DEMONSTRATIO MATHEMATICA
Vol. XVIII No 1 1985

Marian Dawidowski, Bogdan Rzepecki

ON BOUNDED SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS IN BANACH SPACES

Dedicated to the memory of Professor Roman Sikorski

Let I be a real finite compact interval, E a Banach space, and φ an E-valued function defined on $I \times E$. It is well known that neither the continuity, nor even the uniform continuity of φ, does imply the existence of a solution of the Cauchy problem for the differential equation $x' = \varphi(t,x)$. The first who used the measure of noncompactness α as a tool for solving this problem was Ambrosetti [1], who proved (via a fixed point theorem of Schauder's type) the existence theorem under the assumption of uniform continuity of φ assuming in addition that $\alpha(\varphi(t,X)) \leq k\cdot\alpha(X)$ for any $t \in I$ and any bounded subset X of E. Similar results were proved by Szufla [16], Goebel and Rzymowski [9], Cellina [3], Sadovskiĭ [14], Szufla [17], Deimling [7] and many other authors. The bibliography concerning this results is given in [2], [6], [8], [11], [12], [14] and is not necessary to quote it here. The object of the present article is the study of solutions of the semilinear differential equation $x' = A(t)x + f(t,x)$ on the real line $\mathbb{R}$ with the assumption that the linear equation $x' = A(t)x$ possesses an exponential dichotomy and the E-valued function f satisfies on $\mathbb{R}\times E$ some regularity Ambrosetti type condition.

1. Introduction

Throughout this paper, E denote a Banach space with the norm $\|\cdot\|$, L(E) the algebra of continuous linear operators from E into itself with induced standard norm $|\cdot|$, $\mathbb{R}_+$ the half-line $t \geq 0$, and $\mathbb{R}$ the real line.

We shall consider the differential equation

$$(+) \qquad x'(t) = A(t)x(t) + f(t,x(t)),$$

where $t \in \mathbb{R}$, $A(t) \in L(E)$, and f is a E-valued function defined on $\mathbb{R} \times E$.

The purpose of the paper is to prove the existence of bounded solutions of the above equation under the assumption that A possesses an exponential dichotomy and f satisfies some regularity contition expressed in terms of the (Kuratowski-) measure of noncompactness α.

2. Preliminaries

The measure of noncompactness $\alpha(X)$ of a bounded subset X of E is defined as the infimum of all $\varepsilon > 0$ such that there exists a finite covering of X by sets of diameter $\leq \varepsilon$. (Kuratowski [10]). For properties of Kuratowski function α the reader is referred to the monography [8], [11] or [12].

Further, we will use the standard notations. The closure of a set X, its diameter and its closed convex hull be denoted, respectively, by $\overline{X}$, diam X and $\overline{\text{conv}}$ X. For a set $\mathfrak{X}$ of mappings defined on I we write $\mathfrak{X}(t) = \{x(t) : x \in \mathfrak{X}\}$, F[I] will denote the image of I under F. If U and V are subsets of E and t, s are real numbers, then tU + sV is the set of all tu + sv with $u \in U$ and $v \in V$.

Denote by $C(\mathbb{R}, E)$ the set of all continuous functions from $\mathbb{R}$ to E. The set $C(\mathbb{R}, E)$ will be considered as a vector space endowed with the topology of uniform convergence on compact subsets of $\mathbb{R}$.

We shall use the following theorem due to Ambrosetti [1]:

Let I be a compact subinterval of $\mathbb{R}$ and Y a bounded equicontinuous subset of the standard Banach space of continuous functions from I to E. Then

$$\alpha(\bigcup\{Y(t):\ t\in I\}) = \sup\{\alpha(Y(t)):\ t\in I\}.$$

Our result will be proved by the following fixed-point theorem of Schauder type (see [5], [6], [13], [14]):

Let $\mathfrak{X}$ be a closed convex subset of $C(\mathbb{R}, E)$. Let ϕ be a function which assigns to each subset Y of $\mathfrak{X}$ a real number $\phi(Y)\geqslant 0$ with the following properties:

1^o $\phi(Y_1)\leqslant\phi(Y_2)$ whenever $Y_1\subset Y_2$;

2^o $\phi(Y\cup\{y\}) = \phi(Y)$ for all $y\in\mathfrak{X}$;

3^o $\phi(\overline{\mathrm{conv}}\ Y)\leqslant\phi(Y)$;

4^o if $\phi(Y) = 0$ then $\overline{Y}$ is compact.

Assume that $F:\ \mathfrak{X}\longrightarrow\mathfrak{X}$ is a continuous mapping satisfying $\phi(F[Y])<\phi(Y)$ for arbitrary subset Y of $\mathfrak{X}$ with $\phi(Y)>0$. Then F has a fixed point in $\mathfrak{X}$.

3. Main result

Let $A:\mathbb{R}\longrightarrow L(E)$ be strongly measurable and Bochner intergrable on every finite subinterval of $\mathbb{R}$. We suppose that the differential linear equation

$$(*)\qquad x'(t) = A(t)x(t)$$

admits a regular exponential dichotomy (see [4], p.233). Next, denote by G the main Green's function for $(*)$ (see [4], p.240).

Let $f:\ \mathbb{R}\times E\longrightarrow E$ be continuous. Assume that

$$\|f(t,x)\|\leqslant m(t)\quad\text{for}\quad (t,x)\in\mathbb{R}\times E,$$

where m is a locally integrable function on $\mathbb{R}$ with $\sup\left\{\int_t^{t+1} m(s)ds:\ t\in\mathbb{R}\right\}\leqslant M.$

Assume in addition that

$$\alpha(f[I\times X])\leqslant\sup\{g(t):\ t\in I\}\cdot h(\alpha X))$$

for any compact subset I of $\mathbb{R}$ and each bounded subset X of E, where g, h are functions of $\mathbb{R}_+$ into itself such that g is continuous, h is nondecreasing,

$$L = \sup\left\{ \int_{\mathbb{R}} \|G(t,s)\|\, g(s)ds:\ t \in \mathbb{R} \right\} < \infty$$

and $L \cdot h(t) < t$ for $t > 0$.

T h e o r e m . Under the above hypotheses there exists a bounded solution of (+) on $\mathbb{R}$.

P r o o f . We define a mapping F as follows

$$(Fx)(t) = \int_{\mathbb{R}} G(t,s)f(s,x(s))ds \quad \text{for} \quad x \in C(\mathbb{R}, E).$$

According to Lemma IV.3.1 of [4], there exist positive constants N, ν that is

$$\|G(t,s)\| \leqslant N \cdot e^{-\nu|t-s|}$$

for t, s in $\mathbb{R}$. Denote by $\mathcal{X}$ the set of all $x \in C(\mathbb{R}, E)$ such that

$$\|x(t)\| \leqslant K = 2NM(1 - e^{-\nu})^{-1}$$

and

$$\|x(t_1) - x(t_2)\| \leqslant K \cdot \int_{t_1}^{t_2} \|A(s)\| ds + \int_{t_1}^{t_2} m(s)ds$$

for real t, and t_1, t_2 with $t_1 \leqslant t_2$. It is easy to see that $\mathcal{X}$ is closed convex bounded subset of $C(\mathbb{R}, E)$.

Let $x \in \mathcal{X}$. We have

$$\|(Fx)(t)\| \leqslant$$

$$\leqslant N\left(\int_{-\infty}^{t} e^{-\nu(t-s)}m(s)ds + \int_{t}^{\infty} e^{-\nu(s-t)}m(s)ds \right) =$$

$$= N\left(\sum_{k=0}^{\infty} \int_{k}^{k+1} e^{-\nu\sigma}m(t+\sigma)d\sigma + \sum_{k=0}^{\infty} \int_{k}^{k+1} e^{-\nu\sigma}m(t-\sigma)d\sigma \leqslant \right.$$

$$\leq N\left(\sum_{k=0}^{\infty} e^{-\nu k} \int_{t+k}^{t+k+1} m(s)ds + \sum_{k=0}^{\infty} e^{-\nu k} \int_{t-k-1}^{t-k} m(s)ds\right) \leq$$

$$\leq 2NM \cdot \sum_{k=0}^{\infty} e^{-\nu k} = K$$

for $t \in \mathbb{R}$. By Theorem IV.3.2 and Remark IV.3.6 of [4] the function Fx is a solution of the differential equation

$$y'(t) = A(t)y(t) + f(t,x(t))$$

on $\mathbb{R}$. Hence

$$\|(Fx)(t_1) - (Fx)(t_2)\| \leq$$

$$\leq \int_{t_1}^{t_2} \|A(s)(Fx)(s) + f(s,x(s))\| ds \leq$$

$$\leq K \int_{t_1}^{t_2} \|A(s)\| ds + \int_{t_1}^{t_2} m(s)ds$$

whenever $t_1 \leq t_2$. Consequently, $Fx \in \mathcal{X}$.

Let $u,v \in \mathcal{X}$. Let $t \in \mathbb{R}$ and $a > 0$. Then

$$\|(Fu)(t) - (Fv)(t)\| \leq$$

$$\leq N\left(\int_{-\infty}^{t-a} + \int_{t-a}^{t+a} + \int_{t+a}^{\infty}\right) e^{-\nu|t-s|} \cdot \|f(s,u(s)) - f(s,v(s))\| ds \leq$$

$$\leq N \cdot \sup\left\{\|f(s,u(s)) - f(s,v(s))\| : t-a \leq s \leq t+a\right\} \cdot$$

$$\cdot \int_{t-a}^{t+a} e^{-\nu|t-s|} ds + 2N\left(\int_{-\infty}^{t-a} + \int_{t+a}^{\infty}\right) e^{-\nu|t-s|} m(s)ds \leq$$

$$\leq 2N\nu^{-1}(1 - e^{-\nu a})\cdot \sup\Big\{\|f(s,u(s)) - f(s,v(s))\|:\ t-a \leq s \leq t+a\Big\} + Ke^{-\nu a}.$$

It is well known that if $f: \mathbb{R}\times E \longrightarrow E$ is continuous then the operator $x(\cdot) \longmapsto f(\cdot,x(\cdot))$ from $C(\mathbb{R}, E)$ into itself is continuous. Now, from this fact and the above inequality it follows that our F is continuous as a map of $\mathfrak{X}$ into itself.

Let us put: $\Phi(Y) = \sup\{\alpha(Y(t)):\ t\in\mathbb{R}\}$ for a subset Y of $\mathfrak{X}$. By the corresponding properties of α the function Φ satisfy the conditions 1^{o} - 3^{o} listed in Section 2; 4^{o} follows from the Arzela-Ascoli theorem (see [15], Theorem IV.10.1).

Assume that Y is a subset of $\mathfrak{X}$ with $\Phi(Y) > 0$. Let $t\in\mathbb{R}$ be fixed. Let $\varepsilon > 0$ be arbitrary. Choose a number $a > 0$ such that $Ke^{-\nu a} < \varepsilon$. We have

$$\alpha\left(\left\{\int_{-\infty}^{t-a} G(t,s)f(s,y(s))ds:\ y\in Y\right\}\right) \leq$$

$$\leq \operatorname{diam}\left(\left\{\int_{-\infty}^{t-a} G(t,s)f(s,y(s))ds:\ y\in Y\right\}\right) \leq$$

$$\leq 2N\cdot \int_{-\infty}^{t-a} e^{-\nu(t-s)}m(s)ds \leq K\cdot e^{-\nu a} < \varepsilon$$

and analogously

$$\alpha\left(\left\{\int_{t+a}^{\infty} G(t,s)f(s,y(s))ds:\ y\in Y\right\}\right) < \varepsilon .$$

Now, we shall prove that

$$\alpha\left(\left\{\int_{t-a}^{t+a} G(t,s)f(s,y(s))ds:\ y\in Y\right\}\right)\leqslant$$

$$\leqslant h(\alpha(Z))\int_{\mathbb{R}} \|G(t,s)\| g(s)ds,$$

where $Z = \bigcup\left\{Y(s):\ t-a\leqslant s\leqslant t+a\right\}$.

Indeed, for arbitrary $\varepsilon'>0$ there exists a $\delta>0$ such that $|s'-s''|<\delta$ with $s', s''\in[t-a, t]$ or $s', s''\in[t, t+a]$ implies $\|G(t,s') - G(t,s'')\| < \varepsilon'$ and $|g(s') - g(s'')|<\varepsilon'$. Denote by I_i the interval $[t_{i-1}, t_i]$ $(i = 1,2,\dots,2m)$, where

$$t_0 = t-a<t_1<\dots<t_m = t<\dots<t_{2m} = t+a$$

with $t_i-t_{i-1}<\delta$. Let $\sigma_i, \tau_i\in I_i$ be points such that

$$\|G(t,\sigma_i)\| = \sup\left\{\|G(t,s)\|\ :\ s\in I_i\right\},$$

$$g(\tau_i) = \sup\left\{g(s):\ s\in I_i\right\},$$

and let

$$c_1 = \sup\left\{\|G(t,s)\|:\ t-a\leqslant s\leqslant t+a\right\}$$

and $$c_2 = \sup\left\{g(s):\ t-a\leqslant s\leqslant t+a\right\}.$$

By the integral mean value theorem

$$\left\{\int_{t-a}^{t+a} G(t,s)\ f(s,y(s))ds:\ y\in Y\right\}\subset$$

$$\subset\sum_{i=1}^{2m}(t_i-t_{i-1})\ \overline{\mathrm{conv}}\left(\bigcup\left\{G(t,s)f\,[I_i\times Z]:\ s\in I_i\right\}\right).$$

Since

$$\alpha\left(\bigcup\left\{G(t,s)f[I_i\times Z]:\ s\in I_i\right\}\right)\leqslant$$

$$\leqslant \sup\left\{\|G(t,s)\|:\ s\in I_i\right\}\cdot\alpha(f[I_i\times Z])$$

(see Appendix), so

$$\alpha\left(\left\{\int_{t-a}^{t+a} G(t,s)f(s,y(s)):\ y\in Y\right\}\right)\leqslant$$

$$\leqslant\alpha\left(\sum_{i=1}^{2m}(t_i-t_{i-1})\ \overline{\mathrm{conv}}\left(\bigcup\left\{G(t,s)f[I_i\times Z]:\ s\in I_i\right\}\right)\right)\leqslant$$

$$\leqslant\sum_{i=1}^{2m}(t_i-t_{i-1})\ \sup\left\{\|G(t,s)\|:\ s\in I_i\right\}\cdot\alpha(f[I_i\times Z])\leqslant$$

$$\leqslant\sum_{i=1}^{2m}(t_i-t_{i-1})\ \sup\left\{\|G(t,s)\|:\ s\in I_i\right\}\cdot\sup\left\{g(s):s\in I_i\right\}\cdot h(\alpha(Z))=$$

$$= h(\alpha(Z))\cdot\sum_{i=1}^{2m}(t_i-t_{i-1})\|G(t,\sigma_i)\|g(\tau_i)\leqslant$$

$$\leqslant h(\alpha(Z))\cdot\sum_{i=1}^{2m}\int_{I_i}\Big(\|G(t,\sigma_i)-G(t,s)\|g(\tau_i)+$$

$$+\|G(t,s)\|\,|g(\tau_i)-g(s)|+\|G(t,s)\|g(s)\Big)ds\leqslant$$

$$\leqslant h(\alpha(Z))\cdot\left[2a(c_1+c_2)\varepsilon'+\int_{t-a}^{t+a}\|G(t,s)\|g(s)ds\right]$$

and our claim is proved.

Because

$$F[Y](t) \subset \left\{ \int_{-\infty}^{t-a} G(t,s)f(s,y(s))ds:\ y \in Y \right\} +$$

$$+ \left\{ \int_{t-a}^{t+a} G(t,s)f(s,y(s))ds:\ y \in Y \right\} +$$

$$+ \left\{ \int_{t+a}^{\infty} G(t,s)f(s,y(s))ds:\ y \in Y \right\},$$

we obtain

$$\alpha(F[Y](t)) < \varepsilon + h(\alpha(Z)) \int_{\mathbb{R}} \| G(t,s) \| g(s)ds + \varepsilon .$$

Since Y is almost equicontinuous and bounded, we can apply Ambrosetti's result to get

$$\alpha(Z) = \sup\left\{\alpha(Y(s)):\ t-a \leqslant s \leqslant t+a\right\} \leqslant \phi(Y).$$

Hence

$$\alpha(F[Y](t)) < 2\varepsilon + L\cdot h(\phi(Y)),$$

and therefore $\alpha(F[Y](t)) \leqslant L\cdot h(\phi(Y))$.

Consequently, $\phi(F[Y]) \leqslant L\cdot h(\phi(Y))$. Thus all assumptions of our fixed-point theorem are satisfied; F has a fixed point in $\mathfrak{X}$ which ends the proof.

R e m a r k . Let $S_r = \{ x \in E:\ \|x\| \leqslant r \}$. Our result holds whenever f is defined on $\mathbb{R} \times S_r$ and $M < r(1-e^{-\nu})/2N$. Moreover if the condition $\|f(t,x)\| \leqslant m(t)$ is replaced by $\|f(t,x)\| \leqslant M'$ on $\mathbb{R} \times S_r$, then we must assume that $M' < r\nu/2N$.

4. Appendix

The object of this appendix is to derive the following property of the measure of noncompactness α:

If Q is a continuous mapping from a compact interval I to L(E) and W is a bounded subset of E, then

$$\alpha\left(\bigcup\left\{Q(t)W:\ t\in I\right\}\right)\leqslant \sup\left\{\|Q(t)\|:\ t\in I\right\}\cdot\alpha(W).$$

For this purpose we choose $\varepsilon > 0$. Let $\delta = \delta(\varepsilon) > 0$ and $W_i\subset E$ $(i=1,2,\dots,m(\varepsilon))$ be such that

$$\|Q(t') - Q(t'')\| < \left(\sup\left\{\|x\|:\ x\in W\right\}\right)^{-1}\varepsilon \quad \text{for} \quad |t'-t''| < \delta$$

and

$$W = \bigcup_{i=1}^{m} W_i \quad \text{with} \quad \operatorname{diam} W_i < \varepsilon + \alpha(W).$$

Divide the interval I in such a way that $t_1 < t_2 < \dots < t_n$ with $t_{j+1} - t_j < \delta$ $(j=1,2,\dots,n-1)$. Let us put $X_{ij} = \{x\in E:$ there exists a point $w\in W_i$ such that $\|x-Q(t_j)w\| < \varepsilon\}$ for $i=1,2,\dots,m$ and $j=1,2,\dots,n$.

We have

$$\bigcup\left\{Q(t)W:\ t\in I\right\}\subset\bigcup_{i=1}^{m}\bigcup_{j=1}^{n} X_{ij}.$$

If $\|x_k - Q(t_j)w_k\| < \varepsilon$ $(k=1,2)$ with $x_k\in X_{ij}$ and $w_k\in W_i$, then

$$\begin{aligned}\|x_1-x_2\| &\leqslant \|x_1 - Q(t_j)w_1\| + \|Q(t_j)w_1 - Q(t_j)w_2\| + \\ &+ \|Q(t_j)w_2-x_2\| < 2\varepsilon + \|Q(t_j)\|\ \|w_1-w_2\| \leqslant \\ &\leqslant 2\varepsilon + \sup\left\{\|Q(t)\|:\ t\in I\right\}\cdot\operatorname{diam} W_i < \\ &< 2\varepsilon + \left[\varepsilon+\alpha(W)\right]\cdot\sup\left\{\|Q(t)\|:\ t\in I\right\}.\end{aligned}$$

Therefore

$$\alpha\left(\bigcup\left\{Q(t)W:\ t\in I\right\}\right)\leqslant 2\varepsilon + \left[\varepsilon+\alpha(W)\right]\cdot\sup\left\{\|Q(t)\|:\ t\in I\right\},$$

and we have finished.

REFERENCES

[1] A. Ambrosetti: Un teorema di esistenza per le equazioni differenziali negli spazi di Banach, Rend. Sem. Mat. Univ. Padova 39 (1967) 349-360.

[2] J. Banaś, K. Goebel: Measure of noncompactness in Banach spaces, Lect. Notes Pure Applied Math. 60, New York 1980.

[3] A. Cellina: On the existence of solutions of ordinary differential equations in Banach spaces, Funkcial. Ekvac. 14 (1972), 129-136.

[4] J.L. Daleckiĭ, M.G. Krein: Stability of solutions of ordinary differential equations in Banach space, Moscow 1970 (in Russian).

[5] J. Daneš: Some fixed point thecrem, Comment. Math. Univ. Carolinae 9 (1968) 223-235.

[6] J. Daneš: On densifying and related mappings and their application in nonlinear functional analysis, Theory of nonlinear operators, Berlin 1974, 15-56.

[7] K. Deimling: On existence and uniqueness of Cauchy's problem in infinite dimensional Banach spaces, Proc. Colloq. Math. Soc. Janos Bolyai, Vol. 15 (1975), Diff. Eqs., 131-142.

[8] K. Deimling: Ordinary differential equations in Banach spaces, Lect. Notes in Math. 596, Berlin 1977.

[9] K. Goebel, W. Rzymowski: An existence theorem for the equation $x' = f(t,x)$ in Banach space, Bull.Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys., 18 (1970) 367-370.

[10] C. Kuratowski: Sur les espaces completes, Fund. Math. 15 (1930) 301-309.

[11] V. Lakshmikantham, S. Leela: An introduction to nonlinear differential equations in abstract spaces, Pergamon Press, 1980.

[12] R. Martin: Nonlinear operators and differential equations in Banach spaces. New York 1976.

[13] B.N. S a d o v s k i Ĭ : A fixed point principle, Funkcional Anal. i Priložen., 1 (1967) 74-76 (in Russian).
[14] B.N. S a d o v s k i Ĭ : Asymptotically compact and densifying operators, Uspehi Math. Nauk 27 (1972) 81-146 (in Russian).
[15] R. S i k o r s k i : Funkcje rzeczywiste I, Warszawa 1958.
[16] S. S z u f l a : Some remarks on ordinary differential equations in Banach spaces, Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys., 16 (1968) 795-800.
[17] S. S z u f l a : On the existence of solutions of differential equations in Banach spaces, ibid. 30 (1982) 507-515.

INSTITUTE OF MATHEMATICS, A. MICKIEWICZ UNIVERSITY,
60-769 POZNAŃ, POLAND
Received May 30, 1984.