_{(2)}(d_1^{2n-k},x,e_1,\gamma(e_2),\gamma^2(e_3),\dots,\gamma^{k-1}(e_k))$$

where the (2n-k)-ad $\langle d_1^{2n-k}\rangle$ is an inverse of $\langle e_1,\gamma(e_2),\dots$ $\dots,\gamma^{k-1}(e_k)\rangle$ in (G,f). Then $(G,f) = \mathrm{der}^s_{\delta;\underline{c}}(G,g)$ and $\langle e_1^k\rangle$ is an identity k-ad in (G,g) if and only if

$$(11)\quad g(x_1^{k+1}) = f_{(2)}(x_1,\gamma(x_2),\gamma^2(x_3),\dots,\gamma^k(x_{k+1}),d_1^{2n-k}),$$

$$(12)\qquad \delta = \gamma^{-1},$$

(13) $$\langle c_1,\gamma(c_2),\gamma^2(c_3),\ldots,\gamma^{k-1}(c_k)\rangle \underset{f}{=} \underset{f}{=} \underbrace{\langle e_1,\gamma(e_2),\gamma^2(e_3),\ldots,\gamma^{k-1}(e_k)\rangle}_{s+1}$$

for an automorphism γ of (G,f) satisfying (9) and (10).

P r o o f . Let $\langle e_1^k\rangle$ be an s-A-identity k-ad in an (n+1)-group (G,f). Thus there exists a (k+1)-group (G,g) such that $(G,f) = \mathrm{der}^s_{\delta;c}(G,g)$ and $\langle e_1^k\rangle$ is an identity k-ad in (G,g). According to Proposition 1 the operation g is given by (11) where $\gamma = \delta^{-1}$ and the (2n-k)-ad $\langle d_1^{2n-k}\rangle$ is an inverse of the k-ad $\langle\gamma(e_1),\ldots,\gamma^k(e_k)\rangle$ in (G,f). From Lemma 1 it follows that δ is an automorphism of (G,f), whence γ is an automorphism of (G,f) as well. Thus

$$f_{(2)}(\gamma(x_1),\gamma^2(x_2),\ldots,\gamma^{k+1}(x_{k+1}),\gamma(d_1),\gamma(d_2),\ldots,\gamma(d_{2n-k})) =$$

$$= \gamma(f_{(2)}(x_1,\gamma(x_2),\gamma^2(x_3),\ldots,\gamma^k(x_{k+1}),d_1^{2n-k})) = \gamma(g(x_1^{k+1})) =$$

$$= g(\gamma(x_1),\ldots,\gamma(x_{k+1})) = f_{(2)}(\gamma(x_1),\gamma^2(x_2),\ldots,\gamma^{k+1}(x_{k+1}),d_1^{2n-k}),$$

which shows that

(14) $$\langle d_1^{2n-k}\rangle \underset{f}{=} \langle\gamma(d_1),\gamma(d_2),\ldots,\gamma(d_{2n-k})\rangle.$$

Since δ is the inverse of γ, we get

(15) $$\langle d_1^{2n-k}\rangle \underset{f}{=} \langle\delta(d_1),\delta(d_2),\ldots,\delta(d_{2n-k})\rangle.$$

The 2n-ad $\langle d_1^{2n-k},\gamma(e_1),\gamma^2(e_2),\ldots,\gamma^k(e_k)\rangle$ is an identity polyad in (G,f); so the 2n-ad $\langle\delta(d_1),\delta(d_2),\ldots,\delta(d_{2n-k}),\delta\gamma(e_1),\ldots$ $\ldots,\delta\gamma^k(e_k)\rangle$ is also an identity polyad in (G,f). Thus, in view of (15) the k-ad $\langle e_1,\gamma(e_2),\gamma^2(e_3),\ldots,\gamma^{k-1}(e_k)\rangle$ is an inverse of $\langle d_1^{2n-k}\rangle$ in (G,f). Then by (14) we get (9). From the equality

$$x = g(e_1^k, x) = f_{(2)}(e_1, \gamma(e_2), \gamma^2(e_3), \ldots, \gamma^{k-1}(e_k), \gamma^k(x), d_1^{2n-k})$$

and using (9) we obtain (10). Note that

$$f(x_1^{n+1}) = g_{(s+1)}(x_1, \delta(x_2), \ldots, \delta^n(x_{n+1}), c_1^k) =$$

$$= g_{(s)}(x_1, \delta(x_2), \ldots, \delta^{n-1}(x_n), g(\delta^n(x_{n+1}), c_1^k)) =$$

$$= g_{(s)}(x_1, \delta(x_2), \ldots, \delta^{n-1}(x_n), f_{(2)}(\delta^n(x_{n+1}), \gamma(c_1), \gamma^2(c_2), \ldots,$$

$$\ldots, \gamma^k(c_k), d_1^{2n-k})) =$$

$$= g_{(s-1)}(x_1, \delta(x_2), \ldots, \delta^{n-k-1}(x_{n-k}), g(\delta^{n-k}(x_{n-k+1}), \ldots, \delta^{n-1}(x_n),$$

$$, f_{(2)}(\delta^n(x_{n+1}), \gamma(c_1), \gamma^2(c_2), \ldots, \gamma^k(c_k) d_1^{2n-k}))) =$$

$$= g_{(s-1)}(x_1, \delta(x_2), \ldots, \delta^{n-k-1}(x_{n-k}), f_{(4)}(\delta^{n-k}(x_{n-k+1}), \delta^{n-k}(x_{n-k+2}),$$

$$\ldots, \delta^{n-k}(x_n), \delta^{n-k}(x_{n+1}), \gamma^{k+1}(c_1), \gamma^{k+2}(c_2), \ldots, \gamma^{2k}(c_k), \underbrace{d_1^{2n-k}}_{2}) =$$

$$= \ldots = f_{(2s+1)}(x_1^{n+1}, \gamma^n(c_1), \gamma^{n+1}(c_2), \ldots, \gamma^{n+k}(c_k), \underbrace{d_1^{2n-k}}_{s+1}).$$

Thus the k-ad $\langle \gamma^n(c_1), \gamma^{n+1}(c_2), \ldots, \gamma^{n+k}(c_k) \rangle$ is an inverse of the (2ns+n-k)-ad $\langle \underbrace{d_1^{2n-k}}_{s+1} \rangle$ in (G,f). So we have the equality

$$(16) \langle \gamma^n(c_1), \gamma^{n+1}(c_2), \ldots, \gamma^{n+k}(c_k) \rangle \underset{f}{=} \langle \underbrace{e_1, \gamma(e_2), \ldots, \gamma^{k-1}(e_k)}_{s+1} \rangle,$$

which together with (9) gives (13).

Conversely, consider a k-ad $\langle e_1^k \rangle$ in G and an automorphism γ of (G,f) satisfying (9) and (10) where the (2n-k)-ad $\langle d_1^{2n-k} \rangle$ is an inverse of the k-ad $\langle e_1, \gamma(e_2), \gamma^2(e_3), \ldots$ $\ldots, \gamma^{k-1}(e_k) \rangle$ in (G,f). From the definition of $\langle d_1^{2n-k} \rangle$ taking into account (9) we get (14). We may write the equality (10) in the following form

$$(17)\qquad \langle \gamma^k(x), d_1^{2n-k}\rangle \underset{f}{=} \langle d_1^{2n-k}, x\rangle.$$

Define a (k+1)-ary operation g by (11). Then

$$g(g(x_1^{k+1}), x_{k+2}^{2k+1}) = f_{(2)}(f_{(2)}(x_1, \gamma(x_2), \gamma^2(x_3), \ldots$$

$$\ldots, \gamma^k(x_{k+1}), d_1^{2n-k}), \gamma(x_{k+2}), \gamma^2(x_{k+3}), \ldots, \gamma^k(x_{2k+1}) d_1^{2n-k}) =$$

$$= f_{(2)}(x_1, f_{(2)}(\gamma(x_2), \ldots$$

$$\ldots, \gamma^k(x_{k+1}), \gamma^{k+1}(x_{k+2}), d_1^{2n-k}), \gamma^2(x_{k+3}), \ldots, \gamma^k(x_{2k+1}), d_1^{2n-k}) =$$

$$= f_{(2)}(x_1, \gamma(f_{(2)}(x_2^{k+2}, d_1^{2n-k}), \gamma^2(x_{k+3}), \ldots$$

$$\ldots, \gamma^k(x_{2k+1}), d_1^{2n-k}) = g(x_1, g(x_2^{k+2}), x_{k+3}^{2k+1}),$$

which proves that g is a (1,2)associative operation. Thus from Proposition 1 of [3] we infer that (G,g) is a (k+1)-group. Furthermore, $\langle e_1^k\rangle$ is an identity k-ad in this (k+1)-group. Define a mapping δ by (12) and take a sequence $c_1, \ldots, c_k \in G$ that satisfies (13). We claim that $\langle \delta; c_1^k\rangle$ is an A-system over (G,g). Indeed

$$\gamma(g(x_1^{k+1})) = \gamma(f_{(2)}(x_1, \gamma(x_2), \ldots, \gamma^k(x_{k+1}), d_1^{2n-k})) =$$

$$= f_{(2)}(\gamma(x_1), \gamma(\gamma(x_2)), \ldots, \gamma^k(\gamma(x_{k+1})), d_1^{2n-k}) =$$

$$= g(\gamma(x_1), \ldots, \gamma(x_{k+1})),$$

i.e., γ (therefore also δ) is an automorphism of (G,g). Using (9) and (13) we get

$$(18)\qquad \langle c_1, \gamma(c_2), \gamma^2(c_3), \ldots, \gamma^{k-1}(c_k)\rangle \underset{f}{=}$$

$$\underset{f}{=} \langle \gamma(c_1), \gamma^2(c_2), \ldots, \gamma^k(c_k)\rangle.$$

By (18), the definition of g and (14) we have

(19) $$\langle c_1^k \rangle \underset{g}{=} \langle \gamma(c_1),\dots,\gamma(c_k)\rangle,$$

from which one can easily get (7). By (17) we have

(20) $$\langle x, d_1^{2n-k} \rangle \underset{f}{=} \langle d_1^{2n-k}, \delta^k(x)\rangle,$$

which is turn gives

(21) $$\langle \delta^k(x), e_1, \gamma(e_2), \gamma^2(e_3), \dots, \gamma^{k-1}(e_k)\rangle \underset{f}{=}$$

$$\underset{f}{=} \langle e_1, \gamma(e_2), \gamma^2(e_3), \dots, \gamma^{k-1}(e_k), x\rangle$$

(since the k-ad $\langle e_1, \gamma(e_2), \dots, \gamma^{k-1}(e_k)\rangle$ is an inverse of the (2n-k)-ad $\langle d_1^{2n-k}\rangle$ in (G,f). Now we use (21) and (17) to prove (6). In fact,

$$g(\delta^n(x), c_1^k) = f_{(2)}(\delta^n(x), \gamma(c_1), \dots, \gamma^k(c_k), d_1^{2n-k}) =$$

$$= f_{(3)}(\delta^k(\delta^{n-k}(x)), \underbrace{e_1, \gamma(e_2), \gamma^2(e_3), \dots, \gamma^{k-1}(e_k)}_{s+1}, d_1^{2n-k}) =$$

$$= f(e_1, \gamma(e_2), \gamma^2(e_3), \dots, \gamma^{k-1}(e_k), \delta^{n-k}(x), \underbrace{e_1, \gamma(e_2), \dots, \gamma^{k-1}(e_k)}_{s-1}) =$$

$$= \dots = f(\underbrace{e_1, \gamma(e_2), \dots, \gamma^{k-1}(e_k)}_{s}, x) =$$

$$= f_{(3)}(\underbrace{e_1, \gamma(e_2), \dots, \gamma^{k-1}(e_k)}_{s+1}, d_1^{2n-k}, x) =$$

$$= f_{(2)}(c_1, \gamma(c_2), \dots, \gamma^{k-1}(c_k), \gamma^k(x), d_1^{2n-k}) = g(c_1^k, x).$$

Then $\langle \delta ; c_1^k\rangle$ is an A-system over (G,g).

Finally, as in the first part of the proof one shows that

$$g_{(s+1)}(x_1, \delta(x_2), \dots, \delta^n(x_{n+1}), c_1^k) =$$

$$= g_{(s)}(x_1, \delta(x_2), \dots, \delta^{n-1}(x_2), f_{(2)}(\delta^n(x_{n+1}),$$

$$,\gamma(c_1),\gamma^2(c_2),\dots,\gamma^k(c_k),d_1^{2n-k})) =$$

$$= g_{(s)}(x_1,\delta(x_2),\dots,\delta^{n-1}(x_n),f(\delta^n(x_{n+1}),\underbrace{e_1,\gamma(e_2),\dots,\gamma^{k-1}(e_k)}_{s})) =$$

$$= g_{(s-1)}(x_1,\delta(x_2),\dots,\delta^{n-k-1}(x_{n-k}),f(\delta^{n-k}(x_{n-k+1}),\dots,\delta^{n-k}(x_{n+1}),$$

$$,\underbrace{e_1,\gamma(e_2),\dots,\gamma^{k-1}(e_k)}_{s-1})) = \dots = f(x_1^{n+1}),$$

i.e., $\mathrm{der}^s_{\delta;\underline{c}}(G,g) = (G,f)$. This completes the proof of Theorem 1.

So, by Theorem 1 we obtain the complete description of s-A-creating (k+1)-groups of a given (n+1)-group (G,f). Any such (k+1)-group is determined by an appropriate k-ad of G and an appropriate automorphism of (G,f).

Note that in Theorem 1 we require only that $\langle c_1^k \rangle$ satisfies (13), thus we have some freedom in choosing it. For instance, we may do this in the following way.

C o r o l l a r y 5. Let γ be an automorphism of an (n+1)-group (G,f), assume that γ satisfies (9) and (10) for some k-ad $\langle e_1^k \rangle$. If a (k+1)-ary operation g is given by (11), a mapping δ by (12) and a k-ad $\langle c_1^k \rangle$ by the formulas

$$c_1 = f(\underbrace{e_1,\gamma(e_2),\gamma^2(e_3),\dots,\gamma^{k-1}(e_k)}_{s},e_1),\ c_2 = e_2,\dots,c_k = e_k,$$

then $\langle \delta; c_1^k \rangle$ is an s-A-system over (G,g), $\langle e_1^k \rangle$ is an identity k-ad in (G,g) and $\mathrm{der}^s_{\delta;\underline{c}}(G,g) = (G,f)$.

We now use Corollary 5 to show the irregularity of certain conditions.

P r o p o s i t i o n 5. The condition H is (s,k)-irregular for $k \geq 2$.

P r o o f . Let $(G,\cdot) = (S_k,\cdot)$ be the symmetric group of degree k ($k \geq 2$). Form the (n+1)-group $(G,f) = \mathrm{der}_e^n(G,\cdot)$ where e is the neutral element of $(G,\cdot)$ (i.e., $f(x_1^{n+1}) = x_1 \cdot \dots \cdot x_{n+1}$). A mapping $\gamma : G \longrightarrow G$ given by $\gamma(x) = a\cdot x\cdot a^{k-1}$, where a is an element of order k in $(G,\cdot)$, is an automorphism of (G,f). Let $e_i = e$ for $i = 1,\ldots,k$. It is easy to check that the k-ad $\langle e_1^k \rangle = \langle \overset{(k)}{e} \rangle$ and the automorphism γ satisfy the assumption of Corollary 5. Define a $(k+1)$-ary operation g, a mapping δ and a k-ad $\langle c_1^k \rangle$ as in Corollary 5. Then

(22) $$g(x_1^{k+1}) = x_1 \cdot a \cdot x_2 \cdot \ldots \cdot a \cdot x_{k+1},$$

(23) $$\delta(x) = a^{k-1} \cdot x \cdot a,$$

(24) $$c_i = e \quad \text{for} \quad i = 1,\ldots,n.$$

It is evident that the so-defined system $\langle \delta ; c_1^k \rangle$ is an s-H-system over (G,g). Thus (G,f) is H-derived from the group (G,g) and $\langle e_1^k \rangle$ is an identity k-ad in (G,g). On the other hand, let γ' be the identity mapping of S_k onto itself. The k-ad $\langle e_1^k \rangle$ and the automorphism γ also satisfy the assumption of Corollary 5. Define another $(k+1)$-ary operation g', a mapping δ' and a k-ad $\langle c'_1,\ldots,c'_k \rangle$ also satisfying the assumption of Corollary 5. Then

(25) $$g'(x_1^{k+1}) = x_1 \cdot \ldots \cdot x_{k+1},$$

(26) $$\delta'(x) = x,$$

(27) $$c'_i = e \quad \text{for} \quad i = 1,\ldots,n.$$

The system $\langle \delta' ; \underline{c}' \rangle$ is even a PE-system over (G,g) and (G,f) is PE-derived from (G,g). Moreover, $\langle e_1^k \rangle$ is an identity k-ad in (G,g). Thus this s-H-identity k-ad $\langle e,\ldots,e \rangle$ in (G,f) correspondes to two distinct s-H-creating $(k+1)$-groups of (G,f). This completes the proof of Proposition 5.

As the condition A is weaker than H, we have

C o r o l l a r y 6. The condition A is (s,k)-irregular for $k \geqslant 2$.

It is worth while to add that Proposition 4 and Corollary 6 hold for every $s = 1,2,\ldots$, in particular for $s = 1$.

6. A-systems. The binary case

The problem studied in section 5° simplifies considerably in the binary case ($k = 1$). We must treat cases $n>1$ and $n=1$ separately. Assume first $n>1$. In this case formula (11), taking into account (9) and the fact that the $(n-1)$-ad $\langle \overset{(n-2)}{e}, \bar{e}\rangle$ is inverse to e in (G,f), takes the form

$$g(x_1^2) = f(x_1, \overset{(n-2)}{e}, \bar{e}, x_2) \tag{28}$$

where e is the neutral element in the group (G,g). Hence we obtain (cf. Proposition 3 of [4])

C o r o l l a r y 7. If $(G,f) = \mathrm{der}^n_{\delta;c}(G,g)$, then $(G,g) = \overline{\mathrm{Ret}}{}^n_e(G,f)$ where e is the neutral element of the group (G,e).

Note that the above-mentioned binary operation g is the same as that of Proposition 3 of [4].

Corollary 7 is false for $k>1$. As is shown in [4], for every $k>1$ and an appropriate n there exists an $(n+1)$-group H-derived from a $(k+1)$-group which is not a retract of this $(n+1)$-group.

C o r o l l a r y 8. The condition A is $(n,1)$-regular for $n>1$.

Theorem 1 shows that s-A-creating $(k+1)$-groups of a given $(n+1)$-group (G,f) depend on s-A-identity k-ads and some automorphisms of (G,f). In the case of $k = 1$ these groups depend only on n-A-identity elements, while automorphisms appearing in Theorem 1 are determined by these elements.

P r o p o s i t i o n 6. In an $(n+1)$-group (G,f) any element $e \in G$ is an n-A-identity element. Then $(G,f) = \mathrm{der}^n_{\delta;c}(G,g)$, where e is the neutral element of the group (G,g), if and only if $(G,g) = \overline{\mathrm{Ret}}{}^n_e(G,f)$, $\delta(x) = f(e, x, \overset{(n-2)}{e}, \bar{e})$, $c = f(\overset{(n+1)}{e})$.

P r o o f . Let e be an arbitrary element of (G,f) satisfying (9) and (10) (note that the $(n-1)$-ad $\langle \overset{(n-2)}{e}, \bar{e}\rangle$ is an inverse of e in (G,f). Thus, by Theorem 1 the element e is an n-A-identity element in (G,f). The second part of the theorem follows immediately from Theorem 1.

Now we consider the case $n = 1$. Formula (11) takes the form

$$g(x_1^2) = f_{(2)}(x_1,d,x_2), \tag{31}$$

where d is the inverse element of e in (G,f) and e is the neutral element in (G,g).

Corollary 9. If $(G,f) = \mathrm{der}^n_{\delta;c}(G,g)$, then $(G,g) = \mathrm{Ret}_d^{1,2}(G,f)$ where d is the inverse of e in (G,f) and e is the neutral element of the group (G,g).

Corollary 10. The condition A is (1,1)-regular.

Proposition 6 also changes slightly, but the idea of the proof is the same.

Proposition 7. In a group (G,f) any element $e \in G$ is a 1-A-identity element. Then $(G,f) = \mathrm{der}^1_{\delta;c}(G,g)$, where e is the neutral element of the group (G,g), if and only if $(G,g) = \mathrm{Ret}_d^{1,2}(G,f)$, $\delta(x) = f_{(2)}(e,x,d)$, $c = f(\overset{(2)}{e})$ where d is the inverse of e in $G(f)$.

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