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APPROXIMATION OF PERIODIC FUNCTIONS BY THE EULER AND BOREL MEANS OF FOURIER SERIES

*Dedicated to the memory
of Professor Roman Sikorski*

1. Preliminaries

Let L and C be the spaces composed of all 2π-periodic complex-valued functions Lebesgue-integrable on the interval $<0,2\pi>$ and all 2π-periodic complex-valued functions continuous in $<0,2\pi>$, respectively. Introduce in these spaces the usual norms

$$\|f\|_L = \int_0^{2\pi} |f(t)|\,dt \quad \text{if} \quad f \in L,$$

$$\|f\|_C = \sup\{|f(t)| : 0 \le t \le 2\pi\} \quad \text{if} \quad f \in C.$$

Denote by M the set of all bounded functions belonging to L.

Suppose that Φ is a continuous, convex and strictly increasing function in the interval $<0,\infty)$, such that $\Phi(0) = 0$. Given any function $f \in M$, let us denote by $V_\Phi(f;a,b)$ the total Φ-variation of f on the interval $<a,b>$, defined as the upper bound of the set of non-negative numbers

$$\sum_{k=0}^{m-1} \Phi(|f(x_{k+1}) - f(x_k)|)$$

corresponding to all partitions $a \leqslant x_0 < x_1 < \dots < x_m \leqslant b$ of $<a,b>$. The class of all 2π-periodic functions of bounded Φ-variation on $<0,2\pi>$ will be signified by BV_Φ.

Considering a function $f \in M$ and a fixed positive integer n, let us introduce the modulus of variation of f on the interval $<a,b>$

$$v(n;f,a,b) = \sup_{\pi_n} \sum_{k=0}^{n-1} |f(x_{2k+1}) - f(x_{2k})|,$$

where the supremum is taken over all partitions π_n of $<a,b>$ into n non-overlapping intervals $a \leqslant x_0 < x_1 \leqslant x_2 < \dots \leqslant x_{2n-2} < x_{2n-1} \leqslant b$. Write $v(0;f,a,b) = 0$. Some basic properties of this modulus can be found in [2]. For instance, in the case of $f \in BV_\Phi$, the inequality

$$(1) \qquad v(n;f,a,b) \leqslant n\Phi^{-1}\left(\frac{1}{n} V_\Phi(f;a,b)\right) \qquad (n \in N)$$

holds for every interval $<a,b>$. Denoting by $\omega(\delta;f)$ $(\delta \geqslant 0)$ the modulus of continuity of $f \in C$, we have

$$(2) \qquad v(n;f,a,b) \leqslant 2n\omega\left(\frac{b-a}{n}\,;\, f\right)$$

for every integer $n \geqslant 1$ and every interval $<a,b>$ (see also [7]).

Given any function $f \in L$, let $S_n[f]$ $(n+1 \in N)$ be the n-th partial sum of its Fourier series. Denote by $E_{n,q}[f]$ and $B_r[f]$ the Euler and the Borel means of this series which are defined by

$$E_{n,q}[f](x) = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} S_k[f](x) \quad (q > 0,\ n+1 \in N),$$

$$B_r[f](x) = e^{-r} \sum_{k=0}^{\infty} \frac{1}{k!} r^k S_k[f](x) \quad (r > 0).$$

In this paper we shall give some estimates for the rate of convergence of the above means at the points x at which the finite limit

$$(3)\qquad S(f,x) = \lim_{t\to 0} \frac{1}{2}\left\{f(x+t) + f(x-t)\right\}$$

exists. Also, some results concerning the order of uniform approximation of continuous functions by the Euler and Borel means will be deduced.

The symbols c_j, $c_j(q,\dots)$, $j = 0,1,2,\dots$ occuring below will mean some positive absolute constants or positive constants depending only on the indicated parameters $q,\dots$

2. Auxiliary results

Let $(\lambda_k(\varrho))$, $k+1\in N$, be a sequence of non-negative factors defined in a set G of positive numbers, with the accumulation point $+\infty$. Consider the Dirichlet kernels

$$D_k(t) = \frac{\sin\left(k+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t} \qquad (-\infty < t < \infty,\quad k+1\in N)$$

and write

$$(4)\qquad K_\varrho(t) = \frac{1}{A(\varrho)}\sum_{k=0}^{\infty}\lambda_k(\varrho)D_k(t),\quad A(\varrho) = \sum_{k=0}^{\infty}\lambda_k(\varrho).$$

L e m m a 1. Suppose that φ is a complex-valued function measurable and bounded in an interval $\langle 0,\delta\rangle$, $0<\delta\leq\pi$, such that $\varphi(0) = 0$. If, for the kernel defined by (4), there is

$$(5)\qquad \left|\Lambda_\varrho(x)\right| \equiv \left|\int_x^\delta K_\varrho(t)\,dt\right| \leq \frac{c_0}{\varrho x} \quad (0<x\leq\delta,\ \varrho\in G),$$

then

$$\left|\int_{\delta/n}^{\delta} \varphi(t)K_\varrho(t)dt\right| \leq \left(\frac{c_o n}{\delta\varrho} + \frac{\pi}{2}\right) \frac{v(n-1;\varphi,0,\delta)}{n-1} + \left(\frac{c_o n}{\delta\varrho} + \pi\right) \sum_{k=1}^{n-1} \frac{v(k;\varphi,0,k\delta/n)}{k^2} \qquad (\varrho > 1),$$

where $n = \lceil\varrho\rceil \equiv \inf\{j \geq \varrho : j \in N\}$.

P r o o f . Putting $t_k = k\delta/n$ $(k=1,2,\ldots,n)$, we have

$$\int_{\delta/n}^{\delta} \varphi(t)K_\varrho(t)dt = \sum_{k=1}^{n-1} \varphi(t_k) \int_{t_k}^{t_{k+1}} K_\varrho(t)dt + \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} \left\{\varphi(t) - \varphi(t_k)\right\} K_\varrho(t)dt \equiv I_1 + I_2, \quad \text{say.}$$

By the Abel transformation,

$$I_1 = \varphi(t_1) \int_{t_1}^{\delta} K_\varrho(t)dt + \sum_{k=1}^{n-2} \left\{\varphi(t_{k+1}) - \varphi(t_k)\right\} \int_{t_{k+1}}^{\delta} K_\varrho(t)dt.$$

Consequently, in view of (5), we get

$$|I_1| \leq |\varphi(t_1)|\,|\Lambda_\varrho(t_1)| + \sum_{k=1}^{n-2} |\varphi(t_{k+1}) - \varphi(t_k)|\,|\Lambda_\varrho(t_{k+1})| \leq$$

$$\leq \frac{c_o n}{\delta\varrho} \left\{|\varphi(t_1) - \varphi(0)| + \sum_{k=1}^{n-2} |\varphi(t_{k+1}) - \varphi(t_k)| \frac{1}{k+1}\right\}.$$

Applying once more the Abel transformation we obtain

$$|I_1| \leq \frac{c_o n}{\delta\varrho} \left\{|\varphi(t_1)-\varphi(0)| + \sum_{k=1}^{n-3} \sum_{j=1}^{k} |\varphi(t_{j+1})-\varphi(t_j)| \left(\frac{1}{k+1} - \frac{1}{k+2}\right) + \sum_{j=1}^{n-2} |\varphi(t_{j+1})-\varphi(t_j)| \frac{1}{n-1}\right\} \leq$$

$$\leq \frac{c_0 n}{\delta\varrho}\left\{v(1;\varphi,0,t_1) + \sum_{k=1}^{n-3} \frac{v(k;\varphi,0,t_{k+1})}{(k+1)^2} + \frac{v(n-2;\varphi,0,\delta)}{n-1}\right\} \leq$$

$$\leq \frac{c_0 n}{\delta\varrho}\left\{\sum_{k=1}^{n-2} \frac{v(k;\varphi,0,t_k)}{k^2} + \frac{v(n-1;\varphi,0,\delta)}{n-1}\right\}.$$

Since

$$|K_\varrho(t)| \leq \frac{1}{A(\varrho)} \sum_{k=0}^{\infty} \lambda_k(\varrho)\,|D_k(t)| \leq \frac{1}{2\sin\frac{1}{2}t} \leq \frac{\pi}{2t} \qquad (0 < t < \pi),$$

we have

$$|I_2| = \left|\sum_{k=1}^{n-1} \int_0^{\delta/n} \left\{\varphi(t+t_k) - \varphi(t_k)\right\} K_\varrho(t+t_k)dt\right| \leq$$

$$\leq \frac{n\pi}{2\delta} \int_0^{\delta/n} \sum_{k=1}^{n-1} |\varphi(t+t_k) - \varphi(t_k)|\, \frac{1}{k}\, dt =$$

$$= \frac{n\pi}{2\delta} \int_0^{\delta/n} \left\{ \sum_{k=1}^{n-2} \sum_{j=1}^{k} |\varphi(t+t_j) - \varphi(t_j)| \left(\frac{1}{k} - \frac{1}{k+1}\right) + \right.$$

$$\left. + \sum_{j=1}^{n-1} |\varphi(t+t_j) - \varphi(t_j)|\, \frac{1}{n-1}\right\} dt \leq$$

$$\leq \frac{\pi}{2}\left\{\sum_{k=1}^{n-2} \frac{v(k;\varphi,0,t_{k+1})}{k(k+1)} + \frac{v(n-1;\varphi,0,t_{n-1})}{n-1}\right\} \leq$$

$$\leq \pi \sum_{k=2}^{n-1} \frac{v(k;\varphi,0,t_k)}{k^2} + \frac{\pi}{2}\,\frac{v(n-1;\varphi,0,\delta)}{n-1}.$$

Collecting the results we get our thesis.

L e m m a 2. If φ is of bounded Φ-variation on the interval $<0,\delta>$, then

$$\sum_{k=1}^{n-1} \frac{v(k;\varphi,0,k\delta/n)}{k^2} \leqslant 8 \sum_{k=1}^{n-1} \frac{1}{k} \Phi^{-1}\left(\frac{k}{n} V_\Phi(\varphi;0,\frac{\delta}{k})\right) \quad (n \geqslant 2).$$

Moreover, under the assumptions $\lim_{t\to 0^+} \varphi(t) = \varphi(0) = 0$ and

$$(6)' \qquad \sum_{k=1}^{\infty} \frac{1}{k} \Phi^{-1}\left(\frac{1}{k}\right) < +\infty,$$

we have

$$\lim_{n\to\infty} \sum_{k=1}^{n-1} \frac{1}{k} \Phi^{-1}\left(\frac{k}{n} V_\Phi\left(\varphi;0,\frac{\delta}{k}\right)\right) = 0.$$

P r o o f . In view of (1),

$$\sum_{k=1}^{n-1} \frac{v(k;\varphi,0,k\delta/n)}{k^2} \leqslant \sum_{k=1}^{n-1} \frac{1}{k} \Phi^{-1}\left(\frac{1}{k} V_\Phi\left(\varphi;0, \frac{k\delta}{n}\right)\right) \leqslant$$

$$\leqslant 2 \int_{\delta/n}^{\delta} \frac{1}{t} \Phi^{-1}\left(\frac{2\delta}{nt} V_\Phi(\varphi;0,t)\right) dt \leqslant$$

$$\leqslant 2 \sum_{k=1}^{n-1} \frac{1}{k} \Phi^{-1}\left(\frac{4k}{n} V_\Phi\left(\varphi;0, \frac{\delta}{k}\right)\right),$$

and the desired inequality follows. Since φ is right-sidely continuous at the point $t = 0$, we have $\lim_{t\to 0^+} V_\Phi(\varphi;0,t) = 0$. This and the condition (6) imply our assertion by simple calculation.

Suppose now that φ is of class $C \cap BV_\Phi$ and write

$$\Omega_\Phi(\delta;\varphi) = \sup\left\{V_\Phi(\varphi;t',t''): |t' - t''| \leqslant \delta\right\} \quad (0 \leqslant \delta < \infty).$$

Obviously, Ω_Φ is a non-decreasing function of δ and

$$\Phi^{-1}(\Omega_\Phi(\delta;\varphi)) \geq \omega(\delta;\varphi) \quad \text{when} \quad 0 \leq \delta < \infty .$$

As known, for an arbitrary $\varepsilon > 0$ there exists an $\eta > 0$ such that $V_\Phi(\varphi;a,b) < \varepsilon$ if $|b-a| < \eta$ (the proof runs as in [5], Lemma 3). Consequently,

$$\lim_{\delta\to 0+} \Omega_\Phi(\delta;\varphi) = 0. \tag{7}$$

If, in addition, the function Φ satisfies the condition

$$\Phi(2u) \leq \varkappa\Phi(u) \quad (u>0, \varkappa = \text{const.}), \tag{8}$$

then

$$\Omega_\Phi(\delta;\varphi) \leq \varkappa\, w(\delta;V_\Phi(\varphi)) \quad (0 \leq \delta \leq \pi), \tag{9}$$

where $w(\delta;V_\Phi(\varphi))$ denotes the modulus of continuity of the continuous function $V_\Phi(\varphi)$ defined by $V_\Phi(\varphi)(t) = V_\Phi(\varphi;-\pi,t)$ $(t \geq -\pi)$, in the interval $\langle -\pi, \pi \rangle$.

3. Approximation by the Euler means

Given any function $f \in L$ and a fixed point x for which the limit (3) is finite, let us introduce the 2π-periodic function φ_x defined by

$$\varphi_x(t) = \begin{cases} f(x+t) + f(x-t) - 2S(f,x) & \text{when } 0 < |t| \leq \pi, \\ 0 & \text{when } t = 0. \end{cases} \tag{10}$$

It is easy to see that

$$E_{n,q}[f](x) - S(f,x) = \frac{1}{\pi}\int_0^\pi \varphi_x(t)K_n(t)\,dt,$$

with

$$K_n(t) = K_{n,q}(t) = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k}D_k(t) \qquad (q>0,\ n+1 \in N).$$

Our main result concerning the Euler means can be stated as follows.

Theorem 1. (i) Suppose that $f \in L$ and that, at a fixed point x, the limit (3) is finite. If there exists a positive number $\delta \leqslant \pi$ such that f is bounded in the interval $<x-\delta, x+\delta>$, then, for $n \geqslant 2$, we have

$$(11)\quad |E_{n,q}[f](x) - S(f,x)| \leqslant c_1(q,\delta)\frac{v(n-1;\varphi_x,0,\delta)}{n-1} + c_2(q,\delta)\sum_{k=1}^{n-1}\frac{v(k;\varphi_x,0,k\delta/n)}{k^2} + \frac{1}{4\delta}\left(\frac{q^2+2q\cos\delta+1}{q^2+2q+1}\right)^{n/2}\|\varphi_x\|_L.$$

(ii) If $f \in M$, then, at every point x at which the finite limit (3) exists, the estimate (11) with $\delta = \pi$ remains valid. Moreover, the last term on the right of this inequality can be dropped.

Proof. Let us write

$$(12)\quad E_{n,q}[f](x) - S(f,x) = \frac{1}{\pi}\left(\int_0^{\delta/n} + \int_{\delta/n}^{\delta} + \int_{\delta}^{\pi}\right)\varphi_x(t)K_n(t)dt.$$

Since

$$|K_n(t)| \leqslant \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left(k+\frac{1}{2}\right) \leqslant n+\frac{1}{2},$$

we have

$$\left|\int_0^{\delta/n}\varphi_x(t)K_n(t)dt\right| \leqslant \frac{3}{2}\delta\ v\left(1;\varphi_x,0,\frac{\delta}{n}\right).$$

To estimate the second integral on the right-hand side of (12), we shall verify that the kernel K_n satisfies the condition (5) with $\varphi = n$, $c_0 = 2\pi(1+q)$. Indeed, under the assumptions $0 < x \leqslant \delta \leqslant \pi$, $q > 0$ and $n \in N$, we have

$$\left|\int_x^\delta K_n(t)dt\right| \leqslant \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left|\int_x^\delta D_k(t)dt\right| \leqslant$$

$$\leqslant \frac{\pi}{x(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}\frac{q^{n-k}}{k+\frac{1}{2}} \leqslant \frac{2\pi}{x(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}\frac{q^{n-k}}{k+1} =$$

$$= \frac{2\pi}{x(n+1)(1+q)^n}\sum_{k=0}^{n}\binom{n+1}{k+1}q^{n-k} = \frac{2\pi}{x(n+1)(1+q)^n}\sum_{k=1}^{n+1}\binom{n+1}{k}q^{n+1-k} =$$

$$= \frac{2\pi\left\{(1+q)^{n+1} - q^{n+1}\right\}}{x(n+1)(1+q)^n} \leqslant \frac{2\pi(1+q)}{n\,x}\,.$$

Consequently, applying Lemma 1, we get

$$\left|\int_{\delta/n}^{\delta}\varphi_x(t)K_n(t)dt\right| \leqslant \left(\frac{2\pi(1+q)}{\delta} + \frac{\pi}{2}\right)\frac{v(n-1;\varphi_x,0,\delta)}{n-1} +$$

$$+\left(\frac{2\pi(1+q)}{\delta} + \pi\right)\sum_{k=1}^{n-1}\frac{v(k;\varphi_x,0,k\delta/n)}{k^2} \qquad (n \geqslant 2).$$

Finally, let us note that the kernel K_n can be represented in the form

$$K_n(t) = \left(\frac{q^2 + 2q\cos t + 1}{q^2 + 2q + 1}\right)^{n/2}\frac{\sin\left(n\,\theta_t + \frac{1}{2}t\right)}{2\sin\frac{1}{2}t} \qquad (q > 0),$$

where $\theta_t \in (-\pi,\pi)$ is uniquely determined by the following relations

$$q\sin\theta_t = \sin(t - \theta_t),\ \operatorname{sign}\theta_t = \operatorname{sign} t,\ |\theta_t| < |t| \leqslant \pi$$

(see [3], Lemma 1.3). Therefore, if $0<\delta<\pi$, then

$$\left|\int_{\delta}^{\pi}\varphi_x(t)K_n(t)\,dt\right|\leqslant\frac{\pi}{4\delta}\left(\frac{q^2+2q\cos\delta+1}{q^2+2q+1}\right)^{n/2}\|\varphi_x\|_L.$$

Collecting the above results and applying (12) we get the desired assertions with $c_1(q,\delta)\leqslant 2(1+q)/\delta+1/2$, $c_2(q,\delta)\leqslant$ $\leqslant 2(1+q)/\delta+1+3\delta/2\pi$.

Remark 1. Theorem 1 (ii) remains valid for $q=0$, i.e. for the sums $E_{n,0}[f]=S_n[f]$ (see also [7]).

Suppose now that f is of bounded Φ-variation on the interval $\langle x-\delta, x+\delta\rangle$ and introduce the 2π-periodic functions φ_x^+, φ_x^- defined by

$$\varphi_x^{\pm}(t)=\begin{cases} f(x\pm t)-f(x\pm 0) & \text{when } 0<|t|\leqslant\pi,\\ 0 & \text{when } t=0,\end{cases}$$

where $f(x\pm 0)$ denote the one-sided limits of f at the point x. Obviously, in view of (10), $\varphi_x(t)=\varphi_x^+(t)+\varphi_x^-(t)$ and both the functions φ_x^+ and φ_x^- are of bounded Φ-variation on the interval $\langle 0,\delta\rangle$. Moreover, for every interval $\langle a,b\rangle\subset\langle 0,\delta\rangle$ and all positive integers n, we get

$$v(n;\varphi_x,a,b)\leqslant v(n;\varphi_x^+,a,b)+v(n;\varphi_x^-,a,b).$$

Consequently, Theorem 1, Lemma 2 and the inequality (1) yield

Corollary 1. Let $f\in L$ and let there exist a positive number $\delta<\pi$ such that f is of bounded Φ-variation on the interval $\langle x-\delta,x+\delta\rangle$. Then, for $n\geqslant 2$ and $q>0$, we have

$$\left|E_{n,q}[f](x)-\frac{f(x+0)+f(x-0)}{2}\right|\leqslant\frac{1}{4\delta}\left(\frac{q^2+2q\cos\delta+1}{q^2+2q+1}\right)^{n/2}\|\varphi_x\|_L+$$

$$+c_3(q,\delta)\sum_{k=1}^{n-1}\frac{1}{k}\left\{\Phi^{-1}\left(\frac{k}{n}V_\Phi\left(\varphi_x^+;0,\frac{\delta}{k}\right)\right)+\Phi^{-1}\left(\frac{k}{n}V_\Phi\left(\varphi_x^-;0,\frac{\delta}{k}\right)\right)\right\}$$

C o r o l l a r y 2. If $f \in BV_{\phi}$, then

$$\left| E_{n,q}[f](x) - \frac{1}{2}\left\{f(x+0) + f(x-0)\right\}\right| \leq$$

$$\leq c_4(q) \sum_{k=1}^{n-1} \frac{1}{k}\left\{ \phi^{-1}\left(\frac{k}{n} V_{\phi}(\varphi_x^+;0,\frac{\pi}{k})\right) + \phi^{-1}\left(\frac{k}{n} V_{\phi}(\varphi_x^-;0,\frac{\pi}{k})\right)\right\}$$

for every real x and all $n \geq 2$, $q \geq 0$. Under the assumption (8), the function φ_x defined by (10) is of bounded ϕ-variation on $<0,\pi>$, and

$$\left|E_{n,q}[f](x) - \frac{f(x+0) + f(x-0)}{2}\right| \leq c_4(q) \sum_{k=1}^{n-1} \frac{1}{k} \phi^{-1}\left(\frac{k}{n} V_{\phi}(\varphi_x;0,\frac{\pi}{k})\right).$$

R e m a r k 2. Taking in the last estimate $q = 0$ and $\phi(u) = u$ $(u \geq 0)$, we obtain the result due to Bojanic ([1]).

R e m a r k 3. If the function ϕ satisfies the condition (6), then the right-hand sides of the inequalities in Corollaries 1 and 2 converge to zero as $n \to \infty$.

Considering any function $f \in C$ and applying the inequality (2) we observe that

$$v(k;\varphi_x,0,k\pi/n) \leq 2k\omega(\pi/n;\varphi_x) \leq 4k\omega(\pi/n;f) \qquad (1 \leq k \leq n)$$

and

$$\frac{v(n-1;\varphi_x,0,\pi)}{n-1} \leq \frac{2}{n} v(n;\varphi_x,0,\pi) \leq 8\omega\left(\frac{\pi}{n};f\right) \qquad (n \geq 2).$$

Moreover,

$$v(k;\varphi_x,0,k\pi/n) \leq v(k;\varphi_x,0,\pi) \leq 4v(k;f,0,2\pi) \qquad (1 \leq k \leq n).$$

Consequently, the following result analogous to Theorem 1 of [2] can be deduced.

C o r o l l a r y 3. If $f \in C$, then, for every $q \geq 0$ and all integers $n \geq 2$, we have

$$\|E_{n,q}[f]-f\|_C \le c_5(q)\left\{\omega\left(\frac{\pi}{n}; f\right) \sum_{k=1}^{m} \frac{1}{k} + \sum_{k=m+1}^{n-1} \frac{v(k;f,0,2\pi)}{k^2}\right\},$$

m being an arbitrary positive integer not greater than $n-2$.

From the above inequality it follows at once that all estimates given in [2] concerning the rate of uniform convergence of sums $S_n[f]$ remain valid for the sums $E_{n,q}[f]$, $q>0$. For example, we have

$$\|E_{n,q}[f]-f\|_C \le c_6(q)\omega\left(\frac{\pi}{n}; f\right) \log n \quad (n \ge 2,\ q \ge 0).$$

Clearly, this estimate is more precise than the ones obtained by Holland and Sahney in [4] and Singh (see [3], p.32, Remark (2)).

Finally, let us note that Corollary 2 implies

C o r o l l a r y 4. If $f \in C \cap BV_\Phi$, then

$$\|E_{n,q}[f]-f\|_C \le c_7(q) \sum_{k=1}^{n-1} \frac{1}{k} \Phi^{-1}\left(\frac{k}{n}\Omega_\Phi\left(\frac{\pi}{k}; f\right)\right) \quad (n \ge 2,\ q \ge 0).$$

Hence, in view of (7) and under the assumption (6), $\lim_{n\to\infty} E_{n,q}[f](x) = f(x)$ uniformly in $x \in (-\infty, \infty)$. If, in addition, the function Φ satisfies the condition (8), then, in view of (9), we get

$$\|E_{n,q}[f]-f\|_C \le \varkappa c_7(q) \sum_{k=1}^{n-1} \frac{1}{k} \Phi^{-1}\left(\frac{k}{n} w\left(\frac{\pi}{k}; V_\Phi(f)\right)\right) \le$$

$$\le 2\varkappa c_7(q)\left\{\Phi^{-1}\left(\frac{1}{n} V_\Phi(f;-\pi,\pi)\right) + \int_{\pi/n}^{\pi} \frac{1}{t}\Phi^{-1}\left(\frac{1}{nt} w(t;V_\Phi(f))\right)dt\right\}.$$

The last inequality with $q=0$, $\Phi(u)=u$ is equivalent to the Natanson result [6].

4. Approximation by the Borel means

Now, let us consider the Borel means $B_r[f]$ of an arbitrary function $f \in L$, introduced in Section 1. If at a fixed point x the finite limit (3) exists, then

$$B_r[f](x) - S(f,x) = \frac{1}{\pi}\int_0^{\pi} \varphi_x(t) K_r(t)\,dt \quad (r>0),$$

where φ_x is defined by (10) and

$$K_r(t) = e^{-r}\sum_{k=0}^{\infty} \frac{r^k}{k!} D_k(t) = e^{-2r\sin^2 \frac{1}{2}t}\, \frac{\sin(r\sin t + \frac{1}{2}t)}{2\sin\frac{1}{2}t}.$$

It is easy to verify that, for every $r>0$,

$$|K_r(t)| \leqslant r + \frac{1}{2} \qquad (-\infty < t < \infty),$$

$$|K_r(t)| \leqslant \frac{\pi}{2\delta}\, e^{-2r(\delta/\pi)^2} \qquad (0<\delta \leqslant t \leqslant \pi)$$

and

$$\left| \int_x^{\delta} K_r(t)\,dt \right| \leqslant \frac{2\pi}{rx} \quad (0 < x \leqslant \delta \leqslant \pi).$$

Therefore, applying Lemma 1 and arguing similarly as in the proof of Theorem 1 we obtain the following result.

T h e o r e m 2. Let $f \in L$ and let at a fixed point x the limit (3) be finite. If there exists a positive number $\delta \leqslant \pi$ such that f is bounded in the interval $\langle x-\delta, x+\delta\rangle$, then, for $r \geqslant 2$, we have

$$|B_r[f](x) - S(f,x)| \leqslant \left(\frac{3}{\delta} + \frac{1}{2}\right) \frac{v(n-1;\varphi_x,0,\delta)}{n-1} +$$

$$+ \left(\frac{3}{\delta} + \frac{5}{2}\right) \sum_{k=1}^{n-1} \frac{v(k;\varphi_x,0,k\delta/n)}{k^2} + \frac{1}{4\delta}\, e^{-2r(\delta/\pi)^2}\, \|\varphi_x\|_L,$$

where $n = \lceil r \rceil$. In the case $\delta = \pi$, the last term on the right-hand side of the above inequality can be omitted.

From Theorem 2 some estimates for the rate of pointwise and uniform convergence of the Borel means can be deduced as in Section 3. We shall present a few of them.

Corollary 5. Suppose that $f \in BV_\Phi$ and that the condition (6) is fulfilled. Then,

$$\left| B_r[f](x) - \frac{1}{2}\left\{f(x+0) + f(x-0)\right\}\right| \leq$$

$$\leq 31 \sum_{k=1}^{n-1} \frac{1}{k}\left\{ \Phi^{-1}\left(\frac{k}{n} V_\Phi(\varphi_x^+; 0, \frac{\pi}{k})\right) + \Phi^{-1}\left(\frac{k}{n} V_\Phi(\varphi_x^-; 0, \frac{\pi}{k})\right)\right\}$$

for every $x \in (-\infty, \infty)$ and all $r \geq 2$ ($n = \lceil r \rceil$). Hence

$$\lim_{r\to\infty} B_r[f](x) = \frac{1}{2}\left\{f(x+0) + f(x-0)\right\}.$$

If $f \in C \cap BV_\Phi$, then

$$\| B_r[f] - f\|_C \leq 62 \sum_{k=1}^{n-1} \frac{1}{k}\Phi^{-1}\left(\frac{k}{n}\Omega_\Phi\left(\frac{\pi}{k}; f\right)\right) \qquad (r \geq 2)$$

and consequently, $\lim_{r\to\infty} B_r[f](x) = f(x)$, uniformly in $x \in (-\infty, \infty)$.

Corollary 6. If $f \in C$, then

$$\| B_r[f] - f\|_C \leq 26\left\{\omega\left(\frac{\pi}{n}; f\right) \sum_{k=1}^{m} \frac{1}{k} + \sum_{k=m+1}^{n-1} \frac{v(k; f, 0, 2\pi)}{k^2}\right\},$$

where $n = \lceil r \rceil$ and m is an arbitrary positive integer less than $n-2$. In particular,

$$\| B_r[f] - f\|_C \leq c_8\, \omega\left(\frac{\pi}{r}; f\right) \log r \quad (r \geq 2).$$

Moreover,

$$\|B_r[f]-f\|_C \leqslant c_9 \int_0^{\omega(\pi/r;f)} \log \frac{V_\Phi(f;0,2\pi)}{\Phi(t)}\,dt \qquad (r \geqslant 2),$$

provided that $f \in C \cap BV_\Phi$ and $\int_0^1 \log \frac{1}{\Phi(t)}\,dt < \infty$ (see [2]).

REFERENCES

[1] R. Bojanic: An estimate of the rate of convergence for Fourier series of functions of bounded variation, Publications de l'Institut Mathématique 26 (40), (1979) 57-60.

[2] З.А. Чантурия: О равномерной сходимости рядов Фурье, Mat. Sbornik 100, (4) (1976) 535-554.

[3] C.K. Chui, A.S.B. Holland: On the order of approximation by Euler and Taylor means. J. Approximation Theory 39, (1) (1983) 24-38.

[4] A.S.B. Holland, B.N. Sahney: On the degree of approximation by Euler (E,q)-means, Studia Sci. Math. Hungar. 11 (1976) 431-435.

[5] J. Marcinkiewicz: On a class of functions and their Fourier series, Comptes Rendus Soc. Sci. Varsovie 26 (1934) 71-77 (also J. Marcinkiewicz, Collected Papers, PWN Warszawa 1964, 36-41).

[6] Г.И. Натансон: О рядах Фурье непрерывных функций ограниченной вариации, Vestnik Leningrad.Univ. Math. 7 (1972) 154-155.

[7] P. Pych-Taberska: Pointwise approximation by partial sums of Fourier series and conjugate series, Funct. Approximatio Comment. Math. 15 (1985) to appear.

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