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ON THE FIXED POINT PROPERTY OF FINITE ORDERED SETS

Dedicated to the memory of Professor Roman Sikorski

1. Fixed point theorems for ordered sets go back to a theorem found in 1927 by Knaster and Tarski and presented in Knaster [6] (see [6], Lemme (L)). That theorem, stating that if h is a monotone mapping of a family of sets into itself and A a set such that $h(A) \subset A$, then there exists a set $D \subset A$ such that $h(D) = D$, can be regarded as a step in the process of consecutive generalisations of a theorem due to Banach [2] (see [2], Théorème 1), among them Kuratowski [7] and Tarski [12] (in Banach-Tarski [3] the use of the Banach theorem formed an essential ingredient of the apparatus). Sikorski [10] generalized finally the Banach theorem to σ-complete Boolean algebras.

Tarski found in 1939 and published in 1955 (see [13], "the lattice - theoretical fixpoint theorem"; announced in [14]) a generalisation of the Knaster-Tarski theorem to lattices showing, in fact, that every complete lattice has the fixed point property (meaning that each mapping $f: L \to L$ of a complete lattice L into itself such that $f(x) \geq f(y)$ whenever $x \geq y$ has a fixed point x_0, i.e., $f(x_0) = x_0$) and setting the direction for the subsequent research (see [4] for the proof of the converse theorem, i.e., that in the realm of lattices the fixed point property implies completeness and [15] for a generalization to Dedekind - complete structures). Many authors, following the idea of Tarski, have proposed various generalizations of the Tarski theorem and new concepts were developed. We do not intend to go into details here, a survey of the literature on this subject along with a discussion of some new ideas appears in Rival [10]; an exposition and comments on the historical development of the subject will be found in Dugundji and Granas [5]. Let us only mention here a problem that has been posed, viz., to characterise those ordered sets that have the fixed point property (cf. [10], Problem 1).

2. We consider here finite ordered sets. For a finite ordered set P, we consider the set $\mathcal{L}(P)$ of all non-empty chains of elements of P and we define a graph G(P) having $\mathcal{L}(P)$ as the set of vertices. We give in terms of G(P) a condition, sufficient for P to have the fixed point property. We conjecture this condition to be also necessary. As an application, we consider the class of dismantlable finite ordered sets - as shown in Rival [9] (see [9], Corollary 2) those sets have the fixed point property; we show that every dismantlable finite ordered set satisfies our condition.

Following Kuratowski and Mostowski [8], by a graph G on a set of vertices V we mean an antireflexive and symmetric relation $G \subset V^2$ and, for a vertex v, we denote by K(v), resp. L(v), the star of v in G, resp. the antistar of v in G, i.e.,

$$K(v) = \{y \in V \mid (v, y) \in G\} \quad \text{and} \quad L(v) = \{y \in V \mid (v, y) \notin G\};$$

we shall occasionally write K(v, G) to denote the star of v in G when two or more graphs are discussed. We let $G' = V^2 \setminus G$ and we shall say that G' is the complement of the graph G. We shall use occasionally the symbol V(G) to denote the set V of vertices of the graph G and, for a finite set X, we denote by $|X|$ the cardinality of X.

Our terminology concerning fixed point theory follows that of [5]; in particular, a mapping $f: P \to P$ of an ordered set P into itself is said to be isotone if $x \geqslant y$ implies $f(x) \geqslant f(y)$ for each pair x, y of elements of P.

3. To begin with, consider a non-empty finite ordered set P. We define a graph $G(P) \subset \mathcal{L}(P)^2$ by letting

$$G(P) = \{(K, L) \in \mathcal{L}(P)^2 \mid K \cap L = \emptyset\}.$$

We start with the following proposition. For a graph G on a set V, by an endomorphism $h: G' \to G'$ (mod G) of the complement G' of the graph G modulo the graph G we shall mean a mapping h of the set V into itself such that $(v, h(v)) \in G$ for each $v \in V$ and if $v_1 \in L(v_2)$, then $h(v_1) \in L(h(v_2))$ for each pair v_1, v_2 of elements of V (such an h is to be understood as an endomorphism of the relational system $\langle V, G' \rangle$, cf. [8], Ch. II, § 10, satisfying an additional requirement).

P r o p o s i t i o n 1. Let P be a finite ordered set. If there exists an isotone mapping $f: P \to P$ that has no fixed point, then there exists an endomorphism $h: G(P)' \to G(P)$ (mod $G(P)$).

P r o o f . It suffices to observe that if an isotone mapping $f: P \to P$ has no fixed point, then $L \cap f(L) = \emptyset$ for each chain L of elements of P and let $h(L) = f(L)$ for each $L \in \mathcal{L}(P)$.

4. We now consider a graph G on a set V. We show that the problem of existence of an endomorphism of G' modulo G can be reduced to the problem of the existence of an endomorphism $k: G(\min)' \to G(\min)'$ (mod $G(\min)$) for a minimal, in a sense, subgraph $G(\min)$ of the graph G.

For $v \in V$ and for $w \in K(v)$, we shall say that w is an attractor for v if the following condition is satisfied: for each $u \in L(v)$, either $w \in K(u)$ or there exists $z \in K(u)$ such that $w \in L(z)$.

For $v \in V$, we denote by $K^*(v)$ the set of attractors for v, i.e.,

$$K^*(v) = \{ w \in K(v) \mid w \text{ is an attractor for } v \};$$

we shall occasionally write $K^*(v, G)$ when two or more graphs are concerned. The notion of an attractor can be used to state a condition, necessary for the existence of an endomorphism of G' modulo G.

P r o p o s i t i o n 2. If there exists an endomorphism $h: G' \to G' \pmod G$, then $K^*(v) \neq \emptyset$ for every $v \in V$.

P r o o f . Indeed, if there was $v_0 \in V$ with $K^*(v_0) = \emptyset$, then $h(v_0)$ could not be an attractor for v_0, a contradiction.

We shall say that a graph G on a set V satisfies condition (A) if $K^*(v) \neq \emptyset$ for every $v \in V$.

For a graph G on a set V, we consider the set $\mathcal{K}(K) = \{K(v) \mid v \in V\}$ ordered by inclusion $\subset$. We shall say that a vertex $v \in V$ is G-minimal if the star $K(v)$ is a minimal element in the ordered set $\mathcal{K}(G)$. For a subset W of the set V, we define a subgraph $G|W$ by letting

$$G|W = \{(x, y) \in W^2 \mid (x, y) \in G\}.$$

Letting $V^* = \{ v \in V \mid v \text{ is G-minimal}\}$ and $G^* = G|V^*$, we define the subgraph G^* of G. The following theorem shows that G can be reduced to G^* when the problem of the existence of an endomorphism of G' modulo G is concerned.

T h e o r e m 1. An endomorphism $h: G' \to G' \pmod G$ exists if and only if an endomorphism $k: (G^*)' \to (G^*)' \pmod{G\ }$ exists.

P r o o f . We suppose first that an endomorphism $h: G' \to G' \pmod G$ exists and we check that the three following auxiliary statements are true.

S t a t e m e n t I. Let vertices u_1, w_1, u_2, w_2 be such that $u_1 \in L(w_1)$, $K(u_2) \subset K(u_1)$ and $K(w_2) \subset K(w_1)$. Then we have $u_2 \in L(w_2)$.

S t a t e m e n t II. Let vertices v, w_1 and w_2 be such that w_1 is an attractor for v and $K(w_2) \subset K(w_1)$. Then w_2 is an attractor for v.

It should be noted, in particular, that if $w_1 \in K^*(v)$, then there exists a G-minimal vertex $w_2 \in K^*(v)$ with the property that $K(w_2) \subset K(w_1)$.

S t a t e m e n t III. Let vertices v_1, w_1, u_1, v_2, w_2 and u_2 be such that w_1 is an attractor for v_1, w_2 is an attractor for v_2, $K(u_1) \subset K(w_1)$, $K(u_2) \subset K(w_2)$ and $w_1 \in L(w_2)$. Then u_1 is an attractor for v_1, u_2 is an attractor for v_2 and $u_1 \in L(u_2)$.

It should be observed, in particular, that if w_1 is an attractor for v_1, w_2 is an attractor for v_2 and $w_1 \in L(w_2)$, then there exists G-minimal vertices u_1 and u_2 with $K(u_1) \subset \subset K(w_1)$ and $K(u_2) \subset K(w_2)$ such that u_1 is an attractor for v_1, u_2 is an attractor for v_2 and $u_1 \in L(u_2)$.

Returning to the endomorphism $h\colon G' \to G' \pmod G$, for each $v \in V$, we choose a vertex $k(v)$ such that

(i) $\in k(v) \in K(v)$ and $K(k(v)) \subset K(h(v))$;

(ii) $k(v)$ is G-minimal.

This defines $k\colon G' \to G' \pmod G$; it follows from (i) that $(v, k(v)) \in G$ for each $v \in V$ and Statement III along with (i) and (ii) implies that, for a pair v_1, v_2 of vertices of G such that $v_1 \in L(v_2)$, we have $h(v_1) \in L(h(v_2))$ and thus $k(v_1) \in \in L(k(v_2))$. The restriction $k|V^*$ defines an endomorphism $k\colon (G^*)' \to (G^*)' \pmod{G^*}$.

To prove the converse, suppose that there exists an endomorphism $k\colon (G^*)' \to (G^*)' \pmod{G^*}$ and observe that the following statement holds.

S t a t e m e n t IV. Let vertices v, w and w_1 be such that v is an attractor for w and $K(w) \subset K(w_1)$. Then v is an attractor for w_1.

For $v \in V \setminus V^*$, choose a vertex $l(v)$ such that

(iii) $K(l(v)) \subset K(v)$;

(iv) $l(v)$ is G-minimal,

and extend l to V by letting $l(v) = v$ for $v \in V^*$.

Statement IV permits us to define $h: G' \longrightarrow G'$ (mod G) by letting

$$h(v) = k(l(v)) \quad \text{for} \quad v \in V;$$

clearly, (iii) implies that $(v, h(v)) \in G$ for each $v \in V$ and if vertices v_1, v_2 are such that $v_1 \in L(v_2)$, then, by Statement I, $l(v_1) \in L(l(v_2))$ and thus $h(v_1) \in L(h(v_2))$ which completes the proof.

It should be noted that the second part of the proof of Theorem 1 yields the following extension property of endomorphisms.

C o r o l l a r y 1. For each endomorphism $h: (G^*)' \longrightarrow (G^*)'$ (mod G^*) there exists an endomorphism $\tilde{h}: G' \longrightarrow G'$ (mod G) such that $\tilde{h}(v) = h(v)$ for each $v \in V^*$ and $\tilde{h}(V) = h(V^*)$.

To carry the reduction further, consider a graph G over a set V along with an equivalence relation $\mathcal{X}$ defined by letting $v \mathcal{X} w$ if $K(v) = K(w)$ for each pair v, w of elements of V. For each class of equivalence $v_{\mathcal{X}}$, the graph $G|v_{\mathcal{X}}$ is empty (cf. Remark after Statement I in the proof of Theorem 1) and this fact permits us to define a new graph $G_{\mathcal{X}}$ on the set $V_{\mathcal{X}}$ of equivalence classes of $\mathcal{X}$ as follows

$$(v_{\mathcal{X}}, w_{\mathcal{X}}) \in G_{\mathcal{X}} \text{ if and only if } (v, w) \in G.$$

The following proposition permits us to identify in G vertices whose stars are equal when the problem of existence of an endomorphism of G' modulo G is concerned.

P r o p o s i t i o n 3. An endomorphism $h: G' \longrightarrow G'$ (mod G) exists if and only if an endomorphism $k: G'_{\mathcal{X}} \longrightarrow G'_{\mathcal{X}}$ (mod $G_{\mathcal{X}}$) exists.

The proof is obvious.

The tower of a finite graph G is a sequence $G_0, G_1, \ldots, G_k$ of graphs such that $G_0 = G$, $G_{j+1} = (G_j^*)_{\mathcal{X}}$ for $j < k$ and $G_k = (G_k^*)_{\mathcal{X}}$; we denote G_k by the symbol G(min). Theorem 1 along with Proposition 3 implies the following

P r o p o s i t i o n 4. There exists an endomorphism of G' modulo G if and only if there exists an endomorphism of $G(\min)'$ modulo $G(\min)$.

5. We state here a fixed point theorem for ordered sets. Proposition 1 and Proposition 4 imply

T h e o r e m 2. Let P be a finite ordered set. If there is no endomorphism of $G(P)(\min)'$ modulo $G(P)(\min)$, then P has the fixed point property.

Proposition 2 yields

C o r o l l a r y 2. If $G(P)(\min)$ does not satisfy condition (A), then P has the fixed point property; in particular, if $|V(G(P)(\min))| = 1$, then P has the fixed point property.

R e m a r k 1. It may be observed that it follows from the above discussion that each vertex of the graph $G(P)^*$ is a maximal chain of elements of P; in particular, if no element of P is simultaneously minimal and maximal, then no chain of cardinality equal to 1 is a vertex of $G(P)^*$.

6. To give an application of results of Section 5, we consider finite ordered sets having the property of dismantlability (see [1], for the proof that those sets have a stronger fixed point property). We recall that an element a of an ordered finite set P is said to be irreducible if either a has exactly one predecessor or a has exactly one successor and that P is said to be dismantlable if P can be represented as the set $\{a_0, a_1, a_2, \dots, a_n\}$ with a_0 irreducible in P and a_j irreducible in $P \setminus \{a_0, a_1, \dots, a_{j-1}\}$ for $j = 1, 2, \dots, n-1$. We show that every dismantlable finite ordered set P has the property that no endomorphism of $G(P)'$ modulo $G(P)$ exists.

P r o p o s i t i o n 5. If a finite ordered set P is dismantlable, then $|V(G(P)(\min))| = 1$.

P r o o f . By induction $n = |P|$. We begin with the case $n = 2$ and in this case the proposition is manifest. We assume that the proposition is true in the case when $n \leq k$

and consider a dismantlable set P with $|P| = k+1$, i.e., $P = \{a_0, a_1, \dots, a_k\}$, where a_0 is irreducible in P and a_j is irreducible in $P \setminus \{a_0, a_1, \dots, a_{j-1}\}$ for $j = 1,2,\dots,k-1$. Clearly, $P_1 = P \setminus \{a_0\}$ is dismantlable, hence, by the inductive assumption, $|V(G(P_1)(\min))| = 1$. We examine graphs $G(P)^*$ and $G(P_1)^*$; to this end, we consider sets $\mathcal{L}(P)$ and $\mathcal{L}(P_1)$. For a chain $L \subset P_1$ which has the property that $L \cup \{a_0\}$ is a chain of elements of P we let $L^* = L \cup \{a_0\}$; we let $\mathcal{L} = \mathcal{L}(P_1)$ and we denote by $\mathcal{L}_1$ the set $\{L \in \mathcal{L} \mid L^* \text{ is defined}\}$. For $\mathcal{M} \subset \mathcal{L}_1$, we let $\mathcal{M}^* = \{L^* \mid L \in \mathcal{M}\}$. The following sets of chains form a partition of the set $\mathcal{L}(P)$:

(a) the set $\mathcal{L}$;

(b) the set $\mathcal{L}_1^*$;

(c) the set $\{\{a_0\}\}$;

it follows that

(i) $K(L^*, G(P)) = K(L, G(P_1))$ for each $L \in \mathcal{L}_1$;

(ii) $K(L, G(P)) = K(L, G(P_1)) \cup [K(L, G(P_1)) \cap \mathcal{L}_1]^* \cup \{\{a_0\}\}$ for each $L \in \mathcal{L}$, which implies the following two statements.

S t a t e m e n t I. If $L \in \mathcal{L}$ and $L \notin \mathcal{L}_1$, then L is $G(P)$-minimal if and only if L is $G(P_1)$-minimal.

S t a t e m e n t II. If $L \in \mathcal{L}_1$, then L^* is $G(P)$-minimal when L is $G(P_1)$-minimal.

Statements I and II along with Remark 1 and the irreducibility of a_0 imply that the graph $G(P)^*$ can be produced from the graph $G(P_1)^*$ by replacing those L for which L^* is defined with L^* and adding L^* that are $G(P)$-minimal with L not $G(P_1)$--minimal and thus, by Remark 1 again, $G(P)^{**} = G(P_1)^{**}$, which, by the inductive assumption, is sufficient to complete the proof.

7. In Theorem 2 and Corollary 2, above, some conditions for a finite ordered set P sufficient for P to have the fixed point property are stated. We conjecture that the condition stated in Theorem 2 is also necessary.

Conjecture. A finite ordered set P has the fixed point property if and only if there is no endomorphism of G(P)(min) modulo G(P)(min).

It should be observed that the proof of Proposition 5 can be carried on in a more general setting, viz., we shall call a class $\mathcal{P}$ of finite ordered sets decomposable if for each $P \in \mathcal{P}$ there exists an element $a \in P$ such that $P \setminus \{a\} \in \mathcal{P}$; clearly, Proposition 5 remains true for any decomposable class $\mathcal{P}$ with the property that every P in $\mathcal{P}$ has the fixed point property.

The interplay between ordered sets and graphs revealed in Section 4 turns our attention to the question of existence, for a given graph G, of an endomorphism of G modulo G. As shown in Proposition 2 the condition that G satisfy condition (A) is necessary but it is not sufficient as the following example shows.

Example 1. Let $V = \{a, b, c, d, e, f, g, h, i\}$ and $G = \{(a, d), (a, e), (b, f), (b, g), (c, h), (c, i), (d, g), (e, f), (d, i), (e, h), (f, h), (g, i)\}$. The graph G satisfies condition (A) and yet there is no endomorphism of G modulo G.

The following proposition clarifies the role played by condition A. For a set V and a natural number k, we let $[V]^k = \{A \subset V \mid |A| = k\}$. Let G be a graph on a set V; we say that G admits k - selection if, for each $A \in [V]^k$, there exists a mapping $f: A \longrightarrow V$ such that $f(v) \in K^*(v)$ for each $v \in A$ and, for each pair v_1, v_2 of elements of A, if $v_1 \in L(v_2)$, then $f(v_1) \in L(f(v_2))$.

Proposition 6. For each finite graph G we have: G admits 2-selection if and only if G^* satisfies condition (A).

Proof. If G admits no 2-selection consider a pair v_1, v_2 of vertices such that $v_1 \in L(v_2)$ and $w_1 \in K(w_2)$ for each $w_1 \in K^*(v_1)$ and every $w_2 \in K^*(v_2)$; one can suppose that v_1 and v_2 are G-minimal of. Statements I and IV in the proof of

Theorem 1 and thus $K^*(v_1, G^*) = \emptyset = K^*(v_2, G^*)$. To prove the converse, suppose that G admits 2-selection and consider a G-minimal vertex v as well as a G-minimal vertex $w \in K^*(v, G)$. For each G-minimal vertex $u \in L(v)$ choose a G-minimal vertex $\tilde{u}_1 \in K^*(u, G)$; clearly, $\tilde{u} \in L(w)$ and thus $w \in K^*(v, G^*)$ so that G^* satisfies condition (A) and the proposition is proved.

One cannot proceed further along the lines of Proposition 6 as Example 1 shows: the graph G therein admits no 3-selection.

The following problem deserves thus attention not only in the graph - theoretical context - in the light of Theorem 2 its solution would shed light on the structure of finite ordered sets having the fixed point property. By Theorem 1, we can restrict ourselves to graphs whose vertices are all minimal.

P r o b l e m 1. Let G be a finite graph and $G = G(\min)$. What condition on the set $\mathcal{K}(G) = \{K(v) \mid v \in V(G)\}$ is equivalent to the existence of an endomorphism of G' modulo G?

Let us observe that Problem 1 can be stated in terms of choice functions, viz., let X be a finite set and $K = \{K(x) \mid x \in X\}$ a cover of X such that $x \notin K(x)$ for each $x \in X$, if $x \in K(y)$, then $y \in K(x)$ for each pair x, y of elements of X and $K(x) \setminus K(y) \neq \emptyset \neq K(y) \setminus K(x)$ for each pair x, y of elements of X; what condition on K is equivalent to the existence of a choice function h for K such that, for each pair x, y of elements of X, if $x \notin K(y)$, then $k(x) \notin K(h(y))$?

Problem 1 has a counterpart for infinite graphs.

P r o b l e m 2. Characterise those infinite graphs having the property that there exists an endomorphism of the complement modulo the given graph.

Let us finally observe that the results stated above are valid for infinite ordered sets having the property that there exists a natural number k such that each chain of elements of the given set intersects at most k other chains.

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