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JOINT DISTRIBUTIONS AND COMMUTABILITY OF OBSERVABLES

Dedicated to the memory of Professor Roman Sikorski

In the quantum probability theory the σ-field of random events is replaced by the lattice of orthogonal projectors in a separable infinite dimensional Hilbert space H. A countably additive function from this lattices to the unit interval constitutes a state, the non-commutative analogue of a probability measure. The Theorem of Gleason [4] asserts that every state is of the form $\pi \to \mathrm{tr}\, \pi T$, where π runs over all projectors and T is a probability operator on H, i.e. a positive linear operator of unit trace. Conversely, every probability operator determines a state by the Gleason formula. From now onwards let S stand for the set of all states, i.e. all probability operators on H. We shall denote by $\mathcal{T}_1$ the space of all nuclear linear operators acting in H with the norm $\|T\|_1 = \mathrm{tr}(TT^*)^{1/2}$. Of course S is a closed and convex subset of $\mathcal{T}_1$.

In quantum theory to every physical quantity or observable there corresponds a self-adjoint not necessarily bounded linear operator on H. By O we shall denote the set of all observables. Given $A \in O$, the probability distribution of A at the state T is defined for all Borel subsets E of the real line R by the formula $P_T^A(E) = \operatorname{tr} \pi_A(E)T$, where π_A is the projector-valued spectral measure associated with A, i.e. $A = \int_R \lambda \pi_A(d\lambda)$. The characteristic function of P_T^A, i.e. its Fourier transform $\hat{P}_T^A$ is then given by the formula

$$\hat{P}_T^A(\tau) = \operatorname{tr} e^{i\tau A}T \quad (\tau \in R).$$

A system $A_1, A_2, \ldots, A_k$ $(k \geq 2)$ of observables is said to be regular if there exists a dense linear manifold D in H such that for arbitrary real numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ the operator $\sum_{j=1}^{k} \alpha_j A_j$ is well defined on D and is essentially self-adjoint, so that the probability distribution of $\sum_{j=1}^{k} \alpha_j A_j$ at every state T is well defined. Of course, all systems of bounded observables are regular. The set of all regular systems $A = (A_1, A_2, \ldots, A_k)$ of observables will be denoted by $\mathcal{O}_k$. Further, we shall use the following notation. For $a, b \in R^k$ (a,b) will denote the inner product in R^k, $|a| = (a,a)^{\frac{1}{2}}$ and $a \in R^k$ and $A \in \mathcal{O}_k$ $(a, A) = \sum_{j=1}^{k} \alpha_j A_j$ if $a = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $A = (A_1, A_2, \ldots, A_k)$. Of course, $(a, A) \in \mathcal{O}$. In [10] I introduced the concept of the joint probability distribution for $A \in \mathcal{O}_k$. Namely, a Borel probability measure P_T^A on the k-dimensional Euclidean space R^k is said to be the joint probability distribution of the system A of observables at the state T if for every $a \in R^K$ the projection of P_T^A onto the real line defined by $x \to (a,x)$ $(x \in R^k)$ coincides with $P_T^{(a,A)}$. It is clear that the joint probability distribution is uniquely determined provided it exists. Moreover, the characteristic function of P_T^A is given by the formula

$$(1) \qquad \hat{P}_T^A(t) = \operatorname{tr} e^{i(t,A)}T \quad (t \in R^k).$$

Given $A \in \mathcal{O}_k$, by $\mathcal{S}(A)$ we shall denote the set of all states T for which P_T^A exists. It is evident that $T \in \mathcal{S}(A)$ if and only if the function $t \to \operatorname{tr} e^{i(t,A)}T$ $(t \in R^k)$ is continuous and positive definite on R^k. Hence it follows that always $\mathcal{S}(A)$ is a convex and closed in the topology of $\mathcal{T}_1$ subset of $\mathcal{S}$. It may happen that $\mathcal{S}(A)$ is empty.

A relation between the existence of joint probability distribution at every state and the commutability of observables is given by the following statement.

Let $A \in \mathcal{O}_k$. Then $\mathcal{S}(A) = \mathcal{S}$ if and only if A consists of commuting observables, i.e. observables with commuting spectral measures.

For observables with purely point spectrum this statement was proved in [10]. Recently, an elementary proof was given by Ruymgaart [9]. Without any restriction on the spectrum a proof can be found in [5] and [7]. In the more general framework of quantum logics the theorem was proved by Varadarajan [12].

Let I be the unit operator on H. Given $a, b \in R^k$ and $A \in \mathcal{O}_k$ we shall use the notation

$$aA + b = (\alpha_1 A_1 + \beta_1 I, \alpha_2 A_2 + \beta_2 I, \dots, \alpha_k A_k + \beta_k I)$$

where $a = (\alpha_1, \alpha_2, \dots, \alpha_k)$, $b = (\beta_1, \beta_2, \dots, \beta_k)$ and $A = (A_1, A_2, \dots, A_k)$. It is clear that $aA+b \in \mathcal{O}_k$. Moreover,

$$\operatorname{tr} e^{i(t,aA+b)}T = e^{i(t,b)} \operatorname{tr} e^{i(at,A)}T \tag{2}$$

where $a, b, t \in R^k$, $T \in \mathcal{S}$, $t = (\tau_1, \tau_2, \dots, \tau_k)$ and $at = (\alpha_1\tau_1, \alpha_2\tau_2, \dots, \alpha_k\tau_k)$. By formula (1) we have the following lemma.

L e m m a 1. If $a = (\alpha_1, \alpha_2, \dots, \alpha_k) \in R^k$, $\alpha_j \neq 0$ $(j=1,2,\dots,k)$ and $b \in R^k$, then $A \in \mathcal{O}_k$ if and only if $aA+b \in \mathcal{O}_k$ and $\mathcal{S}(A) = \mathcal{S}(aA+b)$.

Let $A \in O_k$. We say that A fulfils the probabilistic commutation condition if there exists a system $B \in O_k$ consisting of commuting observables such that $P_T^A = P_T^B$ for all $T \in S(A)$.

By Lemma 1 and formulas (1) and (2) we have the following simple lemma.

Lemma 2. If $a = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in R^k$, $\alpha_j \neq 0$ $(j=1,2,\ldots,k)$, $b \in R^k$ and $A \in O_k$, then A and $aA+b$ fulfil or do not fulfil the probabilistic commutation condition simultaneously.

Using the method introduced by Ruymgaart in [9] we shall prove the following theorem.

Theorem 1. Let $A \in O_k$ and A consists of one-sided bounded observables with purely point spectrum. Then A fulfils the probabilistic commutation condition.

Proof. If $S(A)$ is empty, then our assertion is obvious. Consequently, we assume that $S(A)$ is non empty. Moreover, by Lemma 2, we may assume without loss of generality that $A = (A_1, A_2, \ldots, A_k)$ where all observables A_j $(j=1,2,\ldots,n)$ are non-negative. Let E_j be the spectrum of A_j, which under our assumptions coincides with the set of all eigen values of A_j. Consequently E_j is at most denumerable and for any $T \in S$ the probability measure $P_T^{A_j}$ is concentrated on E_j. Thus for any $T \in S(A)$ the joint probability distribution P_T^A is concentrated on at most denumerable set $E = E_1 \times E_2 \times \ldots \times E_k$. Hence we get the formula

$$(3) \qquad \hat{P}_T^A(t) = \sum_{e \in E} e^{i(t,e)} P_T^A(\{e\}) \qquad (t \in R^k).$$

Further, for any $a \in R^k$ the probability measure $P_T^{(a,A)}$ is concentrated on the set $(a,E) = \{(a,e) : e \in E\}$ and, by (1) and (3)

$$(4) \qquad \hat{P}_T^{(a,A)}(\tau) = \hat{P}_T^A(\tau a) = \sum_{e \in E} e^{i\tau(a,e)} P_T^A(\{e\}).$$

Let F be the subset of R^k consisting of all elements $a = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ with linearly independent coordinates $\alpha_1, \alpha_2, \ldots, \alpha_k$ over the denumerable field generated by the set $\bigcup_{j=1}^{k} E_j$. It is clear that F is dense in R^k. Moreover for $a \in F$ the mapping $e \to (a,e)$ from E onto (a,E) is one-to-one. Consequently, for $a \in F$ we have the formula

$$P_T^{(a,A)}(\tau) = \sum_{e \in E} e^{i\tau(a,e)} P_T^{(a,A)}(\{(a,e)\}).$$

Taking into account (4) we infer that

$$(5) \qquad P_T^{(a,A)}(\{(a,e)\}) = P_T^A(\{e\}) \quad (T \in S(A),\ e \in E,\ A \in F).$$

Since F is dense in R^k we can find an element $b \in F$ with positive coordinates. Let π be the spectral measure associated with (b,A), i.e. $(b,A) = \int_{(b,E)} \lambda\pi(d\lambda)$. Then for the domain of (b,A) we have the inclusion

$$D((b,A)) \subset D = \left\{ x: \sum_{e \in E} (b,e)^2 \, \|\pi(\{(b,e)\})x\|^2 < \infty \right\}$$

which shows that the set D is dense in H. Using the notation $e = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ we put

$$B_j = \sum_{e \in E} \varepsilon_i \pi(\{(b,e)\}) \quad (j=1,2,\ldots,k).$$

Since $\varepsilon^2\beta^2 \leqslant (b,e)^2$ $(j=1,2,\ldots,k)$ where $\beta>0$ and all coordinates of b are greater than β, we infer that $D(B_j) \supset D$ $(j=1,2,\ldots,n)$ which shows that $B = (B_1, B_2, \ldots, B_k) \in O_k$. The observables $B_1, B_2, \ldots, B_k$ commute with one another and $(a,B) = \sum_{e \in E} (a,e)\, \pi(\{(b,e)\})$ for every $a \in R^k$. Consequently, by (5)

$$\hat{P}_T^{(a,B)}(\tau) = \sum_{e\in E} e^{i\tau(a,e)} \operatorname{tr} \pi(\{(b,e)\})T =$$

$$= \sum_{e\in E} e^{i\tau(a,e)} P_T^A(\{e\}) \quad (T\in S(A)),$$

and, by (4), $\hat{P}_T^{(a,B)} = \hat{P}_T^{(a,A)}$ for every $T\in S(A)$ and $a\in R^k$. This yields the equation $P_T^B = P_T^A$ for all $T\in S(A)$ which completes the proof.

Our next aim is to show that this result cannot be extended to all systems A from O_k. Namely, we shall prove that the pair of canonical observables does not fulfil the condition in question.

Given a subset X of T_1, by $[X]$ we shall denote the linear subspace of T_1 spanned by X.

First we shall prove the following simple lemma.

L e m m a 3. If $A\in O_k$ and A fulfils the probabilistic commutation condition, then

$$S(A) = S\cap [S(A)].$$

P r o o f . Let $A,B\in O_k$, B consists of commuting observables and $P_T^A = P_T^B$ for all $T\in S(A)$. Let S_0 be the set of all operators T from T_1 for which the equation

$$\operatorname{tr} e^{i(t,A)}T = \operatorname{tr} e^{i(t,B)}T \tag{6}$$

holds for all $t\in R^k$. It is clear that S_0 is a linear subspace of T_1 and, by (1), $S(A)\subset S_0$. Consequently, $[S(A)]\subset S_0$. Since for every $T\in S$ the right-hand side of (6) is continuous and positive definite on R^k, we infer that for every $T\in S\cap [S(A)]$ the left-hand side of (6) is also continuous and positive definite on R^k. In other words we have the inclusion $S(A)\supset \supset S\cap [S(A)]$. The converse inclusion is obvious which completes the proof.

From Theorem 1 and Lemma 3 we get the following corollary.

C o r o l l a r y . Let $A \in \mathcal{O}_k$ and A consists of one-sided bounded observables with purely point spectrum. If $[S(A)] = \mathcal{T}_1$, then A consists of commuting observables.

By a pair of canonical observables we mean a pair $C = (P,Q)$ for which there exists a dense linear manifold D in H contained in the domains of P, Q and invariant under P, Q. When restricted to D, the observables P, Q satisfy the Heisenberg commutation relation $PQ - QP = iI$ and the operator $P^2 + Q^2$ is essentially self-adjoint. From von Neumann [8] and Dixmier [2] results if follows that $C \in \mathcal{O}_2$ and the function $t \to \mathrm{tr}\, e^{i(t,C)}T$ $(t \in R^2)$ is continuous for all $T \in \mathcal{T}_1$ ([1], Proposition 3). Put $\tilde{T}(t) = \mathrm{tr}\, e^{i(t,C)}T$ $(t \in R^2,\ T \in \mathcal{T}_1)$. Then, by (1) $\hat{P}^C_T = \tilde{T}$ and, consequently, $T \in S(C)$ if and only if $\tilde{T}$ is positive definite on R^2. Let $\mathcal{T}_2$ be the space of all Hilbert-Schmidt operators on H with the norm $\|T\|_2 = (\mathrm{tr}\, TT^*)^{\frac{1}{2}}$. Obviously, $\mathcal{T}_1 \subset \mathcal{T}_2$ and $\|T\|_2 \leq \|T\|_1$ for $T \in \mathcal{T}_1$. It is well-known ([6], Chapter 5) that the map $T \to \tilde{T}$ $(T \in \mathcal{T}_1)$ extends uniquely to a linear isometric transformation from $\mathcal{T}_2$ onto the space $L^2(R^2)$ of all complex-valued square integrable with respect to the Lebesgue measure functions on R^2 with the norm $\|f\|_2 = ((2\pi)^{-1} \int_{R^2} |f(t)|^2 dt)^{\frac{1}{2}}$. Moreover

$$\tilde{T}^*(t) = \overline{\tilde{T}}(-t) \quad (t \in R^2,\ T \in \mathcal{T}_2). \tag{7}$$

Let $\mathcal{A}$ be the subset of $\mathcal{T}_2$ consisting of all operators T with continuous $\tilde{T}$ vanishing at ∞. The set $\mathcal{A}$ with the norm

$$\|T\| = \|T\|_2 + \max\{|\tilde{T}(t)| \ :\ t \in R^2\}$$

becomes a Banach space. Moreover, we have the inclusion $\mathcal{T}_1 \subset \mathcal{A} \subset \mathcal{T}_2$.

Further, $\mathcal{A}$ is a Banach algebra under the convolution $*$ defined by setting

(8) $$\widetilde{T*U} = \tilde{T}\tilde{U} \quad (T, U \in \mathcal{A})$$

(see [11]). Given, a, $b \in R^2$, $a = (\alpha_1, \alpha_2)$, $b = (\beta_1, \beta_2)$ we put $\Delta(a,b) = \alpha_1\beta_2 - \alpha_2\beta_1$. A complex-valued function f on R^2 is said to be Δ-positive definite if for arbitrary vectors $t_1, t_2, \ldots, t_n \in R^2$ the $n \times n$ matrix $f(t_j - t_k)e^{i/2\Delta(t_j, t_k)}$ is positive definite. An analogue of Bochner's Theorem asserts that $f = \tilde{T}$ for a certain $T \in \mathcal{S}$ if and only if f is Δ-positive definite, continuous at the origin and $f(0) = 1$ ([6], p.243). It is clear that fg is Δ-positive definite whenever f is positive-definite and g Δ-positive definite. Hence and from (8) we get the following lemma.

L e m m a 4. If $T \in \mathcal{S}$, $U \in \mathcal{T}_1$, $\tilde{U}(0) = 1$ and $\tilde{U}$ is positive definite, then $T * U \in \mathcal{S}$.

Further, using (7), we infer that for every pair $T, U \in \mathcal{S}$ the product $\tilde{T}\tilde{U}$ is positive definite. Consequently, by Lemma 4 and formula (8) we get the next lemma.

L e m m a 5. If $T \in \mathcal{S}(C)$ and $U \in \mathcal{S}$, then $T * U \in \mathcal{S}(C)$.

We are now in a position to prove the following lemma.

L e m m a 6. $[\mathcal{S}(C)] = \mathcal{T}_1$.

P r o o f . For every complex number γ with $\mathrm{Re}\,\gamma > 0$ we define the operators G_γ from $\mathcal{T}_2$ by setting $\tilde{G}_\gamma(t) = e^{-\gamma/4|t|^2}$ $(t \in R^2)$. It is known ([6], Chapter 5) that for real $\gamma \geq 1$ G_γ are Gaussian probability operators and, consequently, $G_\gamma \in \mathcal{S}$ $(\gamma \geq 1)$. Since in this case $\tilde{G}_\gamma$ is positive definite on R^2 we have also

(9) $$G_\gamma \in \mathcal{S}(C) \quad (\gamma \geq 1).$$

Moreover, G_γ have a representation

(10) $$G_\gamma = \sum_{n=0}^{\infty} \frac{2}{(\gamma+1)} \left(\frac{\gamma-1}{\gamma+1}\right)^n \Pi_n \quad (\gamma \geq 1)$$

where Π_n are commuting one-dimensional projectors ([6], Chapter 5). Since for every $t \in R^2$ the function

$$\sum_{n=0}^{\infty} \frac{2}{(\gamma+1)} \left(\frac{\gamma-1}{\gamma+1}\right)^{n} \Pi_n(t)$$

is analytic on the half-plane Re $\gamma > 0$ and coincides, by (10), with $\tilde{G}_\gamma(t)$ on the half-line $\gamma \geq 1$, we infer that it coincides with $e^{-\gamma/4|t|^2}$ on the whole half-plane Re $\gamma > 0$. In other words we have the equation (10) for all γ with Re $\gamma > 0$. Put

$$U_\gamma = \frac{4}{(1+\gamma)^2} \sum_{n=0}^{\infty} \left(\frac{\gamma-1}{\gamma+1}\right)^{2n} \Pi_{2n} \qquad (\gamma > 0).$$

Of course, $U_\gamma \in S$ $(\gamma > 0)$ and

(11) $$U_\gamma \quad \frac{\gamma}{1+\gamma} G_\gamma + \frac{1}{1+\gamma} G_{\gamma^{-1}} \qquad (\gamma > 0),$$

which shows that $\tilde{U}_\gamma$ is positive definite. Thus

(12) $$U_\gamma \in S(C) \qquad (\gamma > 0).$$

Let $0 < \gamma < 1$. Then by (9)

(13) $$G_{\gamma^{-1}} \in S(C),$$

which, by (11) and (12), yields $G_\gamma \in \mathcal{T}_1$. Since $\tilde{G}_\gamma$ is positive definite on R^2, $\tilde{G}_\gamma(0) = 1$, we infer, by virtue of Lemma 4, that

(14) $$G_\gamma * T \in S \qquad (0 < \gamma < 1,\ T \in S).$$

Further, by Lemma 5 and formulas (12) and (13), we conclude that for every $T \in S$ and $0 < \gamma < 1$ both operators $U_\gamma * T$ and $G_{\gamma^{-1}} * T$ belong to $S(C)$. Consequently, by (11)

(15) $$G_\gamma * T \in [S(C)] \qquad (0 < \gamma < 1,\ T \in S).$$

It is clear that the functions $\widetilde{G_\gamma * T} = \tilde{G}_\gamma \tilde{T}$ tend uniformly on every compact subset of R^2 to $\tilde{T}$ when $\gamma \to 0$. Since, by (14), $G_\gamma * T$, $T \in S$, this convergence coincides with the convergence in $\mathcal{T}_1$-topology ([11], Chapter 1). Thus, by (15), $T \in [S(C)]$. In other words we have the inclusion $S \subset [S(C)]$ which yields the assertion of the lemma.

We have $S(C) \neq S$ because the canonical observables do not commute with one another. Moreover, it was proved by Fischer [3] that all projectors belonging to $S(C)$ are Gaussian probability operators. On the other hand, by Lemma 6, $S \cap [S(C)] = S$. Thus as a consequence of Lemma 3 we get the following theorem.

T h e o r e m 2. The pair P,Q of canonical observables does not fulfil the probabilistic commutation condition.

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