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THE WEIGHTED (0,2) LACUNARY INTERPOLATION, I

Dedicated to the memory of Professor Roman Sikorski

1. Let $\{x_\nu\}_{\nu=1}^{n}$ be a system of n nodal points (n=1,2,3,...), and let $\{\alpha_{\nu n}\}_{\nu=1}^{n}$ and $\{\beta_{\nu,n}\}_{\nu=1}^{n}$ be arbitrarily chosen real numbers. Balázs and P. Turán, Surányi and P. Turán in their papers [1], [2], [3], dealt extensively on the problems connected with their so-called (0,2) interpolation. Prasad [4] in 1976 established an interpolation polynomial of degree $\leqq 2n-1$ which under suitable conditions converges uniformly to a function belonging to the Zygmund class.

In this paper we are interested in the weighted (0,2) interpolation polynomials, namely those polynomials $Q_m(x)$ constructed such that

$$(1.1)\qquad Q_m(x_\nu) = \alpha_{\nu,n}, \quad \nu=0,1,2,\dots,n+1,$$

$$(1.2)\qquad \left\{\varrho(x)\, Q_m(x)\right\}''_{x=x_\nu} = \beta_{\nu,n}, \quad \nu=1,2,\dots,n,$$

where the weight function $\varrho(x)$ is particularly equal to $(1-x^2)^{\frac{1}{4}}$. This is also in conformity with the weight function chosen by Prasad, namely

$$\rho(x) = (1-x^2)^\lambda, \quad 0 \leqslant \lambda < \frac{3}{2}, \quad \lambda \neq \frac{1}{2}.$$

In addition we have added to requirements (1.1) and (1.2) the following

$$(1.3) \qquad Q_m(-1) = \alpha_0 \qquad \text{and} \qquad Q_m(1) = \alpha_{n+1}.$$

It is needless at this point to emphasize the importance of this type of interpolation polynomial in approximative solution of the boundary value problems of the second order linear differential equations of the type

$$Y''(x) + A(x)Y(x) = 0,$$

$$Y(-1) = \alpha_0; \ Y(+1) = \alpha_{n+1}.$$

It will be observed that we have introduced in (1.3) the extremal nodal points and their corresponding prescribed arbitrary chosen values. Also we succeeded in establishing that the polynomial satisfying the conditions (1.1), (1.2) and (1.3) really exists and it is unique. The question of convergence will be dealt with in our next paper.

2. Let us choose as our $x_{\nu,n}$ the roots of Tchebysheff polynomials

$$(2.1) \qquad X_{\nu,n} = \frac{\cos(2\nu-1)\pi}{2n}, \quad \nu = 1,2,\dots,n; \ n=1,2,\dots$$

$$x_{0,n} = -1, \quad x_{n+1,n} = 1.$$

The Tchebysheff Polynomial

$$(2.2) \qquad T_n(x) = \cos(n \arccos x)$$

satisfies the following differential equation

$$(2.3) \qquad (1-x^2)\, T_n''(x) - x\, T_n'(x) + n^2 T_n(x) = 0 \quad \left(' = \frac{d\cdot}{dx}\right)$$

and -1 and +1 are not roots of $T_n(x)$.

Our task is to contruct a polynomial $Q_m(x)$ of least possible degree satisfying conditions (1.1), (1.2) and (1.3) using the nodes (2.1).

Obviously the degree of our polynomial $m \leq 2n+1$ and it is of the form

$$(2.4) \qquad Q_m(x) = \alpha_0 U_0(x) + \alpha_{n+1} U_{n+1}(x) + \sum_{\nu=1}^{n} \alpha_{\nu,n} U_\nu(x) + \sum_{\nu=1}^{n} \beta_{\nu,n} V_\nu(x),$$

where the polynomials $U_j(x)$ and $V_\nu(x)$, where $j=0,\dots,n+1$, $\nu=1,\dots,n$ are as usual fundamental polynomials of the first and second kind of the weighted (0,2) interpolation belonging to our nodal points (2.1) with the degree $\leq 2n+1$ and having the following interpolation properties:

$$(2.5) \qquad U_j(x_k) = \delta_{jk} = \begin{cases} 0, j \neq k \\ 1, j = k \end{cases} \quad (j,k=0,1,2,\dots,n+1)$$

$$(2.6) \qquad \left\{ \rho(x) U_j(x_j) \right\}''_{x=x_k} = 0, \quad (j,k=0,1,2,\dots n+1;\ x_k=x_{k,n})$$

$$(2.7) \qquad V_j(x_k) = 0, \quad (j=1,2,\dots,n;\ k=0,1,\dots,n+1)$$

$$(2.8) \qquad \left\{ \rho(x) \nu_j(x) \right\}''_{x=x_k} = \delta_{jk}, \quad (j,k=1,2,\dots,n).$$

3. We shall prove the following theorems

T h e o r e m 1. If n is odd, then there in general exists no polynomial $Q_m(x)$ belonging to (2.1) and satisfying the conditions (2.5), (2.6), (2.7) and (2.8).

T h e o r e m 2. If n is even then the polynomial $Q_m(x)$ satisfying the conditions of Theorem 1 exists and is unique.

It is plausible at this juncture to consider an important lemma which will be used in proofs of our theorems.

Lemma 1. If $\rho(x) = \left(1 - x^2\right)^{\frac{1}{4}}$, then

$$(3.1) \qquad \left\{\rho(x)\, T_n(x)\right\}''_{x=x_\nu} = 0 \quad (T_n(x_\nu) = 0,\ \nu=1,2,\dots,n).$$

Proof.

$$\left\{\rho(x)T_n(x)\right\}''_{x=x_\nu} =$$

$$= \left\{\rho''(x)T_n(x) + 2\left[-\frac{1}{2}\,x(1-x^2)^{-\frac{3}{4}}\cdot T_n'(x)\right] + (1-x^2)^{\frac{1}{4}}\,T_n''(x)\right\}_{x=x_\nu} =$$

$$= (1-x_\nu^2)^{-\frac{3}{4}}\left[(1-x_\nu^2)^{\frac{3}{4}}\,\rho''(x_\nu)T_n(x_\nu) - x_\nu T_n'(x_\nu) + (1-x_\nu^2)T_n''(x_\nu)\right].$$

From the differential equation (2.3) and the fact that $T_n(x_\nu) = 0$ we easily see that

$$\left\{\rho(x)T_n(x)\right\}''_{x=x_\nu} = 0, \quad \nu=1,2,\dots,n.$$

and the lemma is proved.

Proof of Theorem 1. Let us consider a special case of our theorem, namely:

Let

$$(3.2) \qquad \alpha_0 = \alpha_1 = \dots = \alpha_n = \alpha_{n+1} = 0$$

and

$$(3.3) \qquad \beta_j = 1, \quad \beta_\nu = 0; \quad \nu=1,2,\dots,j-1,\ j+1,\dots,n.$$

If $R_{2n+1}(x)$ is an interpolation polynomial of degree $\leqslant 2n+1$ satisfying conditions (1.1) and (1.2) with prescribed values (3.2) and (3.3) at the nodes (2.1), then our polynomial must be of the form

$$(3.4) \qquad R_{2n+1}(x) = (1-x^2)T_n(x)g_{n-1}(x)$$

where $g_{n-1}(x)$ is a polynomial of degree $\leq$ n-1. That is

$$\left\{\varrho(x)R_{2n+1}(x)\right\}''_{x=x_\nu} = \left\{\varrho(x)(1-x^2)g_{n-1}(x)\right\}''_{x=x_\nu} =$$

$$= \left[(1-x^2)g_{n-1}(x)\right]_{x=x_\nu}\left[\varrho(x)T_n(x)\right]''_{x=x_\nu} +$$

$$+ 2\left[\varrho(x)T_n(x)\right]'_{x=x_\nu}\left[(1-x^2)g_{n-1}(x)\right]'_{x=x_\nu} +$$

$$+ \left\{\varrho(x)T_n(x)\left[(1-x^2)g_{n-1}(x)\right]''\right\}_{x=x_\nu} = \begin{cases}1, & \nu = j\\ 0, & \nu \neq j\end{cases}, \qquad (1\leq\nu\leq n).$$

That is

$$(2\varrho(x_\nu)T'_n(x_\nu))\left[(1-x_\nu^2)g'_{n-1}(x_\nu) - 2x_\nu g_{n-1}(x_\nu)\right] = \begin{cases}1, & \nu = j\\ 0, & \nu = j,\end{cases}$$

and

$$(1-x_\nu^2)g'_{n-1}(x_\nu) - 2x_\nu g_{n-1}(x_\nu) = \begin{cases}0, & \nu \neq j\\ \dfrac{1}{2\varrho(x_\nu)T'_n(x_\nu)}, & \nu = j.\end{cases}$$

For $x_\nu = x$ we have the equation of the form

$$(3.5) \qquad (1-x^2)g'_{n-1}(x) - 2xg_{n-1}(x) \equiv \frac{l_j(x)}{2\varrho(x_j)T'_n(x_j)}\left[1+a(x-x_j)\right]$$

where as usual

$$l_j(x) = \frac{T_n(x)}{T'_n(x_j)(x-x_j)}$$

is the fundamental polynomial of Lagrange interpolation and a is a constant.

Integrating both sides we have

$$\left[(1-t^2)g_{n-1}(x)\right]_{-1}^{x} \equiv \frac{1}{2T_n'(x_j)\varrho(x_j)} \cdot$$

$$\cdot\left[\int_{-1}^{x} l_j(t)dt + a\int_{-1}^{x}(t-x_j)l_j(t)dt\right] \cdot (1-x^2)g_{n-1}(x) \equiv$$

$$\equiv \frac{1}{2T_n'(x_j)\varrho(x_j)}\left[\int_{-1}^{x} l_j(t)dt + a\int_{-1}^{x}(t-x_j)l_j(t)dt\right].$$

For $x = 1$, we have

$$\int_{-1}^{1} l_j(t)dt = -\frac{a}{T_n'(x_j)}\int_{-1}^{1} T_n(t)dt, \tag{3.6}$$

and for odd n

$$\int_{-1}^{1} T_n(t)dt = 0. \tag{3.7}$$

The relations (3.6), (3.7) imply that $\int_{-1}^{1} l_j(t)dt = 0$, which is not true and hence Theorem 1 is proved.

4. As a consequence of Theorem 1, we can restrict ourselves to the case of even n. The following lemmas are very useful.

L e m m a 2. If n is even and $1 \leq \nu \leq n$, the fundamental polynomial of the first kind $U_\nu(x)$ can explicitly be represented by

$$(4.1)\qquad U_\nu(x) = \frac{1-x^2}{1-x_\nu^2}\, l_\nu(x^2) + $$

$$+ T_n(x)\left\{ A_\nu \int_{-1}^{x} (1-t^2)\left[\frac{l'_\nu(t) - l'_\nu(t) l_\nu(t)}{t - x_\nu}\right] dt + \right.$$

$$\left. + B_\nu \int_{-1}^{x} l_\nu(t)\,dt + C_\nu \int_{-1}^{x} T_n(t)\,dt \right\}$$

where

$$(4.2)\qquad A_\nu = -\frac{1}{(1-x_\nu^2)T'_n(x_\nu)}$$

$$(4.3)\qquad B_\nu = \left\{\frac{1}{1-x_\nu^2} + \frac{3x_\nu^2}{2(1-x_\nu^2)^2} + \frac{\rho'(x_\nu)x_\nu}{\rho(x_\nu)(1-x_\nu^2)} - \frac{\rho''(x_\nu)}{2\rho(x_\nu)}\right\}\frac{1}{T'_n(x_\nu)}$$

$$(4.4)\qquad C_\nu = -\left\{\int_{-1}^{1} \frac{l'_\nu(t) - l'_\nu(x_\nu) l_\nu(t)}{t - x_\nu}\, dt + B_\nu \int_{-1}^{1} l_\nu(t)\,dt\right\} \cdot$$

$$\cdot \left\{\int_{-1}^{1} T_n(t)\,dt\right\}^{-1},$$

where $\int_{-1}^{1} T_n(t)\,dt \neq 0.$

P r o o f . It is trivially true that

$$U_\nu(x_j) = \delta_{\nu j} = \begin{cases} 0, & j \neq \nu \\ 1, & j = \nu \end{cases}, \qquad (j=1,2,\dots,n;\ \nu,n=2,4,\dots)$$

and

$$U_\nu(-1) = U_\nu(1) = 0.$$

Next

$$(4.5)\quad \left\{\rho(x)U_\nu(x)\right\}''_{x=x_j} = \frac{1}{1-x_\nu^2}\left\{\rho(x)(1-x^2)l_\nu(x^2)\right\}''_{x=x_j} +$$

$$+\left\{\rho(x)T_n(x)\int_{-1}^{x}(1-t^2)\frac{l'_\nu(t)-l'_\nu(x_\nu)l_\nu(t)}{t-x_\nu}dt\right\}''_{x=x_j} +$$

$$+B_\nu\left\{\rho(x)T_n(x)\int_{-1}^{x}l_\nu(t)dt\right\}''_{x=x_j} +$$

$$+C_\nu\left\{\rho(x)T_n(x)\int_{-1}^{x}T_n(t)dt\right\}''_{x=x_j} =$$

$$=\frac{1}{1-x_\nu^2}\left\{\rho(x)(1-x^2)l_\nu(x^2)\right\}''_{x=x_j} + A_\nu 2\rho(x_j)T'_n(x_j)(1-x_j^2)\cdot$$

$$\cdot\left\{\frac{l'_\nu(x_j)-l'_\nu(x_\nu)l_\nu(x_j)}{x_j-x_\nu}\right\} + B_\nu\rho(x_j)T'_n(x_j)l_\nu(x_j).$$

Two cases are to be considered, namely:

Case (i). $\nu\neq j$. In this case we have from (4.5) using Lemma 1

$$\left\{\rho(x)U_\nu(x)\right\}''_{x=x_j} = \frac{2}{1-x_\nu^2}\left\{\rho(x_j)(1-x_j)l'_\nu(x_j^2) +\right.$$

$$+2A_\nu\rho(x_j)T'_n(x_j)(1-x_j)\frac{l'_\nu(x_j)}{x_j-x_\nu} =$$

$$\left.=2\lambda(x_j)\frac{1}{1-x_\nu^2}l'_\nu(x_j^2) + A_\nu\frac{T'_n(x_j)l'_\nu(x_j)}{x_j-x_\nu}\right\},$$

where

$$\lambda(x_j) = \rho(x_j)(1-x_j^2).$$

Since $T_n'(x_j) = l_\nu'(x_j)(x_j-x_\nu)T_n'(x_\nu)$ and $A_\nu = -\dfrac{1}{(1-x_\nu^2)T_n'(x_\nu)}$ then $\left\{\varrho(x)U_\nu(x)\right\}''_{x=x_j} = 0.$

Case (ii). $j=\nu$. From (4.3), (4.5) and Lemma 1 we have by applying the l'Hospital rule

$$(4.6)\quad \left\{\varrho(x)U_\nu(x)\right\}''_{x=x_\nu} = \varrho(x_\nu)'' + 2\,\varrho'(x_\nu)\left\{2l_\nu'(x_\nu) - \frac{2x_\nu}{1-x_\nu^2}\right\} +$$

$$+\ \varrho(x_\nu)\left\{2l_\nu''(x_\nu) + 2l_\nu'(x_\nu^2) - \frac{8x_\nu}{1-x_\nu^2}\,l_\nu'(x_\nu) - \frac{2}{1-x_\nu^2}\right\} +$$

$$+\ 2\varrho(x_\nu)T_n(x_\nu')\left\{A_\nu\left[l_\nu''(x_\nu) - l_\nu'(x_\nu)^2\right](1-x_\nu^2) + B_\nu\right\}.$$

Clearly

$$l_\nu'(x_j) = \frac{T_n'(x_j)}{T_n'(x_\nu)(x_j-x_\nu)}$$

and

$$(4.7)\qquad l_\nu'(x_\nu) = \frac{T_n''(x_\nu)}{T_n'(x_\nu)} = \frac{x_\nu}{2(1-x_\nu^2)}\ .$$

Using (4.2) and (4.6) we have

$$\left\{\varrho(x)U_\nu(x)\right\}''_{x=x_\nu} = \varrho''(x_\nu) + 2\varrho(x_\nu')\left\{-\frac{x_\nu}{1-x_\nu^2}\right\} +$$

$$+\ \varrho(x_\nu)^{\bullet}\left\{-\frac{3x_\nu^2}{(1-x_\nu^2)^2} - \frac{2}{1-x_\nu^2}\right\} + 2B_\nu\varrho(x_\nu)T_n'(x_\nu).$$

Applying condition (4.3), the lemma is completely proved.

L e m m a 3. If n is even and $1\le\nu\le n$, then the fundamental polynomial of the second kind $V_\nu(x)$ can be explicitly expressed as

$$(4.8)\quad V_\nu(x) = T_n(x)\left\{\frac{1}{2(1-x_\nu^2)^{\frac{1}{4}}\,T_n'(x_\nu)}\int_{-1}^{x} l_\nu(t)dt + a_\nu \int_{-1}^{x} T_n(t)dt\right\}$$

where

$$(4.9)\quad a_\nu = \left\{-\frac{1}{2(1-x_\nu^2)^{\frac{1}{4}}\,T_n'(x_\nu)}\int_{-1}^{1} l_\nu(t)dt\right\}\left\{\int_{-1}^{1} T_n(t)dt\right\}^{-1}.$$

P r o o f . Obviously $V_\nu(-1) = V_\nu(1) = 0$. Also $V_\nu(x_j) = 0$, $j = 1,2,\ldots,n$. Using Lemma 1 we have

$$\left\{\rho(x)V_\nu(x)\right\}''_{x=x_j} =$$

$$= T_n'(x_j)\rho(x_j)\cdot\frac{2}{2(1-x_\nu^2)^{\frac{1}{4}}\,T_n'(x_\nu)}\left[l_\nu(x_j) + aT_n(x_j)\right] =$$

$$= \delta_{\nu j} = \begin{cases}1, & \nu = j\\ 0, & \nu \neq j\end{cases},\quad \nu, j=1,2,\ldots,n,$$

thus satisfying condition (2.7) and (2.8) so that the lemma is proved.

At this juncture, our next task is to determine the two extremal fundamental polynomials $U_0(x)$ and $U_{n+1}(x)$ such that

$$(4.10)\qquad U_0(1) = U_{n+1}(-1) = 0,$$

$$(4.11)\qquad U_0(-1) = U_{n+1}(1) = 1,$$

and

$$(4.12)\qquad \left\{\rho(x)U_j(x)\right\}''_{x=x_\nu} = 0, \quad (j=0,n+1;\ \nu=1,2,\ldots,n).$$

This is easily settled by the following

L e m m a 4. If n is even, then the extremal fundamental interpolation polynomials $U_0(x)$ and $U_{n+1}(x)$ that satisfy the conditions (4.10), (4.11) and (4.12) can explicitly be written as

$$(4.13)\quad U_0(x) = \frac{1-x}{2} T_n(x^2) - \frac{T_n(x)}{2}\left\{\int_{-1}^{x} (1-t)T_n'(t)dt + A\int_{-1}^{x} T_n(t)dt\right\}$$

where

$$A = \frac{1}{2}\left\{\int_{-1}^{1} (1-t)T_n'(t)dt\right\}\left\{\int_{-1}^{1} T_n(t)dt\right\}^{-1},$$

and

$$(4.14)\quad U_{n+1}(x) =$$
$$= \frac{1+x}{2} T_n(x^2) - \frac{T_n(x)}{2}\left\{\int_{-1}^{x} (1+t)T_n'(t)dt + B\int_{-1}^{x} T_n(t)dt\right\}$$

where

$$B = \frac{1}{2}\left\{\int_{-1}^{1} (1+t)T_n'(t)dt\right\}\left\{\int_{-1}^{1} T_n(t)dt\right\}^{-1}.$$

P r o o f . Clearly $U_0(1) = U_0(x_\nu) = 0$, $\nu = 1,2,\dots,n$ and $U_0(-1) = 1$. However using Lemma 1 we have

$$\left\{\rho(x)U_0(x)\right\}''_{x=x_\nu} = \left\{(1-x_\nu)T_n(x)\right\}'_{x=x_\nu}\left\{\rho(x)T_n(x)\right\}'_{x=x_\nu} -$$
$$- \left\{\rho(x)T_n(x)\right\}'_{x=x_\nu}\left\{(1-x_\nu)T_n'(x_\nu)\right\} = 0.$$

Similarly $\left\{\rho(x)U_{n+1}(x)\right\}''_{x=x_\nu} = 0$ and the lemma is proved.

P r o o f of Theorem 2. Using Theorem 1, Lemmas 2, 3 and 4, the existence of this interpolation polynomial is established.

In dealing with the uniqueness, let $Q_m^*(x)$ be another polynomial satisfying conditions (1.1) and (1.2). Then there exists a polynomial $W(x) = Q_m^*(x) - Q_m(x)$. Surely $W(x)$ satisfies conditions (1.1) and (1.2) with degree $\leq 2n+1$ and can be represented as

$$W(x) = (1-x^2)T_n(x)g_{n-1}(x), \tag{4.15}$$

where $g_{n-1}(x)$ is a polynomial of degree $\leq n-1$. Then

$$\left\{\rho(x)W(x)\right\}''_{x=x_\nu} = \left\{\rho(x)T_n(x)\right\}''_{x=x_\nu} + $$
$$+ 2\rho(x_\nu)T_n'(x_\nu)\left\{(1-x^2)g_{n-1}(x)\right\}'_{x=x_\nu} = 0.$$

That is $\left\{(1-x^2)g_{n-1}(x)\right\}' = C\,T_n(x)$ where C is a constant. Therefore

$$(1-x^2)g_{n-1}(x) = C\int_{-1}^{x} T_n(t)dt.$$

For $x = 1$ we have $0 = C\int_{-1}^{1} T_n(t)dt$. Since n is even $\int_{-1}^{1} T_n(t)dt \neq 0$. That is $C \equiv 0$ and hence $(1-x^2)g_{n-1}(x) \equiv 0$.

Hence from (4.15) it follows $W(x) \equiv 0$. That is $Q_m^*(x) \equiv \equiv Q_m(x)$ and Theorem 2 is completely proved.

If however $f(x) \in C^2[-1,1]$, then an interpolation polynomial $Q_m(f;x)$ of degree $\leq 2n+1$ can be formed such that

$$Q_m(f;x_k) = f(x_k)$$

and

$$\left\{\rho(x)Q_m(f;x_k)\right\}''_{x=x_k} = f''(x_k).$$

The interpolation polynomial for the above function using our nodes (2.1) can be explicitly expressed as

$$Q_m(f;x_k) = f(-1)U_0(x) + f(1)U_{n+1}(x) + \sum_{\nu=1}^{n} f(x_\nu)U_\nu(x) + \sum_{\nu=1}^{n} f'(x_\nu)V_\nu(x).$$

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