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Jan Rogulski

OPERATORS WITH H^k COEFFICIENTS AND GENERALIZED HODGE - DE RHAM DECOMPOSITIONS

Dedicated to the memory of Professor Roman Sikorski

1. Introduction

Roughly speaking, the classical Hodge - de Rham decomposition is the expression of a differentiable form as a sum of exact and coclosed forms. The Sobolev spaces of forms decompose into the direct sums of subspaces of exact and coclosed forms (see [5] and also [3], [7] for the decompositions of spaces of $C^{k+\alpha}$ forms).

The crucial point in the proof of Hodge decomposition lies in the decomposition of space of tensors into the direct sum of image and kernel of laplacian. This property of laplacian is also valid for a wide class of differential (and pseudodifferential) operators, namely, for elliptic operators (see e.g. Palais [10], Thm 7 in Chapter XI). Moreover, the similar behaviour as that of the exterior differentiation d in p-covector bundles can also be noticed for operators with injective symbol acting between vector bundles. D.G. Ebin [4] proved the analogous decomposition with respect to a differential operator D with injective symbol as the Hodge decomposition that was constructed by means of d. He builds the elliptic operator D^*D as a substitute of the laplacian Δ.

The above mentioned theorem for elliptic operators is then applied to prove the generalized Hodge decomposition into a direct sum of im D and ker D^*. Both Authors, Prof. R. Palais and Prof. D. Ebin, deal with the operators with smooth coefficients.

In this paper we give the Hodge-like decomposition for the operators with coefficients of Sobolev class H^k. This generalization is useful for the study of actions of Hilbert-Lie groups onto Hilbert manifolds. The helpfulness of this generalization lies in the fact that it enables to prove that any orbit in the Sobolev space of tensor fields is a smooth submanifold (see [6] and [8] for examples and details). The previous results obtained for operators with smooth coefficients allow to prove that the orbit passing through a smooth element of space of tensor fields is a submanifold.

The generalization of Hodge decomposition for the differential operators with H^k coefficients has also been investigated by M. Cantor who applies the estimates obtained for such operators by Y. Choquet-Bruhat and D. Christodoulou (see e.g. [2] and references given there). However, our approach is quite different and arose from the study of the action of the group of automorphisms of a principal bundle onto the space of connections. We refer also to Cantor's paper [1] for several important examples of Hodge-like decompositions (called: Helmholz decomposition, de Rham decomposition and Berger-Ebin decomposition).

2. Basic notions and notation

Let M be a compact, C^∞, n-dimensional manifold without boundary. We denote by μ a smooth measure on M. Let ξ and η be smooth vector bundles over M provided with fibre metrics $(\, , \,)^\xi$ and $(\, , \,)^\eta$ respectively. For $k \in Z$ we denote by $H^k(\xi)$ and $H^k(\eta)$ the (Hilbert) spaces of sections of Sobolev class H^k of ξ and η respectively. The space of smooth sections of ξ will be denoted by $C^\infty(\xi)$. We refer to Palais [9] for the basic constructions and results concerned with the Sobolev spaces of sections. Let $J^r\xi$, $r \in N \cup \{0\}$, be the r-jet bundle of ξ, i.e. the vector bundle of r-jets of local sections of ξ. We denote by $Hom(J^r\xi, \varrho)$ the vector bundle over M, whose fibre $Hom(J^r\xi, \varrho)_x$, $x \in M$, consists of all endomorphisms of the vector spaces $(J^r\xi)_x$ and ϱ_x. Obviously, $Hom(J^r\xi, \varrho) = (J^r\xi)^* \otimes \varrho$.

2.1. Definition. Any H^m section B of $Hom(J^r\xi, \varrho)$, $m \in Z$, is called a differential operator of order r with H^m coefficients.

It is clear that any differential operator with H^m coefficients, say B, naturally defines a linear map $C^\infty(\xi) \longrightarrow H^m(\varrho)$, which we also denote by B.

2.2. Proposition. Let $m > \frac{n}{2}$ and let $B : C^\infty(\xi) \longrightarrow H^m(\varrho)$ be an r-th order differential operator. Then B extends, for any $-m+r \leq l \leq m+r$, to a continuous linear operator

$$B_l : H^l(\xi) \longrightarrow H^{l-r}(\varrho).$$

Proof. The statement follows from the fact that the multiplication of sections by functions: $C^\infty(M) \times C^\infty(\varrho) \longrightarrow C^\infty(\varrho)$ extends to the continuous bilinear map $H^m(M) \times H^k(\varrho) \longrightarrow H^k(\varrho)$ for any $-m \leq k \leq m$.

2.3. Remarks.

(i) From Rellich's lemma and Prop.2.2 it follows immediately that for B as in 2.2 and for any $l > m+r$ there exists a continuous extension of B to the operator $H^l(\xi) \longrightarrow H^m(\varrho)$.

(ii) If B satisfies the assumptions of 2.2 but is not H^{m+1} differential operator then a continuous extension $H^l(\xi) \longrightarrow H^{l-r}(\varrho)$ for $l < -m+r$ does not exist.

Palais distinguished in ([10] Chapter VIII, § 1) the space $OP_r(\xi, \varrho)$ of linear operators $C^\infty(\xi) \longrightarrow C^\infty(\varrho)$ by the requirement that any operator $B \in OP_r(\xi, \varrho)$ have to continuously extend to a linear operator $H^k(\xi) \longrightarrow H^{k-r}(\varrho)$ for any $k \in Z$. Now, 2.2 and 2.3 imply that a differential operator with H^m coefficients belongs to $OP_r(\xi, \varrho)$ iff it has C^∞ coefficients.

We refer to the mentioned book [10] for the definitions and constructions of spaces $\mathrm{Int}_r(\xi,\varrho)$ and $E_r(\xi,\varrho)$ of operators. We recall only that $\mathrm{Int}(\xi,\varrho)$ denotes the Selley algebra and $E_r(\xi,\varrho)$ is the space of r-th order elliptic operators. The following inclusions hold:

$$(2.4)\begin{cases} \text{(i)} & \mathrm{Diff}_r(\xi,\varrho)\subset \mathrm{Int}_r(\xi,\varrho) \\ \text{(ii)} & OP_{r-1}(\xi,\varrho)\subset \mathrm{Int}_r(\xi,\varrho)\subset OP_r(\xi,\varrho) \\ \text{(iii)} & E_r(\xi,\varrho)\subset \mathrm{Int}_r(\xi,\varrho) \end{cases}$$

Here $\mathrm{Diff}_r(\xi,\varrho)$ denotes the space of r-th order differential operators (with smooth coefficients). The space $E_r(\xi,\varrho)\cap$ $\cap\,\mathrm{Diff}_r(\xi,\varrho)$ consists of all elliptic differential operators of order r (with smooth coefficients). Throughout this paper, an elliptic operator of order r will mean an element of $E_r(\xi,\varrho)$.

Now the fibre metrics $(\ ,\ )^\xi$ and $(\ ,\ )^\varrho$ define the L^2 scalar products in the spaces of sections of ξ and ϱ respectively

$$<v_1,v_2>^\xi := \int_M (v_1(x),v_2(x))^\xi\, d\mu(x),$$

where v_1 and v_2 are sections of ξ. Similarly one defines $<,>^\varrho$ for the sections of ϱ. For any $k\in N\cup\{0\}$ $<\ ,>^\xi$ is the continuous, bilinear, symmetric, positive and nondegenerate form on $H^k(\xi)$, but is not scalar product on $H^k(\xi)$ for $k\in N$. The bilinear form $<,>^\xi$ is then usually called a weak scalar product. The weak scalar products $<,>^\xi$ and $<\ ,>^\varrho$ allow to define a formally adjoint operator B^* for the r-th order differential operator B with H^m coefficients, by the same formula as in $\mathrm{Diff}(\xi,\varrho)$ - case

$$(2.5)\quad <B^*w,\ v>^\xi = <w,\ Bv>^\varrho \text{ for any } v\in C^\infty(\xi) \text{ and } w\in C^\infty(\varrho).$$

It is not hard to show that there exists an uniquely defined differential operator $B^*: C^\infty(\varrho)\longrightarrow H^{m-r}(\xi)$ of order r that satisfies 2.5.

3. Perturbed elliptic operators

Let $L : C^\infty(\xi) \longrightarrow C^\infty(\varrho)$ be an elliptic operator of order r+1, i.e. $L \in E_{r+1}(\xi,\varrho)$. In this section we will consider the perturbations of L by the differential operators of order r with H^m coefficients, namely, the operators of the form L+B, where $B : C^\infty(\xi) \longrightarrow H^m(\varrho)$ is an r-th order differential operator. In general $L+B \notin E_{r+1}(\xi,\varrho)$. More precisely, 2.3 and 2.4 imply that $L+B \in E_{r+1}(\xi,\varrho)$ if and only if $m = \infty$, that is, in the case when B is a differential operator with smooth coefficients ($B \in \mathrm{Diff}_r(\xi,\varrho)$). However L+B has some kind of regularity:

3.1. Proposition. Let $m \geq r \geq 0$ be integers. Let $m > \frac{n}{2}$. If $L \in E_{r+1}(\xi,\varrho)$, and $B : C^\infty(\xi) \longrightarrow H^m(\varrho)$ is an r-th order differential operator then the extension $(L+B)_0 : H^0(\xi) \longrightarrow H^{-r-1}(\varrho)$ exists and $(L+B)_0$ is a Fredholm operator. Moreover, L+B is regular in the following meaning

$$(L+B)_0^{-1}(H^{l-r-1}(\varrho)) \subset H^l(\xi) \quad \text{for any} \quad 0 \leq l \leq m+r+1.$$

Proof. First note that $(L+B)_0 = L_0 + \iota \circ B_0$, where $L_0 : H^0(\xi) \longrightarrow H^{-r-1}(\varrho)$ is a Fredholm operator, $B_0 : H^0(\xi) \longrightarrow H^{-r}(\varrho)$ is continuous and the canonical inclusion map $\iota : H^{-r}(\varrho) \longrightarrow H^{-r-1}(\varrho)$ is a compact operator. Thus $\iota \circ B_0$ is also compact, so $(L+B)_0$ is a Fredholm operator.

For the second part of Proposition we consider the equation $L_0 v + B_0 v = w$, where $w \in \mathrm{im}(L+B)_0 \cap H^{l-r-1}(\varrho)$. Then $L_0 v = w - B_0 v$. If $l = 1$ then $w - B_0 v \in H^{-r}(\varrho)$, so by the regularity of L (cf. [10, Thm. 5 of Chapter XI]) we obtain that $v \in H^1(\xi)$. Now we prove the inclusion in Proposition for $l > 1$ by induction. If it holds for $l = l'-1$, $l \leq m+r$, and $w \in H^{l'-r-1}(\varrho)$ then $v \in H^{l'-1}(\xi)$ and $B_0 v \in H^{l'-r-1}(\varrho)$ (by 2.2), so $(w - B_0 v) \in H^{l'-r-1}(\varrho)$. Thus, by the regularity theorem for elliptic operators mentioned above, we obtain that $v \in H^{l'}(\xi)$, so the inclusion holds for $l+1$. For $l = 0$ the inclusion is trivial.

3.2. Remark. For $l > m+r+1$ we obtain that $(L+B)_0^{-1}(H^{l-r-1}(\varrho)) \subset H^{m+r+1}(\xi)$ and no more, since for $v \in H^{m+r+1}(\xi)$ we have $B_0 v \in H^m(\varrho)$ (see 2.3).

3.3. Corollary. $\ker(L+B)_0 \subset H^{m+r+1}(\xi)$, so $\ker(L+B)_0 = \ker(L+B)_{m+r+1}$.

Using the same arguments as in the proof of 3.1, namely, recalling Rellich's lemma and the fact that the composition of a compact operator and a continuous operator is again a compact operator, one proves that any extension

$$(L+B)_l : H^l(\xi) \to H^{l-r-1}(\varrho)$$

for $0 < l < m+r+1$, is the Fredholm operator. Here the identity $(L+B)_l = L_l + B_{l-1} \circ \iota$, where $\iota : H^l(\xi) \to H^{l-1}(\xi)$, is useful. Since $(L+B)_l$, $l = 0,1,\dots,m+r+1$, are Fredholm operators, they have closed images. In other words

$$(L+B)_0(H^l(\xi)) \subset H^{l-r-1}(\varrho)$$

are closed, finitely codimensional subspaces.

Since $L \in E_{r+1}(\xi,\varrho) \subset OP_{r+1}(\xi,\varrho)$, Proposition 2.2 states that the image of operator

$$(\text{graph}(L+B)_0) \cap (H^l(\xi) \times H^{-r-1}(\varrho))$$

is a subspace of $H^{l-r-1}(\varrho)$, $l = 0,1,\dots,m+r+1$. On the other hand, Proposition 3.1 says that the domain of operator

$$(\text{graph}(L+B)_0) \cap (H^0(\xi) \times H^{l-r-1}(\varrho))$$

is a subspace of $H^l(\xi)$. Hence the conjunction of these statements is equivalent to the following equality

$$\text{graph}(L+B)_0 \cap (H^0(\xi) \times H^{l-r-1}(\varrho)) =$$

$$= \text{grap}(L+B)_0 \cap (H^l(\xi) \times H^{-r-1}(\varrho))$$

and this subspace of $H^l(\xi)\times H^{l-r-1}(\eta)$ is simply the graph of $(L+B)_l$, $0<l\leq m+r+1$.

Now we will prove a simple but useful lemma on algebraic decompositions of a Banach space.

3.4. Lemma. Let X be a Banach space and let $<\,,\,>$ be a weak scalar product on X (i.e. a continuous, symmetric and strictly positive bilinear form). We assume that there exist two linear subspaces X_1, $X_2\subset X$ such that $X = X_1+X_2$ and X_1, X_2 are $<\,,\,>$-orthogonal. Then both subspaces X_1 and X_2 are closed in X, $X_1\cap X_2 = \{0\}$, i.e. the decomposition $X = X_1\oplus X_2$ is topological. Moreover, X_1 and X_2 are closed in the weak topology defined by $<\,,\,>$.

Proof. Let us denote $X_1^\perp := \{y\in X \mid <y,y'> = 0$ for any $y'\in X_1\}$. Since $X_1^\perp$ is the intersection of kernels of the continuous functionals on X

$$X_1^\perp = \bigcap_{y'\in X_1} \ker(X\ni y\longmapsto <y,y'>\in R)$$

then $X_1^\perp$ is closed in X, and is also closed with respect to the weak topology. By the assumption $X_2\subset X_1^\perp$. Now let $y\in X_1^\perp$. We have $y = y_1+y_2$, where $y_1\in X_1$ and $y_2\in X_2$. Of course $<y_1,y> = 0$. On the other hand $<y_1,y> = <y_1,y_1> + <y_1,y_2> = <y_1,y_1>$, so $<y_1,y_1> = 0$. It means that $y_1 = 0$, so $y = y_2\in X_2$. Thus we proved that $X_1^\perp\subset X_2$, so $X_1^\perp = X_2$. Replacing the indices 1 and 2 in the above considerations, we obtain that $X_2^\perp = X_1$, so the lemma follows.

Now we give the theorem on decompositions defined by the perturbed elliptic operators. This theorem extends the result presented in ([10], Thm. 7 of Chapter XI).

3.5. Theorem. Let $m>\frac{n}{2}+r$ and $m\geq 2r$. Let us take $L\in E_{r+1}(\xi,\eta)$ and let $B: C^\infty(\xi)\longrightarrow H^m(\eta)$ be an r-th order differential operator with H^m coefficients. Then for $0\leq l\leq m$

$$H^l(\eta) = \ker(L^*+B^*)_0\oplus \operatorname{im}(L+B)_{l+r+1}.$$

Both subspaces in the above decomposition are closed and orthogonal with respect to the weak scalar product $<,>^{\varrho}$.

P r o o f . First note that $L^{*}\in E_{r+1}(\varrho,\xi)$ and $B^{*} : C^{\infty}(\varrho)\longrightarrow H^{m-r}(\xi)$ is an r-th order differential operator H^{m-r} coefficients, where $m-r>\frac{n}{2}$ and $m-r\geqslant r$. So $L^{*}+B^{*}$ continuously extends to the operator $(L^{*}+B^{*})_{0} : H^{0}(\varrho)\longrightarrow H^{-r-1}(\xi)$ (cf. 3.1). We know by Proposition 3.1 that $(L^{*}+B^{*})_{0}$ is a Fredholm operator, so in particular it has a closed image. Then the (topologically) adjoint operator $(L^{*}+B^{*})_{0}' : H^{-r-1}(\xi)'\longrightarrow H^{0}(\varrho)'$ has also the closed image and, moreover,

$$\text{(x)} \qquad \operatorname{im}(L^{*}+B^{*})_{0}' = (\ker(L^{*}+B^{*})_{0})^{\perp},$$

where the space on the right side is the space of all functionals in $H^{0}(\varrho)'$ vanishing on $\ker(L^{*}+B^{*})_{0}$. We identify $H^{0}(\varrho)'$ and $H^{0}(\varrho)$ by $<,>^{\varrho}$ (applying the Riesz theorem). Since $<,>^{\xi}$ extends to the duality on $H^{-r-1}(\xi)\times H^{r+1}(\xi)$, one obtains an isomorphism $H^{r+1}(\xi)\longrightarrow H^{-r-1}(\xi)'$. Then, composing $(L^{*}+B^{*})_{0}'$ with the above isomorphism and the Riesz isomorphism, we obtain the continuous operator $H^{r+1}(\xi)\longrightarrow H^{0}(\varrho)$. This operator is simply equal $(L+B)_{r+1}$. Thus, the equality (x) and the above remarks imply the following decomposition

$$H^{0}(\varrho) = \ker(L^{*}+B^{*})_{0}\oplus \operatorname{im}(L+B)_{r+1},$$

where both subspaces are closed and $<,>^{\varrho}$ -orthogonal.

Now, applying 3.3 to $L^{*}+B^{*}$ instead of L+B and changing m into m-r, we obtain that $\ker(L^{*}+B^{*})_{0} = \ker(L^{*}+B^{*})_{m+1}$. In particular, $\ker(L^{*}+B^{*})_{0}\subset H^{l}(\varrho)$ for any $0\leqslant l\leqslant m$.

Let $P : H^{0}(\varrho)\longrightarrow H^{0}(\varrho)$ be the orthogonal projector onto the subspace $\ker(L^{*}+B^{*})_{0}$ (defined by the above decomposition of $H^{0}(\varrho)$). Let $w\in H^{l}(\varrho)\subset H^{0}(\varrho)$. Then

$$(w-Pw)\in \operatorname{im}(L+B)_{r+1}\subset \operatorname{im}(L+B)_{0}$$

and also $(w-Pw)\in H^l(\varrho)$ (since $Pw\in H^{m+1}(\varrho)\subset H^l(\varrho)$). By virtue of Proposition 3.1, we get the inclusion

$$(L+B)_o^{-1}(w-Pw)\subset H^{l+r+1}(\xi),$$

so $(w-Pw)\in \mathrm{im}(L+B)_{l+r+1}$. Thus we obtain the decomposition:

$$H^l(\varrho) = \ker(L^*+B^*)_o + \mathrm{im}(L+B)_{l+r+1}.$$

By Lemma 3.4 both subspaces on the right side are closed in $H^l(\varrho)$ and $<,>^{\varrho}$-weakly closed.

4. Generalized Hodge decompositions

Throughout this section D denotes a differential operator of order $s\geqslant 1$ with injective symbol ($D\in \mathrm{Diff}_s(\xi,\varrho)$). Then $L := D^*D$ is the elliptic operator, $L\in E_{2s}(\xi,\xi)$. Let $C : C^\infty(\xi)\longrightarrow H^k(\varrho)$ be a differential operator of order $(s-1)$ with H^k coefficients. In this section we will deal with the operator $D+C : C^\infty(\xi)\longrightarrow H^k(\varrho)$. The purpose of this section is to prove the theorem giving a decomposition of $H^l(\varrho)$ analogous to the Hodge decomposition. In our considerations the operator D+C will play the same role as the exterior differentiation d in the Hodge theory.

Let us notice that $(D+C)^* = D^*+C^*$ is the operator with $H^{k-(s-1)}$ coefficients. Hence, by virtue of Proposition 2.2, for $k>\frac{n}{2}+(s-1)$ the extension $(D+C)_k^* : H^k(\varrho)\longrightarrow H^{k-s}(\xi)$ does exist and the composition

$$\mathcal{L} := (D+C)_k^*\circ(D+C) : C^\infty(\xi)\longrightarrow H^{k-s}(\xi)$$

is the well defined differential operator with H^{k-s} coefficients. We have

$$(D+C)_k^*\circ(D+C) = D^*D + (D_k^*C+C^*D+C_k^*C),$$

so denoting by

$$B := D_k^*C + C^*D + C_k^*C$$

we obtain the equality

$$\mathcal{L} = L + B,$$

where L is an elliptic operator and $B = C^{\infty}(\xi) \to H^{k-s}(\xi)$ is a differential operator of order $r = 2s-1$ with H^m coefficients ($m = k-s$). Now, applying Theorem 3.5 for the operator $\mathcal{L}$, we obtain

4.1. P r o p o s i t i o n . Let $k > \frac{n}{2} + 3s-1$ and $k \geqslant 5s-2$. Then for any l satisfying the inequality $s \leqslant l \leqslant k$ we have

$$H^{l-s}(\xi) = \ker(\mathcal{L}^*)_0 \oplus \operatorname{im} \mathcal{L}_{l+s}.$$

P r o o f . If we substitute $m = k-s$ and $r = 2s-1$ in Theorem 3.5 then we obtain this statement for $\mathcal{L} = L+B$ defined above.

4.2. R e m a r k . As is easily seen, $\mathcal{L}^* = \mathcal{L}$. Thus, by Corollary 3.3, we know that $\ker(\mathcal{L}^*)_0 = \ker \mathcal{L}_0 \subset H^{k+s}(\xi)$. Hence the decomposition in Proposition 4.1 can be rewritten in the following form

$$H^{l-s}(\xi) = \ker \mathcal{L}_0 \oplus \operatorname{im} \mathcal{L}_{l+s} = \ker \mathcal{L}_{l+s} \oplus \operatorname{im} \mathcal{L}_{l+s}.$$

The subspace $\ker \mathcal{L}_0$ is finite dimensional, since $\mathcal{L}_0$ is a Fredholm operator (see 3.1).

4.3. L e m m a . Let $k > \frac{n}{2} + 3s-1$ and $k \geqslant 5s-2$. Then for any $s \leqslant l \leqslant k$

$$\operatorname{im} \mathcal{L}_{l+s} = \operatorname{im}(D^*+C^*)_l.$$

P r o o f . It is clear that $\mathcal{L}_{l+s} = (D^*+C^*)_l(D+C)_{l+s}$. Thus

$$\operatorname{im} \mathcal{L}_{l+s} \subset \operatorname{im}(D^*+C^*)_l.$$

Since for any $w \in H^l(\rho)$ and for any $v \in H^{l+s}(\rho)$

$$\langle (D^*+C^*)_l w,\ v \rangle^{\xi} = \langle w,\ (D+C)_{l+s} v \rangle^{\rho},$$

the subspaces $\mathrm{im}(D^*+C^*)_l$ and $\ker(D+C)_{l+s}$ are $<,>^\xi$-orthogonal. We observe that for any $v \in \ker \mathcal{L}_{l+s}$

$$0 = <v, \mathcal{L}_{l+s}v>^\xi = <v, (D^*+C^*)_l(D+C)_{l+s}v> =$$

$$= <(D+C)_{l+s}v, (D+C)_{l+s}v>, \text{ so } (D+C)_{l+s}v = 0.$$

Thus $\ker \mathcal{L}_{l+s} \subset \ker(D+C)_{l+s}$. The inverse inclusion: $\ker \mathcal{L}_{l+s} \supset \ker(D+C)_{l+s}$ is trivial. Hence $\ker \mathcal{L}_{l+s} = \ker(D+C)_{l+s}$. Since $\mathrm{im}(D^*+C^*)_l$ is weakly orthogonal to $\ker(D+C)_{l+s} = \ker \mathcal{L}_{l+s}$, it follows by 4.2 that

$$\mathrm{im}(D^*+C^*)_l \subset \mathrm{im}\, \mathcal{L}_{l+s}.$$

Composing the above inclusion with the inverse inclusion obtained before, we prove the lemma.

Now we are ready to state the main result of this paper:

4.4. Theorem. Let k,s and l be integers satisfying the following inequalities: $k > \frac{n}{2} + 3s-1$, $k \geq 5s-2$, $s \leq l \leq k$. We assume that $C : C^\infty(\xi) \longrightarrow H^k(\eta)$ is a differential operator of order s-1 with H^k coefficients and $D : C^\infty(\xi) \longrightarrow C^\infty(\eta)$ is a differential operators of order s with injective symbol. Then

$$H^l(\eta) = \mathrm{im}(D+C)_{l+s} \oplus \ker(D^*+C^*)_l.$$

Both subspaces on the right side are closed in $H^l(\eta)$. Moreover, they are $<,>^\eta$-orthogonal and closed in the weak topology defined by $<,>^\eta$.

Proof. It is clear that

$$H^l(\eta) = (D^*+C^*)_l^{-1}(\mathrm{im}(D^*+C^*)_l).$$

Hence, by Lemma 4.3, we obtain

$$H^l(\eta) = (D^*+C^*)_l^{-1}(\mathrm{im}\, \mathcal{L}_{l+s}) = (D^*+C^*)_l^{-1}((D^*+C^*)_l(\mathrm{im}(D+C)_{l+s})) =$$

$$= \mathrm{im}(D+C)_{l+s} + \ker(D^*+C^*)_l.$$

It is easy to verify that $\operatorname{im}(D+C)_{l+s}$ and $\ker(D^*+C^*)_l$ are $<,>^{\varrho}$-orthogonal. Now, applying Lemma 3.4, we prove our assertion.

4.5. Corollary. The operator $(D+C)_{l+s}: H^{l+s}(\xi) \longrightarrow H^l(\varrho)$ has the closed image.

We point out that there exist closed subspaces in $H^l(\varrho)$ (for $l \geq 1$), even finite dimensional, such that $<,>^{\varrho}$-orthogonal subspaces to them are not complementary. It is also possible to construct a continuous linear operator $H^{l+s}(\xi) \longrightarrow H^l(\varrho)$ that has a closed image, but this image does not admit a weakly orthogonal complementary subspace in $H^l(\varrho)$.

Concluding, we would like to mention that the result presented in Theorem 4.4 can be easily improved. For instance, the same decomposition we obtain for any $D \in \operatorname{Int}_s(\xi, \varrho)$ such that $L := D^*D$ is an elliptic operator. On the other hand, the generalized Hodge decomposition given in Theorem 4.4 seems to be sufficient for many applications.

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INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW,
00-661 WARSZAWA
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