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Julian Klukowski, Maciej Zworski

ON THE REPRESENTATION OF P_0-LATTICES BEING P-ALGEBRAS

Dedicated to the memory
of Professor Roman Sikorski

The notion of a P_0-lattice of finite order was introduced first by T. Traczyk [2] in 1963. G. Epstein and A. Horn [1] used this concept for some new generalizations of Post algebras. They discovered P_1- and P_2-lattices in this way.

On the other hand T. Traczyk and W. Zarębski [3] and W. Zarębski [4] introduced generalized P_0-, P_1- and P_2-lattices of order ω^+.

In the present paper P_0-lattices which are P-algebras (called P_0P-lattices) will be examined. The theorem about the monotonic representation of P_0P-lattices is given in section 2. In section 3 it is shown that a P_0P-lattice L generates the Boolean algebra B^{n-1} for certain n if and only if L is a Post algebra.

1. Preliminaries

Let L be a distributive lattice with the least element 0 and the greatest element 1; $x \cup y$ and xy denote the join and the meet of elements $x,y \in L$. The center B of L is the Boolean sublattice of all complemented elements of L. The complement of $b \in B$ is denoted by $\bar{b}$. The greatest element $z \in L (z \in B)$

such that $xz \leq y$, if it exists, is denoted by $x \to y (x \Rightarrow y)$. If $x \to y (x \Rightarrow y)$ exists for any $x, y \in L$ then L is called a Heyting algebra (a B-algebra). In particular $1 \Rightarrow x$ is denoted by $!x$. The least Boolean element greater than x, if it exists, is denoted by $x!$; A B-algebra is called a P-algebra if $(x \Rightarrow y \cup y \Rightarrow x) = 1$ or, equivalently, if

$$z \Rightarrow (x \cup y) = (z \Rightarrow x) \cup (z \Rightarrow y) \tag{1}$$

is satisfied in it. If there exists an ascending sequence

$$0 = e_0 \leq e_1 \leq \ldots \leq e_{n-1} = 1 \tag{2}$$

where n is an integer ≥ 2 such that every $x \in L$ can be written in the form

$$x = \bigcup_{i=1}^{n-1} b_i e_i, \quad b_i \in B \tag{3}$$

then L is a P_0-lattice. In this case we write $L = (e_0, \ldots, e_{n-1}, B)$. The chain (2) whose union with the center B generates L is called the chain base for L. The order of L is the smallest number of elements in a chain base of L.

Every $x \in L = (e_0, \ldots, e_{n-1}, B)$ has a monotonic representation

$$x = \bigcup_{i=1}^{n-1} x_i e_i, \quad x_i \in B, \quad x_1 \geq x_2 \geq \ldots \geq x_{n-1}. \tag{4}$$

A P_0-lattice $L = (e_0, \ldots, e_{n-1}, B)$ which is a Heyting algebra and satisfies: $(e_{i+1} \to e_i) = e_i$ for $i = 0, 1, \ldots, n-2$ is called a P_1-algebra.

A P_1-algebra $L = (e_0, \ldots, e_{n-1}, B)$ such that $e_i \Rightarrow x$ exists for every $x \in L$, $i = 0, 1, \ldots, n-1$, is called a P_2-algebra.

2. <u>The monotonic representation in a P_0P-lattice</u>

Notice that if a P_0-lattice L has the property that $e_i \Rightarrow e_j$ exists for every i, j, then L is a P-algebra and a Heyting algebra (see [1] th.3.1 and th.4.2).

L e m m a 2.1. Let L be a P_0P-lattice with the center B. Then

(i) $$(x \cup y) \Longrightarrow z = (x \Longrightarrow z)(y \Longrightarrow z)$$

(ii) $$(z \Longrightarrow xy) = (z \Longrightarrow x)(z \Longrightarrow y)$$

(iii) $$bx \Longrightarrow (c \cup y) = \bar{b} \cup c \cup (x \Longrightarrow y) \text{ for } b,c \in B$$

(iv) $$(xy \Longrightarrow z) = (x \Longrightarrow z) \cup (y \Longrightarrow z)$$

(v) $$(x \Longrightarrow y)(y \Longrightarrow z) \leqslant (x \Longrightarrow z)$$

(vi) $$x! = \overline{x \Longrightarrow 0}$$

(vii) $$(x \cup y)! = x! \cup y!; \quad (xy)! = x!y!$$

P r o o f . To prove (i), (ii), (iii) it suffices to observe that those properties hold in B-algebras (see [1]).

We now prove (iv). Let $x = \bigcup_{i=1}^{n-1} x_i e_i$ and $y = \bigcup_{i=1}^{n-1} y_i e_i$ be monotonic representations of x and y. It is known that $xy = \bigcup_{i=1}^{n-1} x_i y_i e_i$ is a monotonic representation of xy. By (i) and (iii) we obtain

$$xy \Longrightarrow z = \left(\bigcup_{i=1}^{n-1} x_i y_i e_i \right) \Longrightarrow z = \bigcap_{i=1}^{n-1} (x_i y_i e_i \Longrightarrow z) =$$

$$= \bigcap_{j=1}^{n-1} (\overline{x_i y_i} \cup (e_i \Longrightarrow z)) = \bigcap_{j=1}^{n-1} (\bar{x}_i \cup \bar{y}_i \cup (e_i \Longrightarrow z)).$$

Easy calculation shows that if $a_1 \leqslant \dots \leqslant a_{n-1}$ and $c_1 \geqslant \dots \geqslant c_{n-1}$, then $\bigcap_{i=1}^{n-1} (a_i \cup c_i) = a_1 \cup \bigcup_{i=2}^{n-1} a_i c_{i-1} \cup c_{n-1}$. If, in addition, $b_1 \leqslant \dots \leqslant b_{n-1}$ then

$$\bigcap_{i=1}^{n-1} (a_i \cup b_i \cup c_i) = a_1 \cup b_1 \cup \bigcup_{i=2}^{n-1} ((a_i \cup b_i) c_i) \cup c_{n-1} =$$

$$= \bigcap_{i=1}^{n-1} (a_i \cup c_i) \cup \bigcap_{j=1}^{n-1} (b_i \cup c_i).$$

Therefore

$$xy \Longrightarrow z = \bigcap_{i=1}^{n-1} (\bar{x}_i \cup \bar{y}_i \cup (e_i \Longrightarrow z)) =$$

$$= \bigcap_{i=1}^{n-1} (\bar{x}_i \cup (e_i \Longrightarrow z)) \cup \bigcap_{i=1}^{n-1} (\bar{y}_i \cup (e_i \Longrightarrow z)) = (x \Longrightarrow z) \cup (y \Longrightarrow z)$$

because $\bar{x}_j \leqslant \bar{x}_k$, $\bar{y}_j \leqslant \bar{y}_k$ and $(e_j \Longrightarrow z) \geqslant (e_k \Longrightarrow z)$ for $j \leqslant k$.

For (v) we have: if $a \leqslant (x \Longrightarrow y)(y \Longrightarrow z)$ then $ax \leqslant y$ and $ay \leqslant z$. Hence $ax \leqslant ay \leqslant z$, thus $a \leqslant (x \Longrightarrow z)$.

To prove (vi) note that $x(x \Longrightarrow 0) = 0$ and thus $x \leqslant \overline{x \Longrightarrow 0}$. If $x \leqslant b$ then $\bar{b}x = 0$ for $b \in B$, so $\bar{b} \leqslant (x \Longrightarrow 0)$ and $\overline{x \Longrightarrow 0} \leqslant b$.

(vii) follows directly from (vi), (i) and (iv).

L e m m a 2.2. If $L = (e_0, \dots, e_{n-1}, B)$ is a P_0P-lattice, then

(i) $(x \Longrightarrow 0)(e_i \Longrightarrow x) \leqslant (y \Longrightarrow 0) \cup (e_i \Longrightarrow y)$

(ii) $x!(e_i \Longrightarrow y) \cup y!(e_i \Longrightarrow x) \leqslant x!(e_i \Longrightarrow x) \cup y!(e_i \Longrightarrow y)$

for every x,y L and $i = 0,1,\dots,n-1$.

P r o o f .(i). By Lemma 2.1 (v) we obtain

$$(e_i \Longrightarrow x)(x \Longrightarrow 0) \leqslant (e_i \Longrightarrow 0) \leqslant (e_i \Longrightarrow y) \leqslant (y \Longrightarrow 0) \cup (e_i \Longrightarrow y).$$

(ii) $(x!(e_i \Longrightarrow y) \cup y!(e_i \Longrightarrow x))\overline{(x!(e_i \Longrightarrow x) \cup y!(e_i \Longrightarrow y))} =$

$= (x!(e_i \Longrightarrow y) \cup y!(e_i \Longrightarrow x))(\bar{x}! \cup \overline{e_i \Longrightarrow x})(\bar{y}! \cup \overline{e_i \Longrightarrow y}) =$

$= x!(e_i \Longrightarrow y)(\overline{e_i \Longrightarrow x})\bar{y}! \cup y!(e_i \Longrightarrow x)\bar{x}!(\overline{e_i \Longrightarrow y}) = 0.$

The last equality holds by (i).

T h e o r e m 2.3. Let $L = (e_0, \dots, e_{n-1}, B)$ be a P_0P-lattice. Then every $x \in L$ can be written in the form

$$(*) \quad x = \bigcup_{i=1}^{n-1} D_i(x)e_i, \quad \text{where} \quad D_i(x) = x!(e_i \Longrightarrow x),\ i=1,2,\dots,n-1$$

and the following properties hold:

(i) $$D_1(x) \geq D_2(x) \geq \dots \geq D_{n-1}(x)$$

(ii) $$D_i(x \cup y) = D_i(x) \cup D_i(y)$$

(iii) $$D_i(xy) = D_i(x) D_i(y)$$

(iv) $$D_i(b) = b \quad \text{for} \quad b \in B$$

(v) $D_i(e_j) = e_j!$ for $i \leq j$ and $D_i(e_j) = e_j!(e_i \Longrightarrow e_j)$ for $i > j$.
In particular $D_{n-1}(e_j) = !e_j$.

P r o o f . Let $x = \bigcup_{i=1}^{n-1} x_i e_i$ be a monotonic representation of x. Of course $x_i e_i \leq x$ for $i = 1,2,\dots,n-1$. Thus $x_i \leq (e_i \Longrightarrow x)$ and $x_i e_i \leq (e_i \Longrightarrow x) e_i$. Therefore

$$x = \bigcup_{i=1}^{n-1} x_i e_i \leq \bigcup_{i=1}^{n-1} (e_i \Longrightarrow x) e_i$$

and

$$x = (x!)x \leq x! \bigcup_{i=1}^{n-1} (e_i \Longrightarrow x) e_i = \bigcup_{i=1}^{n-1} D_i(x) e_i .$$

On the other hand $(e_i \Longrightarrow x) e_i \leq x$. Thus

$$\bigcup_{i=1}^{n-1} (e_i \Longrightarrow x) e_i \leq x \quad \text{and} \quad \bigcup_{i=1}^{n-1} D_i(x) e_i =$$

$$= \bigcup_{i=1}^{n-1} x!(e_i \Longrightarrow x) e_i \leq x \; x! = x.$$

Therefore $x = \bigcup_{i=1}^{n-1} D_i(x) e_i$.

It is easy to see that (i), (iv) and (v) hold. It remains to show (ii) and (iii).

We prove (ii). By Lemma 2.1 (vii), the definition of a P-algebra (1) and Lemma 2.2 we obtain

$$D_i(x \cup y) = (x \cup y)!(e_i \Longrightarrow (x \cup y)) = (x! \cup y!)((e_i \Longrightarrow x) \cup (e_i \Longrightarrow y)) =$$

$$= x!(e_i \Longrightarrow x) \cup y!(e_i \Longrightarrow y) \cup x!(e_i \Longrightarrow y) \cup y!(e_i \Longrightarrow x) =$$

$$= x!(e_i \Longrightarrow x) \cup y!(e_i \Longrightarrow y) = D_i(x) \cup D_i(y).$$

Now, we prove (iii), by Lemma 2.1 (ii), (vii) we obtain

$$D_i(xy) = (xy)!(e_i \Longrightarrow xy) = x!y!(e_i \Longrightarrow x)(e_i \Longrightarrow y) = D_i(x)D_i(y)$$

and this completes the proof.

Theorem 2.4. Let $L = (e_0, \dots, e_{n-1}, B)$ be a P_0P-lattice and B^{n-1} be a direct power of a Boolean algebra B. Then there exists a (0,1)-lattice monomorphism from L to B^{n-1}.

Proof. If $x = \bigcup_{i=1}^{n-1} D_i(x)e_i$ is a representation (∗) of element x, then we define $h : L \longrightarrow B^{n-1}$ by $h(x) = (D_1(x), D_2(x), \dots, D_{n-1}(x))$. By Theorem 2.3 we obtain $h(x \cup y) = h(x) \cup h(y)$, $h(xy) = h(x)h(y)$, $h(0) = [0]$, $h(1) = [1]$ where $[b]$ stands for $(b, b, \dots, b)$ for $b \in B$. Obviously $h(\bar{b}) = [\bar{b}]$.

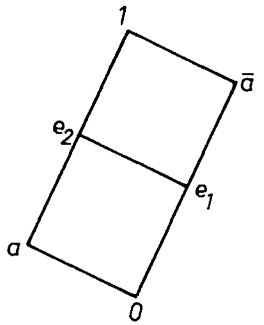
By this theorem we can consider every P_0P-lattice as the sublattice of some monotonic elements of B^{n-1}. ($(b_1, b_2, \dots, b_{n-1})$ is said to be a monotonic element if $b_1 \geq b_2 \geq \dots \geq b_{n-1}$).

Observe that in a Post algebra, which is in particular a P_0P-lattice, the representation (∗) is the usual monotonic representation of element in a Post algebra, which is known to be unique.

The representation $x = \bigcup_{i=1}^{n-1} x_i e_i$ is said to be the highest monotonic representation of x, provided that $x_i \geq y_i$ for any monotonic representation $x = \bigcup_{i=1}^{n-1} y_i e_i$. If the highest monotonic representation exists, then x_i is denoted by $D_i^h(x)$. From [1] it is known that in P_0P-lattices the highest monotonic representation exists and $D_i^h(x \cup y) = D_i^h(x) \cup D_i^h(y)$, $D_i^h(xy) = D_i^h(x)D_i^h(y)$, but $D_i^h(b)$ is not equal to b in general. If P_0P-lattice is a P_2-lattice (the condition $(e_{i+1} \to e_i) = e_i$ is satisfied) then $D_i^h(b) = b$ (see [1] and [4]). It is also known that in every P_0P-lattice of order m there exists a unique chain base $f_0, f_1, \ldots, f_{m-1}$ such that L is a P_2-lattice, so we can introduce the highest monotonic representation in this base which satisfies $D_i^h(b) = b$ for $b \in B$.

Anyway, the advantage of representation (*) is that for a given P_0P-lattice $L = (e_0, \ldots, e_{n-1}, B)$, we can directly represent elements in B^{n-1} such that $D_i(b) = b$ for $b \in B$, even if the unique base $f_0, f_1, \ldots, f_{m-1}$ and the order of L are unknown.

E x a m p l e



This P_0P-lattice is not a P_2-lattice because of $\bar{a}\, e_2 \leq e_1$, but it is not true that $\bar{a} \leq e_1$. Observe that in the highest representation $\bar{a} \leftrightarrow (1, \bar{a}, \bar{a})$ so $D_1^h(\bar{a}) = 1$. By representation (*) we obtain $b \leftrightarrow (b,b,b)$ for $b \in B = \{0,1,a,\bar{a}\}$ and $e_1 \leftrightarrow (\bar{a},\bar{a},0)$, $e_2 \leftrightarrow (1,1,a)$.

If we find the base of this P_0P-lattice, such that P_0P-lattice will be a P_2-lattice, we will get $f_0 = 0$, $f_1 = e_2$, $f_2 = 1$. Now, in the highest representation we obtain

$$b \leftrightarrow (b,b) \quad \text{for} \quad b \in B, \quad e_1 \leftrightarrow (\bar{a},0) \quad e_2 \leftrightarrow (1,a).$$

3. The Boolean algebra generated by P_0P-lattice

Lemma 3.1. A P_0P-lattice $L = (e_0,\dots,e_{n-1},B)$ is a Post algebra of order n if and only if $D_i(e_j) = 1$ for $i \leq j$ and $D_i(e_j) = 0$ for $i > j$.

Proof. If a P_0P-lattice L is a Post algebra of order n then $(e_i \Longrightarrow e_j) = 0$ for $i > j$ (see [2]) and in particular $(e_i \Longrightarrow 0) = 0$. Then $e_i! = 1$. Obviously $(e_i \Longrightarrow e_j) = 1$ for $i \leq j$, so $D_i(e_j) = e_j!$ and $(e_i \Longrightarrow e_j)$ is equal to 1 for $i \leq j$ and to 0 for $i > j$. If a P_0P-lattice is not a Post algebra of order n, then there exists some i and $0 \neq b \in B$ such that $(e_i \Longrightarrow e_{i-1}) = b$. Hence $b\, e_i \leq e_{i-1}$ and by th.2.4 $D_i(b)D_i(e_i) \leq D_i(e_{i-1})$ so $b\, D_i(e_i) \leq D_i(e_{i-1})$. Therefore, either $D_i(e_i) < 1$ or $D_i(e_i) = 1$ and $D_i(e_{i-1}) \geq b \neq 0$.

Lemma 3.2. The only chain E_n: $e_0 = [0] \leq e_1 \leq \dots \leq e_{n-1} = [1]$ of monotonic elements $\in B^{n-1}$ which together with the diagonal of B^{n-1} (denoted by $[B]$) generates B^{n-1}, is the chain F_n: $e_0 = [0]$, $e_1 = (1,0,\dots,0,0)$ $\dots, e_{n-2} = (1,1,\dots,1,0)$, $e_{n-1} = [1]$.

Proof. Observe that $(0,\dots,0,b_i,0,\dots,0) = [b]\,\overline{e_{i-1}}e_i$ for $i = 1,2,\dots,n-1$, so $F_n \cup [B]$ generates B^{n-1}. Suppose now that $E_n \cup [B]$ generates B^{n-1}. Then every elements $x \in B^{n-1}$ can be written in the form

$$(1) \qquad x = \bigcup_{i,j=0}^{n-1} e_i\bar{e}_j[b_{ij}] = \bigcup_{i>j} e_i\bar{e}_j[b_{ij}], \qquad i,j \in \{0,1,\dots,n-1\}.$$

In particular we get an element $x = (1,\dots,1,0)$ in this form. Suppose that $e_i = (e_i^1,\dots,e_i^{n-1})$ for $i = 0,1,\dots,n-1$. For the last two coordinates we obtain

$$\bigcup_{i>j} e_i^{n-2}\, \overline{e_j^{n-2}}\, b_{ij} = 1,$$

$$\bigcup_{i>j} e_i^{n-1}\, \overline{e_j^{n-1}}\, b_{ij} = 0.$$

From the second equality we get

$$\bigcup_{j=0}^{n-2} \overline{e_j^{n-1}}\, b_{n-1j} = 0, \quad \text{so} \quad \bigcup_{j=0}^{n-2} \overline{e_j^{n-2}}\, b_{n-1j} = 0.$$

Then $1 \leqslant \bigcup_{i>j}^{n-2} e_i^{n-2}\, \overline{e_j^{n-2}}\, b_{ij} \leqslant e_{n-2}^{n-2} \bigcup_{i>j}^{n-2} b_{ij}$ because $e_i^{n-2} \leqslant e_{n-2}^{n-2}$ for $i = 1,2,\dots,n-2$ and $\overline{e_j^{n-2}} \leqslant \overline{e_0^{n-2}} = 1$ for $j = 0,1,\dots,n-3$, so

(2) $\qquad e_{n-2}^{n-2} = 1$ and $e_{n-2} = \left(1,1,\dots 1, e_{n-2}^{n-1}\right)$.

Observe that if $E_n \cup [B]$ generates B^{n-1}, then obviously $E'_n : e'_i = (e_i^1,\dots,e_i^{n-2})$, $i = 0,\dots,n-1$, with the diagonal of B^{n-2} generates B^{n-2}. But by (2) $e'_{n-2} = e'_{n-1} = (1,1,\dots,1)$, so the chain $e'_0, e'_1,\dots,e'_{n-2}$ is the chain E_{n-1}.

Applying (2) for E_{n-1} and so on, we obtain

$$(3) \qquad E_n: \begin{array}{l} e_0 = (0,0,\dots,0,0) \\ e_1 = (1, e_1^2,\dots,e_1^{n-2}, e_1^{n-1}) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ e_{n-2} = (1,1,\dots,1,e_{n-2}^{n-1}) \\ e_{n-1} = (1,1,\dots,1,1). \end{array}$$

For $n = 3$, let $E_3 = \left\{ e_0 = (0,0),\ e_1 = (1,e_1^2),\ e_2 = (1,1)\right\}$ be the chain such that $E_3 \cup [B]$ generates B^2. Then an element $x = (0,1) \in B^2$ can be written in the form

$$\bigcup_{i>j} e_i \bar{e}_j [b_{ij}] = (0,1), \quad i,j \in \{0,1,2\}$$

Hence

$$b_{20} \cup b_{10} = 0,$$

$$b_{20} \cup \overline{e_1^2} b_{21} \cup e_1^2 b_{10} = 1.$$

Thus $\overline{e_1^2}\, b_{21} = 1$ and $\overline{e_1^2} = 1$, so $e_1^2 = 0$ and $E_3 = F_3$. Suppose that for $k \geqslant 3$ the only chain E_k, such that $E_k \cup [B]$ generates B^{k-1} is F_k. By (3), the chain E_{k+1} has the form

$$\text{(4)} \qquad E_{k+1} : \begin{array}{l} e_0 = (0,0,\dots,0,0) \\ e_1 = (1,e_1^2,\dots,e_1^{k-1},e_1^k) \\ \dots\dots\dots\dots\dots\dots \\ e_{k-1} = (1,1,\dots,1,e_{k-1}^k) \\ e_k = (1,1,\dots,1,1). \end{array}$$

If $E_{k+1} \cup [B]$ generates B^k then $E'_{k+1} = E_k$ with diagonal of B^{k-1} generates B^{k-1}.

By the inductive assumption $E_k = F_k$, so

$$\text{(5)} \qquad E_{k+1} = \begin{array}{l} e_0 = (0,0,\dots,0,0) \\ e_1 = (1,0,\dots,0,0) \\ \dots\dots\dots\dots\dots\dots \\ e_{k-1} = (1,1,\dots,1,e_{k-1}^k) \\ e_k = (1,1,\dots,1,1). \end{array}$$

Express the element $x = (0,0,\dots,0,1) \in B^k$ in the form

$$\bigcup_{i>j} e_i \bar{e}_j [b_{ij}] = (0,0,\dots,0,1), \quad i,j \in \{0,1,\dots,k\}.$$

For the last two coordinates we obtain

$$\bigcup_{j=0}^{k-2} b_{k-1j} \cup \bigcup_{j=0}^{k-2} b_{kj} = 0$$

and

$$e_{k-1}^{k} \bigcup_{j=0}^{k-2} b_{k-1j} \cup \bigcup_{j=0}^{k-2} b_{kj} \cup \overline{e_{k-1}^{k}}\, b_{kk-1} = 1.$$

Hence $\overline{e_{k-1}^{k}} = 1$ and $e_{k-1}^{k} = 0$, so $E_{k+1} = F_{k+1}$ which completes the proof.

Theorem 3.3. A P_oP-lattice $L = (e_0, e_1, \ldots, e_{n-1}, B)$ generates B^{n-1} if and only if L is a Post algebra. (It is understood that L is a (0,1)-sublattice of B^{n-1} as in Theorem 2.4).

Proof. If L is a Post algebra, then by Lemma 3.1 the constans $e_1, \ldots, e_{n-1}$ must be as follows $e_1 = (1,0,\ldots,0), \ldots, e_{n-1} = (1,1,\ldots,1)$ and $B = [B]$ of course. Then by Lemma 3.2 L generates B^{n-1}. On the other hand, if L is not any Post algebra, then by Lemma 3.1 there exists the constant $e_i \neq (\underbrace{1,\ldots,1}_{i},0 \ldots 0)$. Then by Lemma 3.2 L does not generate B^{n-1}.

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INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW,
00-661 WARSZAWA;
ADDRESS OF THE SECOND COAUTHOR: 4 AMES St., CAMBRIDGE MA,
02 139 U.S.A.
Received May 30, 1984.