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BOOLEAN ALGEBRAS WHOSE IDEALS ARE DISJOINTLY GENERATED

Dedicated to the memory of Professor Roman Sikorski

In this paper we define classes of Boolean algebras (BA's for short) by imposing conditions on the generation of ideals. Demanding all ideals of a ring to be finitely generated, leads to the important class of Noetherian rings. For BA's, however, this gives nothing new, since it leads to the class of finite algebras. Trying again, we demand all ideals of a BA to be countably generated. The class obtained that way includes all countable algebras and some more. For a reason that will become clear in section 1 we call them Lindelöf algebras.

Starting with a countable one it is easy to find a set of pairwise disjoint generators for a given ideal. Consequently, the Lindelöf algebras belong to the class of BA's all ideals of which are disjointly generated. These are the main subject of the paper.

There is a different approach to new classes of BA's from the topological side. It is based on the observation that subspaces of Boolean spaces do not necessarily have the properties which one is used to from well-behaved spaces. For example they may fail to be normal. This suggests the consideration

of BA's whose Stone spaces satisfy some topological property hereditarily.

The class we are interested in can be obtained in both ways. It turns out that each ideal of a BA is disjointly generated iff its Stone space is hereditarily paracompact. The reader is supposed to have a working knowledge of Boolean algebra theory. Especially the basic facts about Stone duality (correspondence of ideals and open sets, points and ultrafilters, etc.) are used throughout the paper without further explanation. They can be found in [5].

Most of our results are more or less easy consequences of well-known topological theorems. One more recent of them is due to Šapirovskii and has not yet found its way into the monographs. For the readers convenience we give it with full proof. The other essential topological results are all quoted from the monograph of Engelking [4]. The reader is supposed to have it at hand. Our topological terminology is in accordance with that book.

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0. Notational conventions:

$\mathcal{A}, \mathcal{B}, \mathcal{C}$ denote Boolean algebras; X, Y, Z their respective Stone spaces. If useful, points of X are treated as ultrafilters of $\mathcal{A}$. However, in most cases we consider $\mathcal{A}$ as the algebra of clopen subsets of X.

We use 0, 1, $\wedge$, $\vee$, - for the Boolean operations. a-b means the same as $a \wedge -b$. If we deal with an algebra of sets, then the usual notation $\cap$, $\cup$, $\setminus$ is also used. $\alpha, \beta, \gamma, \ldots$ denote ordinals; $\varkappa$, λ cardinals; and i, j, k, ... natural numbers. $|C|$ stands for the cardinality of the set C.

1. Dualization and definitions

Theorem 1.1. Let $I \subset \mathcal{A}$ be an ideal and $U \subset X$ the corresponding open set. Then the following conditions are equivalent:

(1) I is countably generated.

(2) The subspace U has the Lindelöf property.

(3) U is an F_σ in X.

If one of these conditions is satisfied, then I has a pairwise disjoint, countable set of generators.

P r o o f . (1) ⟶ (2). A countable union of compact sets is always Lindelöf.

(2) ⟶ (3). U is even a countable union of compact sets.

(3) ⟶ (1). Suppose $U = \bigcup_{n<\omega} F_n$ where each F_n is closed in X, hence compact. It is easy to find clopen sets A_n such that $F_n \subset A_n \subset U$. Then $U = \bigcup_{n<\omega} A_n$. So I is generated by $\{A_n | n<\omega\}$. Setting $B_o = A_o$, $B_{n+1} = A_{n+1} \setminus \bigcup_{m\leq n} B_m$, we arrive at a countable disjoint set of generators for I.

T h e o r e m 1.2. Let $I \subset \mathfrak{A}$ be an ideal, $U \subset X$ the corresponding open set. I has a disjoint set of generators iff the subspace U is paracompact.

P r o o f . If I is disjointly generated, then U is the disjoint union of compact clopen subspaces and, therefore, paracompact ([4],5.1.30)..

Suppose now that U is paracompact. Being locally compact it can be represented as a union of pairwise disjoint open subsets each of which has the Lindelöf property ([4], 5.1.27). By 1.1 each of the parts is a disjoint union of clopen sets and so is U.

Applied to all ideals of a given algebra the preceding theorems yield

C o r o l l a r y 1.3. The following conditions are equivalent:

(1) Each ideal of $\mathfrak{A}$ is countably generated.

(2) Each open subspace of X is Lindelöf.

(3) X is perfectly normal (i.e., normal and all closed subsets are G_δ).

C o r o l l a r y 1.4. Each ideal of $\mathfrak{A}$ is disjointly generated iff each open subspace of X is paracompact.

It is an easy exercise to see that in both cases the restriction to open subspaces is unnecessary, i.e. X is even hereditarily Lindelöf resp. paracompact. We use the words paracompact (pc) algebra and Lindelöf algebra in accordance with the corollaries.

Note that a BA is Lindelöf iff it is pc and satisfies the ccc. Clearly, all countable BA's are Lindelöf, hence pc. Next we give an alternative characterization of the Lindelöf property. It spares us the consideration of yet another class of BA's.

P r o p o s i t i o n 1.5. A BA is Lindelöf iff each of its ideals is generated by a chain.

P r o o f . One direction is trivial because every countably generated ideal can be generated by a chain.

Suppose each ideal of $\mathfrak{A}$ is generated by a chain. First we prove that $\mathfrak{A}$ satisfies the ccc. Otherwise, there were a family $(a_\alpha)_{\alpha<\omega_1}$ of non-zero pairwise disjoint elements. Let $B\subset\mathfrak{A}$ be a chain generating the same ideal. Then all sets $S(b)=\{\alpha \mid a_\alpha\wedge b\neq 0\}$ are finite, all $\alpha<\omega_1$ fall in some $S(B)$ with $b\in B$, and $S(b)\subset S(c)$ for $b\leqslant c$. We conclude that ω_1 would be the union of an increasing sequence of finite sets, which is not true.

Every linear ordering has a cofinal well-ordered subset. Each cofinite subchain generates the same ideal. Consequently, all ideals of $\mathfrak{A}$ are generated by well-ordered chains. This implies that $\mathfrak{A}$ is Lindelöf since every well-ordered chain in a ccc algebra is countable.

Applied to one-point sets 1.3.(3) implies that the Stone spaces of Lindelöf algebras are first countable. This is not true for pc algebras in general, but we have

P r o p o s i t i o n 1.6. If $\mathfrak{A}$ is pc, then X is a Fréchet space (i.e., $x\in\bar{C}\setminus C$ implies the existence of a sequence $c_n\in C$ converging to x).

P r o o f . Consider $C\subset X$ and $x\in\bar{C}\setminus C$. The prime ideal corresponding to $X\setminus\{x\}$ is disjointly generated. Therefore, we find pairwise disjoint non-empty clopen U_α, $\alpha<\varkappa$ such that

$X \setminus \{x\} = \bigcup_{\alpha<\varkappa} U_\alpha$. C must meet infinitely many U_α, since each finite union of them is closed and does not contain x. So we can choose different $\alpha_n < \varkappa$ and $c_n \in C \cap U_{\alpha_n}$. It is then easy to see that $(c_n)_{n<\omega}$ converges to x.

Example 1.7. (1) Infinite complete algebras cannot be paracompact, for, in their Stone spaces, each converging sequence is eventually constant.

(2) If $\mathfrak{A}$ is an uncountable subalgebra of a free one, then $\mathfrak{A}$ is not pc. This follows from Efimov's result that dyadic Fréchet spaces are metrizable ([3], Theorem 26, stronger results are in [4], 3.12.12).

In connection with our notion of a pc algebra it is natural to ask for strongly (weakly) paracompact algebras (with the obvious definition).

Strong paracompactness gives nothing new because every disjoint cover is star-finite.

On the other hand, there are weakly paracompact BA's that are not pc (see example 6.4 below). In the presence of the ccc the two notions, however, coincide. Indeed, every point-finite open cover of a locally compact ccc space is countable ([1], Proposition 1).

2. Operations on pc algebras

Theorem 2.1. The class of pc BA's is closed under homomorphic images and subalgebras.

Proof. If $\mathfrak{B}$ is a homomorphic image of $\mathfrak{A}$, then Y is a subspace of X. If X is hereditarily paracompact, then so is Y. If $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, then there is a closed mapping f of X onto Y. Consider an open subset U of Y. $V = f^{-1}(U)$ is open in X, hence paracompact. $f|V$ remains a closed mapping, which implies the paracompactness of U by virtue of the Michael theorem ([4], 5.1.33).

Proposition 2.2. Let $\mathfrak{A}$ be pc and satisfy the $\varkappa$-chain condition (i.e., every disjoint subset of $\mathfrak{A}$ has power $<\varkappa$). If $\mathfrak{B}$ is either a subalgebra or a homomorphic image of $\mathcal{A}$, then $\mathcal{B}$ satisfies the $\varkappa$-chain condition, too. In particular, if $\mathcal{A}$ is Lindelöf, then $\mathcal{B}$ is Lindelöf.

P r o o f . The assertion on subalgebras is obvious. Consider $\mathcal{B} = \mathcal{A}/I$ for some ideal I and suppose, by contradiction, the existence of elements a_α, $\alpha < \varkappa$ such that $a_\alpha \notin I$, but $a_\alpha \wedge a_\beta \in I$, for all $\alpha \neq \beta$.

Let J be the ideal generated by $\{a_\alpha | \alpha < \varkappa\}$ and fix a disjoint set of generators $\{b_\beta | \beta < \lambda\}$ for J. λ must be less than $\varkappa$, since $\mathcal{A}$ satisfies the $\varkappa$-chain condition. Every b_β is covered by finitely many a_α. Consequently, λ of them suffice to generate J. Without loss of generality we can assume that $\{a_\alpha | \alpha < \lambda\}$ generates J. Then we find $\alpha_1, \ldots, \alpha_n < \lambda$ such that $a_\lambda \leq a_{\alpha_1} \vee \ldots \vee a_{\alpha_n}$.

Therefore, $a_\lambda = (a_{\alpha_1} \wedge a_\lambda) \vee \ldots \vee (a_{\alpha_n} \wedge a_\lambda)$ and, since each $a_{\alpha_i} \wedge a_\lambda$ belongs to I, so does a_λ, contradiction.

This proposition shows again that uncountable free BA's cannot be pc, because they are ccc, but have non-ccc homomorphic images. The same holds for the power-set algebra on ω. On the other hand, this algebra is isomorphic to the direct product of countably many copies of the two-element algebra. We conclude that paracompactness is not preserved by (infinite) direct products.

P r o p o s i t i o n 2.3. Finite direct products and arbitrary weak direct products of pc algebras are pc.

P r o o f . The assertion concerning finite products is obvious. Suppose $(\mathcal{A}_\alpha)_{\alpha < \varkappa}$ is a family of pc algebras with Stone spaces X_α. Denote their weak product by $\sum_{\alpha < \varkappa} \mathcal{A}_\alpha$ and its Stone space by X. Then X is homeomorphic to the one-point compactification of $\bigoplus_{\alpha < \varkappa} X_\alpha$. It is an easy exercise to see that X is hereditarily paracompact provided all X_α are.

Before continuing, the reader should recall the concept of a superatomic BA. This property can be characterized in terms of a transfinite sequence of ideals $I_\alpha(\mathcal{A})$ attached to each BA in the following way:

$I_0(\mathcal{A}) = \{0\}$,
$I_{\alpha+1}(\mathcal{A})$ = the ideal generated by

$$\{a \in \mathcal{A} \mid a/I_\alpha(\mathcal{A}) \text{ is an atom of } \mathcal{A}/I_\alpha(\mathcal{A})\},$$

$I_\gamma(\mathcal{A}) = \bigcup_{\alpha<\gamma} I_\alpha(\mathcal{A})$, for limit ordinals γ.
$\mathcal{A}$ is superatomic iff $I_\alpha(\mathcal{A}) = \mathcal{A}$ for some α. The first of such α is an invariant of $\mathcal{A}$.

Now we use the operations considered so far to produce many paracompact algebras.

Theorem 2.4. For each infinite cardinal $\varkappa$ there are $2^\varkappa$ non-isomorphic pc BA's of power $\varkappa$.

Proof. We start with an auxiliary construction formerly used by Paljutin [10].

Suppose $\mathcal{A}, \mathcal{B}$ are BA's; p,q ultrafilters of $\mathcal{A}$ resp. $\mathcal{B}$. Then $\{(a,b) \in \mathcal{A} \times \mathcal{B} \mid a \in p \text{ iff } b \in q\}$ is easily seen to be a subalgebra of $\mathcal{A}\times\mathcal{B}$, which we denote by $(\mathcal{A},p) \vee (\mathcal{B},q)$. To visualize this operation topologically, consider p and q as points of the Stone spaces X resp. Y. The Stone space of $(\mathcal{A},p) \vee (\mathcal{B},q)$ is what topologists call a bouquet, the factor space of $X \oplus Y$ resulting from the identification of p and q. Being a subalgebra of their product, the bouquet of two pc algebras is pc.

Put $\mathcal{B}_0 = \{0,1\}$, $\mathcal{B}_{\alpha+1} = \sum_{i<\omega} \mathcal{C}_i$, with each $\mathcal{C}_i = \mathcal{B}_\alpha$,
$\mathcal{B}_\gamma = \sum_{\alpha<\gamma} \mathcal{B}_\alpha$, for a limit ordinal γ.

A simple induction shows that each $\mathcal{B}_\alpha$ is a superatomic pc algebra of power $\leq |\alpha| + \omega$ with $I_\alpha(\mathcal{B}_\alpha)$ prime. Denote the corresponding ultrafilter by q_α. Let $\mathcal{A}$ denote the countable atomless BA and fix some ultrafilter p of $\mathcal{A}$. To prove the theorem, let $S \subset \varkappa$ be an arbitrary subset. From the algebra $\mathcal{A}_S = \sum_{\alpha<\varkappa} \mathcal{L}_\alpha$, where

$$\mathcal{L}_\alpha = \begin{cases} (\mathcal{A},p) \vee (\mathcal{B}_\alpha, q_\alpha), & \text{if } \alpha \in S \\ \mathcal{A}, & \text{otherwise.} \end{cases}$$

$\mathcal{O}\!\mathcal{l}_S$ is pc and has power $\varkappa$. Moreover, it is not hard to see that S can be recovered from the isomorphism type of $\mathcal{O}\!\mathcal{l}_S$. (Imagine the Stone space and note that the ordinal α is a topological invariant of the point resulting from the identification of p and q_α).

Example 2.5. Let SAPC denote the smallest class of BA's containing the two-element one and being closed under finite direct products and arbitrary weak direct products. Then SAPC is identical with the class of all superatomic paracompact algebras.

(Hint: Let $\mathcal{L}$ be superatomic and pc. Consider the ideal $I = \{b \in \mathcal{L} \mid b=0 \text{ or for all } 0 < a \leq b \ \ \mathcal{L}|a \in \mathrm{SAPC}\}$. Show that $\mathcal{L}/I$ cannot contain an atom. Then $I = \mathcal{L}$, consequently $\mathcal{L} \in \mathrm{SAPC}$).

3. Two more examples

Example 3.1. Let $\varkappa$ be a cardinal. We already know that there are $2^\varkappa$ pc algebras of cardinality $\varkappa$. By now, all our examples have many countable principal ideals. Next we construct a pc algebra $\mathcal{L}$ such that for all $b \in \mathcal{L} \setminus \{0\}$ $|\{a \in \mathcal{L} \mid a \leq b\}| = \varkappa$. Namely, $\mathcal{L}$ is the subalgebra generated by all sets $U_n(f) = \{g \in \varkappa^\omega \mid g|n = f|n\}$ in the power-set algebra on $\varkappa^\omega$. f and n run over $\varkappa^\omega$ and ω, respectively. There is no difficulty in checking that two generators are either disjoint or one contains the other. More precisely, $U_n(f) \subset U_m(g)$ iff $n \geq m$ and $f|m = g|m$. From this it easily follows that each element of $\mathcal{L}$ can be written as a finite union of elements of the form

$$U_n(f) \setminus \bigcup_{i=1}^{m} U_{n_i}(g_i)$$

with $n_i \geq n$ and $g_i|n = f|n$. (We allow $n = 0$ and $m = 0$). Elements of that form will be called primitive.

To prove paracompactness, consider an ideal $I \subset \mathcal{L}$. Call an element of I prime of size n if it is of the form $U_n(f) \setminus \bigcup_{i=1}^{m} U_{n+1}(g_i)$ and there is no $U_n(f) \setminus \bigcup_{i=1}^{m-1} U_{n+1}(h_i)$ in I. (Again we allow $m = 0$).

Note that two prime elements of the same size are either identical or disjoint. We claim that I is generated by its prime elements. To see this, consider a primitive element $U = U_n(f) \setminus \bigcup_{i=1}^{m} U_{n_i}(g_i)$ belonging to I. By induction on $k = \max\{n_i\} - n$ we prove that U is covered by a finite union of prime elements. In case $k = 1$ we have $U = U_n(f) \setminus \bigcup_{i=1}^{m} U_{n+1}(g_i)$, and it is easy to see that U is contained in one prime element. For $k > 1$ U can be written as $\left[U_n(f) \setminus \bigcup_{i=1}^{m} U_{n+1}(g_i)\right] \cup \bigcup_{j=1}^{m} \left[U_{n+1}(g_j) \setminus \bigcup_{i=1}^{m} U_{n_i}(g_i)\right]$ and the induction hypothesis applies to each element in brackets.

Put $A_0 = \emptyset$ and choose, by induction, A_{n+1} to be a maximal pairwise disjoint set of prime elements of size n disjoint to all elements of $\bigcup_{j \leq n} A_j$. It is then easily seen that $\bigcup_{n<\omega} A_n$ is a pairwise disjoint set generating I.

Example 3.2. (CH) In [14] Shelah used CH to produce a very peculiar BA. In the following we describe a rough version of his method, which guarantees the Lindelöf property only. It has the advantage of being easily visualized and it keeps one essential part of what is done in [14].

We start with the countable atomless BA $\mathfrak{a}$ conceived as the algebra of clopen subsets of X. Let $U_\alpha, \alpha < \omega_1 = 2^\omega$, be an enumeration of all open subsets of X. Choose, by induction, points $x_\alpha, \alpha < \omega_1$ such that

$$x_\alpha \notin \bigcup_{\beta<\alpha} \operatorname{Fr} U_\beta \cup \{x_\beta \mid \beta < \alpha\}.$$

This is possible by the Baire Category Theorem, since each $\operatorname{Fr} U_\beta$ ($=\bar{U}_\beta \setminus U_\beta$) is nowhere dense. Now fix regular open sets C_α such that $\bar{C}_\alpha \setminus C_\alpha = \{x_\alpha\}$. (To find one, represent $X \setminus \{x_\alpha\}$ as $\bigcup_{n<\omega} V_n$ with V_n non-empty, disjoint, clopen, and put $C_\alpha = \bigcup_{n<\omega} V_{2n}$). Let us note that all the delicacy of Shelah's construction lies in the proper choice of the C_α.

Denote by $\mathscr{B}$ the subalgebra of the regular-open algebra on X (consult [5, §4] for the definition) generated by $\mathfrak{A}$ and $\{C_\alpha | \alpha < \omega_1\}$. This is the desired algebra. First we check that each element of $\mathscr{B}$ can be written as

$$\bigvee_{i=1}^{l} a_i \vee \bigvee_{j=1}^{m} (b_j \wedge C_{\alpha_j}) \vee \bigvee_{k=1}^{n} (c_k - C_{\alpha_k})$$

where a_i, b_j, c_k are elements of $\mathfrak{A}$. This is based on the observation that each element of $\mathscr{B}$ has (as a regular open set) a boundary consisting of finitely many x_α's, which can be separated by elements of $\mathfrak{A}$. Notice that $C_\alpha \wedge a$ and $a - C_\alpha$ are both clopen, hence elements of $\mathfrak{A}$, whenever $x_\alpha \notin a$.

As an example we give a representation of $C_\alpha \wedge C_\beta \wedge - C_\gamma$ in the desired form. Choose $a, b, c \in \mathfrak{A}$ pairwise disjoint, with $a \vee b \vee c = 1$ and $x_\alpha \in a$, $x_\beta \in b$, $x_\gamma \in c$. Then we have

$$\begin{aligned} C_\alpha \wedge C_\beta \wedge - C_\gamma &= [(C_\alpha \wedge C_\beta \wedge - C_\gamma) \wedge a] \vee [(C_\alpha \wedge C_\beta \wedge - C_\gamma) \wedge b] \vee \\ &\quad \vee [(C_\alpha \wedge C_\beta \wedge - C_\gamma) \wedge c] = \\ &= [C_\alpha \wedge (C_\beta \wedge a) \wedge (a - C_\gamma)] \vee [C_\beta \wedge (C_\alpha \wedge b) \wedge (b - C_\gamma)] \vee \\ &\quad \vee [C_\alpha \wedge c) \wedge (C_\beta \wedge c) \wedge - C_\gamma]. \end{aligned}$$

This is the desired form, since all expressions in parenthesis are elements of $\mathfrak{A}$.

Let $I \subset \mathscr{B}$ be an ideal and $U \subset X$ the open set corresponding to $I \cap \mathfrak{A}$. We claim that I is generated by the countable set $(I \cap \mathfrak{A}) \cup \{a \wedge C_\alpha \in I \mid a \in \mathfrak{A},\ x_\alpha \in \mathrm{Fr}\ U\} \cup \{a - C_\alpha \in I \mid a \in \mathfrak{A},\ x_\alpha \in \mathrm{Fr}\ U\}$. To see this, consider an element $D \in I$ of the form $b \wedge C_\alpha$ or $b - C_\alpha$ with $x_\alpha \notin \mathrm{Fr}\ U$. Regarded as an open set, D is contained in U, consequently, $\bar{D} \subset \bar{U}$. But $\bar{D} \subset D \cup \{x_\alpha\}$ and $x_\alpha \notin \bar{U} \setminus U$, hence even $\bar{D} \subset U$. Since $\bar{D}$ is compact, there is a clopen a with $\bar{D} \subset a \subset U$. Clearly, $a \in I \cap \mathfrak{A}$ and $D \leq a$ in the sense of $\mathscr{B}$. A similar argument shows that $\mathscr{B}/I$ is countable whenever I is a dense ideal. This property implies retractiveness (see section 6 below).

4. Chains in paracompact algebras

An important class of BA's arises from linear orderings. Let $(A,<)$ be a linear ordering. The set of all finite unions of left-closed, right-open intervals $[a_1,b_1) \cup \ldots \cup [a_n,b_n)$, with $a_i,b_i \in A \cup \{\pm\infty\}$, forms together with $\emptyset$ a BA under the set-theoretical operations. It is called the interval algebra on A and denoted by $\mathcal{J}(A)$. The reader may consult [9] for details on interval algebras. There he will find a proof of the important fact that the topology of the Stone space of an interval algebra can be generated by a linear ordering.

Theorem 4.1. Let $(A,<)$ be a linear ordering. $\mathcal{J}(A)$ is pc iff each convex subset U of A contains a countable subset, which is cofinal and coinitial in U. In particular, if $(A,<)$ is cco, then $\mathcal{J}(A)$ is pc.

Proof. If the condition on the order is violated, then either ω_1 or its reverse can be (order-theoretically) embedded into A. Then ω_1+1 can be (topologically) embedded into the Stone space of $\mathcal{J}(A)$. But this is impossible if it is Fréchet. To prove sufficiency, consider an ideal $I \subset \mathcal{J}(A)$. Put $U = \bigcup \{[a,b) \mid [a,b) \in I\} \subset A$. For $a,b \in U$ define $a \sim b$ iff $[\min(a,b),\max(a,b)) \in I$. Obviously, $\sim$ is an equivalence relation whose classes are convex subsets of A. We want to find pairwise disjoint intervals $[a_\alpha,b_\alpha) \in I$ such that each $[a,b) \in I$ is covered by a finite number of them.

If $[a,b) \in I$, then $[a,b)$ is contained in one equivalence class under $\sim$. This shows that we can handle each of these classes separately, that is, we can assume that U itself consists of just one class.

Several cases are possible. The others being similar, we confine ourselves to the one in which U has no first but a last element, say b. It splits into two subcases according to whether $[-\infty,b)$ belongs to I or not. $b \in U$ implies the existence of some $d > b$ (possibly ∞) with $[b,d) \in I$. $b = \max U$, hence $(b,d) = \emptyset$. In the first subcase we find $U = [-\infty,d)$ and I is even principal. If $[-\infty,b) \notin I$, then we take, by assumption, a strictly decreasing coinitial sequence a_n.

Clearly, $U = [a_0,d) \cup \bigcup_{n<\omega} [a_{n+1},a_n)$, and these are the desired intervals.

R e m a r k 4.2. (1) It may seem that we have proved $\mathcal{J}(A)$ to be even Lindelöf. This is not true, since $\sim$ can have uncountably many equivalence classes. Even if A is ccc (i.e., every family of pairwise disjoint open intervals is countable), $\mathcal{J}(A)$ need not be Lindelöf.

(Hint: Take the lexicographic order of reals $\times\{0,1\}$).

(2) If, on the other hand, $\mathcal{J}(A)$ is ccc, then A must be ccc nad thus $\mathcal{J}(A)$ is Lindelöf. There is a much stronger result in this respect. From Theorem 2 in [8] it follows that each ccc subalgebra of an interval algebra is Lindelöf.

The interval algebra on the reals is Lindelöf and has cardinality 2^ω. Next we show that this is the maximum.

P r o p o s i t i o n 4.3. (1) If $\mathcal{A}$ is a pc interval algebra, then $|\mathcal{A}| \leq |X| \leq 2^\omega$.
(2) Every chain in a pc algebra has power at most 2^ω.
(3) Every well-ordered chain in a pc algebra is countable.

P r o o f . (1) $|\mathcal{A}| \leq |X|$ is independent of paracompactness. X is a linearly ordered Fréchet space, hence first countable. Since it is also compact, $|X| \leq 2^\omega$ follows from Arhangel'skiǐ's Theorem ([4], 3.1.29).
(2) follows immediately from (1) and 2.1.
(3) is an immediate consequence of 4.1 and 2.1.

So far, all our examples of pc algebras have had many countably generated ultrafilters. The next theorem shows that the reason is not our lack of imagination. The proof given below is due to the referee. It is much simpler than the original one that proved density only.

T h e o r e m 4.4. If $\mathcal{A}$ is pc, then the set FC of countably generated ultrafilters (i.e., points of countable character) is dense and of the second category in X.

The following argument is used twice in the proof of 4.4 and, therefore, we separate it as

L e m m a 4.5. If $G \subset X$ is a closed, nowhere dense G_δ, then $G \subset FC$.

P r o o f . Consider $x \in G$ and choose families $(A_\alpha)_{\alpha<\varkappa}$ and $(B_n)_{n<\omega}$ of disjoint, clopen, non-zero subsets of X such that $X \setminus G = \bigcup_{n<\omega} B_n$ and $X \setminus \{x\} = \bigcup_{\alpha<\varkappa} A_\alpha$. Since each B_n is covered by a finite number of A_α's, the sets $S(n) = \{\alpha \mid B_n \cap A_\alpha \neq \emptyset\}$ are all finite. Moreover, all A_α intersect some B_n, because G is nowhere dense. This implies $\varkappa = \bigcup_{n<\omega} S(n)$. So $\varkappa$ must be countable, as was to be shown.

P r o o f of 4.4. First we show that FC is dense in X. Consider any clopen $U \subset X$. If U contains an isolated point, then $FC \cap U \neq \emptyset$. Otherwise, consider a maximal chain C of clopen, non-void subsets of U. By 4.1 C has a countable co-initial subchain. Therefore, $\emptyset \neq G = \bigcap C$ is a closed G_δ. Since U contains no isolated point and C is maximal, G must be nowhere dense. By the lemma, $G \subset FC$, hence $FC \cap U \neq \emptyset$. If FC were a first-category set, then one could find a closed G_δ, say H, disjoint from FC. Since FC is dense, H would be nowhere dense. But then $H \subset FC$, by the lemma, a contradiction.

R e m a r k 4.6. Every basic clopen subspace U of X meets the assumption of the theorem. Therefore, $FC \cap U$ is of the second category in U, and FC is even of the second category at each point of X.

5. Cardinal functions

Many cardinal functions for BA's have been introduced and extensively studied in recent years. Some of the results obtained so far (e.g. 1.7 (2) or 4.3) can be restated in terms of cardinal functions. In this section we want to add some new ones. The key to these results is a strong topological theorem due to Šapirovskiĭ (Theorem 2.1 of [13]). Since it has not yet found its way into the monographs and the original paper is not too widespread, we give the full proof of the special case we need.

T h e o r e m 5.1. Suppose $\mathfrak{A}$ is po and satisfies the λ-chain condition for some λ with $cf(\lambda) > \omega_1$. Then each dense subset C of X has a subset D which is still dense and has cardinality less than λ.

P r o o f . Suppose $\bar{C} = X$. For $\alpha < \omega_1$ we choose inductively subsets $C_\alpha \subset C$ and families B_α of pairwise disjoint clopen subsets of X such that:

(1) $|C_\alpha| < \lambda$ and $|B_\alpha| < \lambda$

(2) $C_\alpha \subset C_\beta$ for $\alpha < \beta$

(3) $X \setminus \bar{C}_\alpha = \bigcup B_\alpha$

(4) If K is a finite subfamily of $\bigcup_{\alpha<\beta} B_\alpha$ and $\bigcap K \neq \emptyset$, then $C_\beta \cap \bigcap K \neq \emptyset$.

Suppose C_α, B_α are constructed for $\alpha < \beta < \omega_1$. (1) and $\beta < \omega_1 < \mathrm{cf}(\lambda)$ imply $|\bigcup_{\alpha<\beta} B_\alpha| < \lambda$. Therefore, the number of finite subfamilies K to be considered at step β is less than λ. Since $\bigcap K$ is open and C is dense, we can choose $c_K \in \bigcap K \cap C$ provided that $\bigcap K$ is not empty. Put $C_\beta = \bigcup_{\alpha<\beta} C_\alpha \cup \{c_K \mid K \subset \subset \bigcup_{\alpha<\beta} B_\alpha,\ \text{finite},\ \bigcap K \neq \emptyset\}$. Since A is pc, there is a disjoint family B_β of non-empty clopen sets such that $X \setminus \bar{C}_\beta = \bigcup B_\beta$. The λ-chain condition yields $|B_\beta| < \lambda$. The set $D = \bigcup_{\alpha<\omega_1} C_\alpha$ is a subset of C of cardinality less than λ. To prove $\bar{D} = X$, suppose, by contradiction, the existence of some $x \in X \setminus \bar{D}$. Clearly, $x \in X \setminus \bar{C}_\alpha$, hence $x \in U_\alpha$, for some $U_\alpha \in B_\alpha$. By construction, $\bar{C}_\alpha \subset X \setminus U_\alpha$. Since X is Fréchet, we have $\bar{D} = \bigcup_{\alpha<\omega_1} \bar{C}_\alpha$, hence $\bar{D} \subset \bigcup_{\alpha<\omega_1} (X \setminus U_\alpha)$. By compactness, a finite union is sufficient to cover $\bar{D}$, say $\bar{D} \subset (X \setminus U_{\alpha_1}) \cup \ldots \cup (X \setminus U_{\alpha_n})$. Fix some $\beta > \alpha_1, \ldots, \alpha_n$ and consider $K = \{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$. $\bigcap K$ is not empty, since it contains x. Therefore, at stage β of the construction we chose $c_K \in C \cap \bigcap K$. Thus $D \cap \bigcap K \neq \emptyset$, contradicting $\bar{D} \subset X \setminus \bigcap K$.

From 5.1. we now obtain results for the following cardinal functions:

$c(\mathfrak{A}) = \sup \{ |B| \mid B \text{ is a disjoint subsets of } \mathfrak{A} \}$ (cellularity)

$d(\mathfrak{A}) = \inf \{ |\mathcal{B}| \mid \mathcal{B} \text{ is a dense subalgebra of } \mathfrak{A} \}$ (density)

T h e o r e m 5.2. If $\mathfrak{A}$ is pc, then $\mathfrak{A}$ contains a disjoint subset of cardinality $c(\mathfrak{A})$, i.e., the supremum is attained.

P r o o f . This is clear if $c(\mathfrak{A})$ is a successor or ω. For $c(\mathfrak{A})$ singular, the property does not depend on paracompactness and is known (cf. [6], Theorem 3.1). Suppose now that $c(\mathfrak{A})$ is a regular limit cardinal. Then $cf(c(\mathfrak{A})) = c(\mathfrak{A}) > \omega_1$. If $c(\mathfrak{A})$ were not attained, then 5.1 would yield a dense subset of X of some cardinality less than $c(\mathfrak{A})$. But this is impossible.

T h e o r e m 5.3. If $\mathfrak{A}$ is pc, then $c(\mathfrak{A}) \leq |\mathfrak{A}| \leq c(\mathfrak{A})^{\omega}$.

P r o o f . $c(\mathfrak{A}) \leq |\mathfrak{A}| \leq |X|$ holds for all BA's. To prove $|X| \leq c(\mathfrak{A})^{\omega}$, suppose first that $c(\mathfrak{A})$ is uncountable. Then $cf(c(\mathfrak{A})^{+}) > \omega_1$ and 5.1 yields a dense subset D of X with $|D| \leq c(\mathfrak{A})$. From the fact that X is Fréchet we conclude that $|X| = |\bar{D}| \leq |D|^{\omega} \leq c(\mathfrak{A})^{\omega}$. Consider now the Lindelöf case, i.e., $c(\mathfrak{A}) = \omega$. 5.1 can be applied to $\lambda = \omega_2$ and yields a dense subset of cardinality less than ω_2. As in the first case, $|X| \leq \omega_1^{\omega} = \omega^{\omega} = c(\mathfrak{A})^{\omega}$.

T h e o r e m 5.4. If $\mathfrak{A}$ is pc and $c(\mathfrak{A}) \geq \omega_1$, then $c(\mathfrak{A}) = d(\mathfrak{A})$. If $\mathfrak{A}$ is Lindelöf, then $d(\mathfrak{A}) \leq \omega_1$.

P r o o f . We prove the first statement and indicate the changes in the Lindelof case in brackets. 5.1 yields a dense subset D of X with $|D| = c(\mathfrak{A})$ $[|D| \leq \omega_1]$. For each $x \in D$ fix a family B_x of clopen sets such that $\{x\} = \bigcap B_x$ and $|B_x| \leq c(\mathfrak{A})$ $[|B_x| \leq \omega]$. It is easily seen that $\bigcup_{x \in D} B_x$ generates a dense subalgebra $\mathcal{L}$ of $\mathfrak{A}$. Consequently, $d(\mathfrak{A}) \leq |\mathcal{L}| \leq |D| \cdot \sup |B_x| \leq c(\mathfrak{A}) \cdot c(\mathfrak{A}) = c(\mathfrak{A})$. $[d(\mathfrak{A}) \leq |\mathcal{L}| \leq |D| \cdot \omega \leq \omega_1]$. The inverse inequality $c(\mathfrak{A}) \leq d(\mathfrak{A})$ is obvious and does not depend on paracompactness.

R e m a r k 5.5. In the previous proof we have established that $d(\mathfrak{A})$ is equal to the density character of X provided that $\mathfrak{A}$ is pc.

T h e o r e m 5.6. If $\mathfrak{A}$ is pc, then $\mathfrak{A}$ has exactly $2^{c(\mathfrak{A})}$ ideals.

P r o o f . A family of $c(\mathfrak{A})$ pairwise disjoint elements of $\mathfrak{A}$ gives, obviously, rise to $2^{c(\mathfrak{A})}$ different ideals. On the other hand, each ideal is generated by $\leqslant c(\mathfrak{A})$ elements. Therefore, their number is bounded by $|\mathfrak{A}|^{c(\mathfrak{A})} \leqslant c(\mathfrak{A})^{\omega \cdot c(\mathfrak{A})} = 2^{c(\mathfrak{A})}$.

R e m a r k s 5.7. (1) Consider some cardinal $\varkappa > 2^{\omega}$ and let $\mathfrak{A}$ denote the free BA of cardinality $\varkappa$. Then $c(\mathfrak{A}) = \omega$, $|\mathfrak{A}| = d(\mathfrak{A}) = \varkappa$, and $\mathfrak{A}$ has $2^{\varkappa}$ ideals. This shows that 5.3, 5.4, and 5.6 do not hold for non-pc algebras.

(2) A non-pc counterexample to 5.2 is the free product of the sequence $\mathfrak{A}_{\varkappa}$, $\varkappa < \lambda$, where λ is a regular limit cardinal and $\mathfrak{A}_{\varkappa}$ the algebra of finite and cofinite subsets of $\varkappa$. For details see ([6], example 6.5).

(3) The second statement of 5.4 cannot be improved in ZFC. If there is a Suslin line, then its interval algebra is Lindelöf, but does not have a countable dense subalgebra. On the other hand, MA+$\neg$CH implies that each perfectly normal, compact space is separable (cf. [6], chapter 5), so each Lindelöf algebra has a countable dense subalgebra.

6. Normality and retractiveness

Recall that a topological space is normal iff each pair of disjoint closed sets can be separated by open neighbourhoods. Every Stone space is compact, hence normal, but this neet not be true of subspaces. Normality considerations were already used implicitly in Boolean algebra theory but never made explicit. The reason may be that this notion does not have a nice description in Boolean algebraic terms.

We call an ideal $I \subset \mathfrak{A}$ normal iff the corresponding open subspace $U \subset X$ is normal. We say that $\mathfrak{A}$ is normal provided that all its ideals are normal (i.e., if X is a hereditarily normal space). Finally, an extension $\mathfrak{A} \subset \mathfrak{B}$ will be called normal iff each ideal of $\mathfrak{A}$ generates a normal ideal in $\mathfrak{B}$.

Topologists have established many relations between paracompactness and normality of topological spaces. We use some of them to find connections between our Boolean algebraic notions.

T h e o r e m 6.1. $\mathfrak{A}$ is pc iff all its extensions are normal. In particular, all pc algebras are normal.

P r o o f . Suppose $\mathfrak{A}$ to be pc, let $\mathfrak{A}\subset\mathfrak{B}$ be an extension, and consider an ideal $I\subset\mathfrak{A}$. Let J be the ideal generated by I in $\mathfrak{B}$. Since I is disjointly generated, so is J. Therefore, the open subspace of the Stone space of $\mathfrak{B}$ corresponding to J is a sum of compact spaces, hence normal. If every extension of $\mathfrak{A}$ is normal, then, in particular, so is the standard extension $\mathfrak{A}\subset\mathfrak{A}*\mathfrak{A}$ (free product). Consider an ideal I of $\mathfrak{A}$ and the corresponding open subset U of X. In $\mathfrak{A}*\mathfrak{A}$ I generates an ideal which the open set $U\times X\subset X\times X$ corresponds to. (Recall that the Stone space of $\mathfrak{A}*\mathfrak{A}$ is canonically homeomorphic to $X\times X$). The open subspace $U\times X$ is normal by assumption. As normality is always inherited by closed subspaces, $U\times\bar{U}$ is normal, too. The paracompactness of U now follows from Tamano's Theorem ([4], 5.1.38).

R e m a r k 6.2. Every order topology is hereditarily normal ([4], 2.7.5), which implies that every interval algebra is normal. 4.3 immediately yields examples of normal non-pc BA's. The problem becomes more delicate if we restrict our attention to ccc algebras. Under CH the answer is known. In [7] there is an example of a locally compact, zero-dimensional, (even hereditarily) separable, hereditarily normal, non-Lindelöf space. Its one-point compactification is the Stone space of a normal, ccc, non-pc BA.

T h e o r e m 6.3. The following conditions are equivalent:

(1) $\mathfrak{A}$ is Lindelöf.

(2) $\mathfrak{A}*\mathfrak{B}$ is pc (Lindelöf) for each countable BA $\mathfrak{B}$.

(3) $\mathfrak{A}*\mathfrak{B}$ is pc (Lindelöf) for some countable BA $\mathfrak{B}$.

(4) $\mathfrak{A}*\mathfrak{B}$ is normal for each countable BA $\mathfrak{B}$.

(5) $\mathfrak{A}*\mathfrak{B}$ is normal for some countable Ba $\mathfrak{B}$.

P r o o f . (1) $\longrightarrow$ (2). Let $I\subset\mathfrak{A}*\mathfrak{B}$ be an ideal. For each $b\in\mathfrak{B}$ consider the ideal $I_b=\{a\in\mathfrak{A}\mid a*b\in I\}\subset\mathfrak{A}$. Since $\mathfrak{A}$ is Lindelöf, each I_b has a countable set of generators, say J_b. I is obviously generated by the set $\{a*b\mid a\in J_b,\ b\in\mathfrak{B}\}$, which is countable.

(5) ⟶ (1): This is the Boolean translation of Katetov's result stating that for $X \times Y$ to be normal it is necessary that X is perfectly normal provided that Y contains a converging sequence ([4], 2.7.15). All the other implications are either obvious or immediate consequences of 6.1.

Example 6.4. Let $\mathcal{A}$ and $\mathcal{L}$ be the algebras of finite and cofinite subsets of ω and ω_1, respectively. $\mathcal{L}$ is not Lindelöf, thus $\mathcal{A} * \mathcal{L}$ is not normal, let alone pc. But the Stone space of $\mathcal{A} * \mathcal{L}$ is hereditarily wekly paracompact ([4], 5.3.B). Denote by $\mathcal{A}_n$ the subalgebra of $\mathcal{A}$ generated by the first n atoms and let $\mathcal{L}_\alpha$ denote the subalgebra of $\mathcal{L}$ generated by $\{\{\beta\} \mid \beta < \alpha\}$. $\mathcal{A} * \mathcal{L}_\alpha$ is countable, hence Lindelöf. $\mathcal{A}_n * \mathcal{L}$ is paracompact, as it is isomorphic to $\mathcal{L}^{n+1}$. Obviously $\mathcal{A} * \mathcal{L} = \bigcup_{n<\omega} (\mathcal{A}_n * \mathcal{L}) = \bigcup_{\alpha < \omega_1} (\mathcal{A} * \mathcal{L}_\alpha)$. We conclude that countable unions of pc algebras and uncountable unions of Lindelöf algebras need not be pc. On the other hand, it is obvious that a countable union of Lindelöf algebras is Lindelöf again.

A Boolean algebra $\mathcal{A}$ is said to be retractive iff for each epimorphism $g: \mathcal{A} \to \mathcal{L}$ there is a monomorphism $f: \mathcal{L} \to \mathcal{A}$ such that $g \circ f = \mathrm{id}$. Details on retractive BA's can be found in [11] and [12]. In the language of Stone spaces we have the following equivalent definition:

$\mathcal{A}$ is retractive iff for each closed $F \subset X$ there is a retraction $f: X \to F$, i.e., a continuous map which is identical on F.

Many proofs of non-retractiveness (e.g. in [12]) made implicit use of the following

Proposition 6.5. Retractive BA's are normal.

Proof. Let $U \subset X$ be open and let F, G be closed (in U) subsets of U with $F \cap G = \emptyset$. By definition, there are F', G' closed in X such that $F = F' \cap U$ and $G = G' \cap U$. Consider a retraction $f: X \to F' \cup G'$. An easy argument shows that $U \cap f^{-1}(F' \setminus G')$ and $U \cap f^{-1}(G' \setminus F')$ are the desired neighbourhoods separating F and G in U.

Next we give an example showing that normality does not imply retractiveness in general.

E x a m p l e 6.6. Let $\mathfrak{A}$ denote the interval algebra on the reals and $\mathfrak{B}$ the finite-cofinite algebra on ω. By 6.1, $\mathfrak{A}*\mathfrak{B}$ is normal. The non-retractiveness of $\mathfrak{A}*\mathfrak{B}$ follows from a general theorem in [12], but can be rapidly seen directly: Identify Y, the Stone space of $\mathfrak{B}$, with $\omega+1$. Each real number r will be considered as the point (i.e., ultrafilter) $\{a\in\mathfrak{A} \mid r\in a\}$ of X. Let r_n, $n<\omega$ be an enumeration of the rationals and denote by F the closed subset $X\times\{\omega\}\cup$ $\cup\{(r_n,n)\mid n<\omega\}$ of $X\times Y$. Suppose there were a retraction $f: X\times Y \to F$. For each n there must be some $\varepsilon_n>0$ such that $f([r_n,r_n+{}_n)\times\{n\}) = \{(r_n,n)\}$. There will be some real number r_ω such that we have $r_n<r_\omega<r_n+\varepsilon_n$ for infinitely many n (Remember your first lessions of calculus!). As the point (r_ω,ω) belongs to F, there must be a $\delta>0$ and a natural number N such that $f([r_\omega,r_\omega+\delta)\times\{i>N\})\subset[r_\omega,r_\omega+1)\times Y$. Take any $n>N$ such that $r_n<r_\omega<r_n+\varepsilon_n$. Then, on the one hand, $f((r_\omega,n))=$ $=(r_n,n)$ and, on the other hand, $f((r_\omega,n))\in[r_\omega,r_\omega+1)\times Y$. This is a contradiction, since $r_n\notin[r_\omega,r_\omega+1)$.

7. Open problems

In spite of considerable efforts ([11], [12]) the question whether the free product of two uncountable BA's can be retractive is still open. The same problem is interesting for normal and pc algebras. In the following we list the more or less obvious observations that one can make.

(1) $\mathfrak{A}*\mathfrak{B}$ normal implies $\mathfrak{A},\mathfrak{B}$ Lindelöf. This follows from the already mentioned result of Katetov ([4], 2.7.15).

(2) $\mathfrak{A}*\mathfrak{A}$ Lindelöf implies $\mathfrak{A}$ countable. Indeed, if $\mathfrak{A}*\mathfrak{A}$ is Lindelöf, then the diagonal is a G_δ in $X\times X$. By a theorem of Šneider ([4], 4.2.B), X must be metrizable.

(3) $\mathfrak{A}*\mathfrak{A}*\mathfrak{A}$ normal implies $\mathfrak{A}$ countable. This is a combination of (1) and (2).

(4) (MA+⌉CH) $\mathfrak{A}*\mathfrak{A}$ paracompact implies $\mathfrak{A}$ countable. This follows from (1) and (2), because under MA+⌉CH the ccc is multiplicative ([6], Theorem 5.5).

P r o b l e m 7.1. Can the free product of two uncountable BA's be paracompact (normal)? The Stone space of a homomorphic image of $\mathfrak{A}$ can be embedded into X. Therefore, homomorphic images of normal algebras are normal.

P r o b l e m 7.2. Is every subalgebra of a normal algebra normal again?

We have already mentioned (4.2.(2) and 6.2) that ccc subalgebras of interval algebras are Lindelöf, whereas this is not necessarily true for normal ones. Retractiveness lies inbetween these properties, and this motivates

P r o b l e m 7.3. Is there a retractive, ccc, non-Lindelöf BA?

The interval algebra on the reals shows that there are pc algebras $\mathfrak{A}$ with $c(\mathfrak{A}) = \omega$ and $|\mathfrak{A}| = \omega^{\omega}$. Taking the direct product with the algebra of finite and cofinite subsets of ω_1 we obtain a pc algebra $\mathfrak{B}$ with $c(\mathfrak{B}) = \omega_1$ and $|\mathfrak{B}| = \omega_1^{\omega}$. (This example has been pointed out to me by J.D. Monk). For bigger cardinals we ask

P r o b l e m 7.4. For which cardinals $\varkappa$ are there pc algebras of cellularity $\varkappa$ and power $\varkappa^{\omega}$?

The number of non-isomorphic pc algebras of cellularity $\varkappa$ is bounded below by $2^{\varkappa}$ (proof of 2.4) and above by $2^{\varkappa^{\omega}}$ (5.3). In the Lindelöf case the upper bound is attained, since there are enough subalgebras of the interval algebra on the reals ([2], §4). For bigger cardinals it would be interesting to know

P r o b l e m 7.5. How many non-isomorphic pc algebras of cellularity $\varkappa$ are there?

The last problem has been suggested by the referee in connection with his improvement of 4.4.

P r o b l e m 7.6. Does there exist a hereditarily paracompact Boolean space in which the set of all points with uncountable character is of the second category?

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