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THE ESSENTIAL SPECTRA OF D-COMMUTING SYSTEMS

Dedicated to the memory of Professor Roman Sikorski

1. Introduction

Let X and Y be two complex Hilbert spaces. Let L(X,Y) denote the set of all closed linear operators on linear submanifolds of X, with values in Y. For L(X,X), we shall simply write L(X). For any operator $A \in L(X,Y)$, D(A), N(A) and R(A) denote respectively, the domain, the null space and the range of A. Let $\gamma(A)$, the reduced minimum modulus of A, be defined by the formula

$$(1.1)\quad \gamma(A) = \sup\left\{\gamma \geq 0 : \|Ax\| \geq \gamma \|(I-P_{N(A)})x\|, \quad x \in D(A)\right\},$$

where $P_{N(A)}$ denotes the orthogonal projection of X onto N(A), provided that $A \neq 0$.

Let $\{X^p\}_{p \in Z}$ (where Z is the ring of integers) be a family of Hilbert spaces, and let $S^p \in L(X^p, X^{p+1})$ be a family of operators such that $R(S^p) \subset N(S^{p+1})$, for all $p \in Z$. Let the following sequence represent the complex of Hilbert spaces

$$(1.2)\qquad \ldots \xrightarrow{S^{p-1}} X^p \xrightarrow{S^p} X^{p+1} \xrightarrow{S^{p+1}} \ldots$$

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Let $\{H^p(X,S)\}_{p\in Z}$ denote the cohomology of the complex $(X,S) = (X^p,S^p)_{p\in Z}$, that is, $H^p(X,S) = N(S^p)/R(S^{p-1})$, $p\in Z$. Let the $\dim(H^p(X,S))$ denote the algebraic dimension of the linear space $H^p(X,S)$.

We recall that a complex $(X,S) = (X^p,S^p)_{p\in Z}$ is said to be Fredholm if $\inf\{\gamma(S^p)\} > 0$, $\dim(H^p(X,S)) < \infty$ for each $p\in Z$, and $H^p(X,S) \neq 0$ only for a finite number of indices.

Next, we recall some definitions [5] related to the commuting systems of linear operators.

Let $s = (s_1,\dots,s_n)$ be a system of n indeterminates, and let $\Lambda[s]$ be the exterior algebra over the complex field C, generated by $s_1,\dots,s_n$. Then, for any integer p, $0<p<n$, $\Lambda^p[s]$ is the space of all exterior forms of degree p in $s_1,\dots,s_n$. For any Hilbert space X, $\Lambda[s,X]$ (resp. $\Lambda^p[s,X]$) will denote the tensor product $X\otimes\Lambda[s]$ (resp. $X\otimes\Lambda^p[s]$). In the case of two Hilbert spaces X and Y, there is a natural identification of the space $\Lambda[s,X]\otimes\Lambda[t,Y]$ with the space $\Lambda[(s,t),X\bar{\otimes}Y]$, where $t = (t_1,\dots,t_n)$ is another system of indeterminates. Let the operator $H_j:\Lambda[s]\to\Lambda[s]$, $H_j\xi = s_j\wedge\xi$, $\xi\in\Lambda[s]$, satisfy the relation

$$H_jH_k + H_kH_j = 0,\quad j,k=1,\dots,n.$$

D e f i n i t i o n 1.1. An n-tuple $S = (S_1,\dots,S_n)\subset L(X)$ is said to be a D-commuting system if there exists a dense subspace D of X in $\bigcap_{j=1}^{n} D(S_j)$ with the following properties:

(i) the restriction $\hat{\delta}_S = (S_1\otimes H_1 + \dots + S_n\otimes H_n)|\Lambda[s,D]$ is closable,

(ii) if δ_S is the canonical closure of $\hat{\delta}_S$, then $R(\delta_S)\subset N(\delta_S)$.

D e f i n i t i o n 1.2. A D-commuting system $S = (S_1,\dots,S_n)\subset L(X)$ is said to be singular (resp. non-singular) if $R(\delta_S)\neq N(\delta_S)$ (resp. $R(\delta_S) = N(\delta_S)$).

D e f i n i t i o n 1.3. Let $S = (S_1,\dots,S_n)$ be a D-commuting system associated to a complex of Hilbert spaces $(\Lambda^p[s,X], \delta_S^p)_{p=0}^n$, where $\delta_S^p = \delta_S \mid \Lambda^p[s,X] \cap D(\delta_S)$. Then the operator S is said to be Fredholm if the corresponding complex is Fredholm.

D e f i n i t i o n 1.4. The joint spectrum of a D-commuting system $S = (S_1,\dots,S_n) \subset L(X)$ is the set of those points $\lambda \in C^n$ such that $S - \lambda I$ is singular, denoted by $\sigma_D(S,X)$.

D e f i n i t i o n 1.5. The joint essential spectrum of a D-commuting system $S = (S_1,\dots,S_n) \subset L(X)$ is the set of those points $\lambda \in C^n$ such that $S - \lambda I$ is not Fredholm, denoted by $\sigma_{e_D}(S,X)$.

The main aim of this note is to establish an inclusion relation for the joint essential spectrum of a D-commuting system $\tilde{S} = (\tilde{S}_1,\dots,\tilde{S}_n)$ by applying the spectral results of C. Grosu and F. Vasilescu [5] for the tensor product of two D-commuting systems, where $\tilde{S}_j \in L(\tilde{X})$, for $\tilde{X} = X_1 \bar{\otimes} \dots \bar{\otimes} X_n$ and $X_1,\dots,X_n$ are Hilbert spaces. The obtained spectral inclusion includes some interesting results as special cases.

2. Some supporting lemmas

L e m m a 2.1. (Grosu and Vasilescu [5]). Let (X,S) and (Y,T) be two Fredholm complexes of Hilbert spaces. Then their tensor product $(X \bar{\otimes} Y, S \bar{\otimes} T)$ is Fredholm.

L e m m a 2.2. (Grosu and Vasilescu [5]). Let (X,S) and (Y,T) be two complexes of Hilbert spaces. Then their tensor product $(X \bar{\otimes} Y, S \bar{\otimes} T)$ is Fredholm and exact iff either (X,S) or (Y,T) is Fredholm and exact.

L e m m a 2.3. (Grosu and Vasilescu [5]). Let $S = (S_1,\dots,S_n) \subset L(X)$ be a D_S-commuting system and let $T = (T_1,\dots,T_n) \subset L(Y)$ be a D_T-commuting system. Then

$$S \bar{\otimes} T = (S_1 \bar{\otimes} I_Y,\dots,S_n \bar{\otimes} I_Y, I_X \bar{\otimes} T_1,\dots,I_X \bar{\otimes} T_n) \subset L(X \bar{\otimes} Y)$$

is a $D_S \otimes D_T$-commuting system.

Lemma 2.4. Let $S = (S_1,\dots,S_n) \subset L(X)$ be a D_S-commuting system, and let $T = (T_1,\dots,T_n) \subset L(Y)$ be D_T-commuting system. Then

$$\sigma_{e_{D_S \otimes D_T}}(S \bar{\otimes} T, X \bar{\otimes} Y) \subset \sigma_{e_{D_S}}(S,X) \times \sigma_{D_T}(T,Y) \cup \sigma_{D_S}(S,X) \times \sigma_{e_{D_T}}(T,Y).$$

Proof. The proof follows from a combination of Lemma 2.1 and Lemma 2.2.

Remark 2.5. Under the conditions as in Lemma 2.4, we conjecture the following inclusion relation

$$\sigma_{e_{D_S}}(S,X) \times \sigma_{D_T}(T,Y) \cup \sigma_{D_S}(S,X) \times \sigma_{e_{D_T}}(T,Y) \subset \sigma_{e_{D_S \otimes D_T}}(S \bar{\otimes} T, X \bar{\otimes} Y).$$

Lemma 2. (Grosu and Vasilescu [5]). Let $S = (S_1,\dots,S_n) \subset L(X)$ be a D_S-commuting system, and let $T = (T_1,\dots,T_n) \subset L(Y)$ be a D_T-commuting system. Then

$$\sigma_{D_S \otimes D_T}(S \bar{\otimes} T,\ X \bar{\otimes} Y) = \sigma_{D_S}(S,X) \times \sigma_{D_T}(T,Y),$$

and if S and T are Fredholm, then $S \bar{\otimes} T$ is Fredholm.

3. The essential spectrum $\sigma_{e_D}(\tilde{S},\tilde{X})$

Let $X_1,\dots,X_n$ be Hilbert spaces and let $S_j \in L(X_j)$, for $j = 1,\dots,n$, be densely-defined operators. Let $\tilde{X} = X_1 \bar{\otimes} \dots \bar{\otimes} X_n$ be the completion of the algebraic tensor product $X_1 \otimes \dots \otimes X_n$ with respect to the canonical Hilbert norm, and let $\tilde{S}_j \in L(\tilde{X})$ be the canonical closure of the operator

$$I_1 \otimes I_2 \otimes \dots \otimes I_{j-1} \otimes S_j \otimes I_{j+1} \otimes \dots \otimes I_n.$$

Then $\tilde{S} = (\tilde{S}_1,\dots,\tilde{S}_n)$ is a D-commuting system, for $D = D(S_1) \otimes \dots \otimes D(S_n)$.

T h e o r e m 3.1.

$$\sigma_{e_D}(\tilde{S};\tilde{X}) \subset \left(\sigma_{e_{D(S_1)}}(S_1,X_1) \times \sigma_{D(S_2)}(S_2,X_2) \times \ldots \times \sigma_{D(S_n)}(S_n,X_n)\right) \cup$$

$$\cup \left(\sigma_{D(S_1)}(S_1,X_1) \times \sigma_{e_{D(S_2)}}(S_2,X_2) \times \sigma_{D(S_3)}(S_3,X_3) \times \ldots\right.$$

$$\left.\ldots \times \sigma_{D(S_n)}(S_n,X_n)\right) \cup \ldots \cup \left(\sigma_{D(S_1)}(S_1,X_1) \times \ldots\right.$$

$$\left.\ldots \times \sigma_{D(S_{n-1})}(S_{n-1},X_{n-1}) \times \sigma_{e_{D(S_n)}}(S_n,X_n)\right).$$

C o r o l l a r y 3.2. For $n = 2$, Theorem 3.1 reduces to a special case of Lemma 2.4, that is,

$$\sigma_{e_{D(S_1)\otimes D(S_2)}}((S_1 \bar{\otimes} I_2,\ I_1 \bar{\otimes} S_2),\ X_1 \bar{\otimes} X_2) \subset$$

$$\subset \left(\sigma_{e_{D(S_1)}}(S_1,X_1) \times \sigma_{D(S_2)}(S_2,X_2) \cup \sigma_{D(S_1)}(S_1,X_1) \times \sigma_{e_{D(S_2)}}(S_2,X_2)\right).$$

P r o o f of Theorem 3.1. The proof follows from an inductive argument. For $n = 2$, the theorem reduces to a special case of Lemma 2.4.

Let us assume that the theorem holds for any $n-1$ operators, for $n \geq 3$. Then, if B_j denotes the canonical closure of the operator

$$I_1 \otimes \ldots \otimes S_j \otimes \ldots \otimes I_{n-1}, \quad (j=1,\ldots,n-1),$$

we obtain

$$\tilde{S} = (B_1 \bar{\otimes} I_n, \ldots, B_{n-1} \bar{\otimes} I_n,\ I_1 \bar{\otimes} \ldots \bar{\otimes} I_{n-1} \bar{\otimes} S_n).$$

If $B = (B_1,\ldots,B_{n-1})$, $D(B) = D(S_1) \otimes \ldots \otimes D(S_{n-1})$ and $\tilde{X}_{n-1} = X_1 \bar{\otimes} \ldots \bar{\otimes} X_{n-1}$, then Lemma 2.4, Lemma 2.7 and the induction hypothesis imply that

$$\sigma_{e_{D(B)\otimes D(S_n)}}(B\bar{\otimes} S_n,\ \tilde{X})\subset\Big(\sigma_{e_{D(B)}}(B,\tilde{X}_{n-1})\times$$

$$\times\,\sigma_{D(S_n)}(S_n,X_n)\Big)\cup\Big(\sigma_{D(B)}(B,\tilde{X}_{n-1})\times\sigma_{e_{D(S_n)}}(S_n,X_n)\Big)\subset$$

$$\subset\Big[\Big(\sigma_{e_{D(S_1)}}(S_1,X_1)\times\sigma_{D(S_2)}(S_2,X_2)\times\ \dots\ \times\sigma_{D(S_{n-1})}(S_{n-1},X_{n-1})\Big)\cup$$

$$\cup\Big(\sigma_{D(S_1)}(S_1,X_1)\times\sigma_{e_{D(S_2)}}(S_2,X_2)\times\sigma_{D(S_3)}(S_3,X_3)\times\ \dots$$

$$\dots\times\sigma_{D(S_{n-1})}(S_{n-1},X_{n-1})\Big)\cup\ \dots\ \cup\Big(\sigma_{D(S_1)}(S_1,X_1)\times\ \dots$$

$$\dots\times\sigma_{D(S_{n-2})}(S_{n-2},X_{n-2})\times\sigma_{e_{D(S_{n-1})}}(S_{n-1},X_{n-1})\Big)\Big]\times\sigma_{D(S_n)}(S_n,X_n)\cup$$

$$\cup\Big(\sigma_{D(S_1)}(S_1,X_1)\times\ \dots\times\sigma_{D(S_{n-1})}(S_{n-1},X_{n-1})\Big)\times\sigma_{e_{D(S_n)}}(S_n,X_n).$$

This completes the proof.

C o n j e c t u r e 3.3. Under the conditions as in Theorem 3.1, we conjecture the following inclusion relation which seems to be true

$$\Big(\sigma_{e_{D(S_1)}}(S_1,X_1)\times\sigma_{D(S_2)}(S_2,X_2)\times\ \dots\ \times\sigma_{D(S_n)}(S_n,X_n)\Big)\cup\ \dots$$

$$\dots\cup\Big(\sigma_{D(S_1)}(S_1,X_1)\times\ \dots$$

$$\dots\times\sigma_{D(S_{n-1})}(S_{n-1},X_{n-1})\times\sigma_{e_{D(S_n)}}(S_n,X_n)\Big)\subset\sigma_{e_D}(\tilde{S},\tilde{X}).$$

R e m a r k 3.4. I believe strongly that Conjecture 3.3 must be true, and I guess that it may follow from some combination of Taylor's technique [7] and Fialkow's approach [4].

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