DEMONSTRATIO MATHEMATICA
Vol. XVII No 2 1984

Lothar v. Wolfersdorf, Janina Wolska - Bochenek

A COMPOUND RIEMANN - HILBERT PROBLEM FOR HOLOMORPHIC FUNCTIONS WITH NONLINEAR BOUNDARY CONDITION

1. Introduction

Linear compound Riemann-Hilbert problems for holomorphic functions, i.e. problems with a Riemann-Hilbert condition on the boundary of the considered region and a conjugacy condition (Riemann or Hilbert condition) on an inner contour in the region, have been investigated in several papers, so by I.S.Rogozhina [6] and Lu Chien Ke [2]. The case of a non-linear conjugacy condition for generalized analytic functions in the sense of I.N.Vekua was considered by J.Wolska-Bochenek [9], whereas in the papers of A.Mamourian [3], [4], linear compound problems are treated for general elliptic complex equations of first order. In the recent paper of Fang Ainong [1] problems with linear conjugacy and Riemann-Hilbert condition for a class of strongly nonlinear elliptic complex equations of first order are dealt with. A forthcoming paper of Wen Guo-chun [7] treats a compound problem with a shift for nonlinear elliptic complex equations of first order.

In the present paper a special class of compound problems for holomorphic functions in the unit disk with a nonlinear Riemann-Hilbert condition is considered. Reducing the problems in a known way to the corresponding nonlinear Riemann-Hilbert problems and utilising the results of [8] for the Riemann-Hilbert problems, we obtain existence theorems for the compound problems.

2. Statement of problems

Let $D: |z| < 1$ be the unit disk in the complex z plane with the boundary $S: |t| = 1$, $t = e^{is}$ $(-\pi \leq s \leq \pi)$. Let L be a simple closed smooth contour lying in D with variable point τ. We denote by D^- the inner region bounded by L, and by D^+ the doubly-connected region situated inside S and outside L. As positive direction on L we take the clockwise direction, and as positive direction on S the counter-clockwise one such that the domain D^+ lies on the left of S and L.

Problem E

It is required to determine a sectionally holomorphic function $\Phi(z) = \varphi(z) + i\,\psi(z)$ in D with continuous limiting values $\Phi^+(\tau)$, $\Phi^-(\tau)$ on L and continuous boundary values $\Phi(t) = \varphi(t) + i\,\psi(t)$ on S satisfying the conjugacy condition

$$(1) \qquad \Phi^+(\tau) = G\,\Phi^-(\tau) + g(\tau) \quad \text{on} \quad L,$$

the nonlinear boundary condition

$$(2) \qquad \psi(t) + F(s,\varphi(t)) = f(s) \quad \text{on} \quad S,$$

and the additional condition

$$(3) \qquad \varphi(1) = k \quad \text{in} \quad t = 1.$$

In Problem E_1 and E_2 the additional condition is

$$(3a) \qquad \psi(0) = c \quad \text{in} \quad z = 0$$

and

$$(3b) \qquad \varphi(0) = d \quad \text{in} \quad z = 0,$$

respectively, where it is supposed that the origin $z = 0$ lies in D^-.

We make the following basic assumptions on the data:

(i) $G = G_1 + i\,G_2$ is, in general, a complex constant different from zero; $g(\tau) \in H_\mu(L)$, $0 < \mu < 1$, is a given Hölder continuous function on L.

(ii) $F(s,\varphi)$ is a real-valued continuous function on $[-\pi,\pi]\times\mathbb{R}$ which is 2π-periodic in s and possesses a continuous partial derivative F_φ and a partial derivative F_s satisfying the Carathéodory conditions and an estimation of the form

$$|F_s(s,\varphi)| \leqslant E(s) \in L_\varrho(S), \quad \varrho > 1,$$

for values φ from bounded intervals of $\mathbb{R}$.

(iii) $f(s)$ is a real-valued absolutely continuous 2π-periodic function on $[-\pi,\pi]$ possessing a derivative $f'(s) \in L_\varrho(S)$, $\varrho > 1$.

(iv) k, c, d are given real constants.

3. Reduction to a nonlinear Riemann-Hilbert problem

We represent the unknown function $\Phi(z)$ in the form

$$\Phi(z) = X(z)\,[w(z) + w_0(z)], \tag{4}$$

where

$$X(z) = \begin{cases} \frac{1}{G} & \text{in } D^- \\ 1 & \text{in } D^+ \end{cases} \tag{5}$$

and

$$w_0(z) = u_0(z) + iv_0(z) = \frac{1}{2\pi i}\int_L \frac{g(\tau)}{\tau - z}\,d\tau. \tag{6}$$

Then the new unknown function $w(z) = u(z) + iv(z)$ is holomorphic in D, continuous in $\bar{D}$ and satisfies the boundary condition

$$v(t) + F(s, u(t) + u_0(t)) = f(s) - v_0(t) \quad \text{on } S \tag{7}$$

and one of the additional conditions

$$u(1) = k - u_0(1) \equiv k_0 \quad \text{in } t = 1 \tag{8}$$

and

(9a) $G_1\, v(0) - G_2\, u(0) = c|G|^2 - G_1\, v_0(0) + G_2\, u_0(0)$ in $z=0$,

(9b) $G_1\, u(0) + G_2\, v(0) = d|G|^2 - G_1\, u_0(0) - G_2\, v_0(0)$ in $z=0$,

in problems E and E_1, E_2, respectively. The continuity of $w(z)$ across the inner contour L easily follows from the well-known Plemelj formulae (cf. [5]) applied to the Cauchy integral in (6). Problems E_1 and E_2 reduce to Riemann-Hilbert problems of similar type, where the case of a real (imaginary) constant G in Problem E_1 corresponds to the case of an imaginary (real) constant G in Problem E_2. In the sequel we restrict ourselves to Problems E and E_1.

4. Problem E

The Riemann-Hilbert problem (7), (8) has the form of Problem P_1 of [8]. Theorem 2 of [8] implies the following one.

Theorem 1. If for some p with $1 < p < \varrho$ there exists $R \geqslant |k_0| + (2\pi)^{\frac{1}{q}} C_R$, q being the exponent conjugate to p and

$$C_R = (2\pi)^{\frac{2}{K}} \left[M + r_R\right] \left\{1 + 2A_r(\cos K\gamma_R)^{-\frac{2}{K}}\right\}, \tag{10}$$

where $K = \dfrac{2p\varrho}{\varrho - p}$,

$$M = \left\| f' - \frac{dv_0}{ds} \right\|_\varrho , \tag{11}$$

$\|\cdot\|_\varrho$ denoting the norm in $L_\varrho(S)$,

$$r_R = \sup_{\substack{s \in [-\pi,\pi] \\ |u| \leqslant R}} \left\| F_s(s, u+u_0(t)) + F_\varphi(s, u+u_0(t))\, \frac{du_0}{ds} \right\|_\varrho \tag{12}$$

and

$$(13)\qquad 2\gamma_R = \max_{\substack{s\in[-\pi,\pi]\\ |u|\leq R}} [m(s,u)] - \min_{\substack{s\in[-\pi,\pi]\\ |u|\leq R}} [m(s,u)]$$

with

$$(14)\qquad m(s,u) = \arctan F_\varphi(s,u+u_0(t)),$$

then Problem E possesses a solution $\Phi(z)\in C_\lambda(\overline{D^+})\cap C_\mu(\overline{D^-})$, $\lambda = \min\left(\frac{1}{q},\mu\right)$, with boundary values $\Phi'(t)\in L_p(S)$. If additionally the derivative $F_\varphi(s,\varphi)$ satisfies a Hölder condition, the function $\Phi(z)$ is the only Hölder continuous sectionally holomorphic solution of Problem E. Here A_r denotes the norm of the Hilbert transformation

$$(15)\qquad (Hu)(s) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{i\sigma})\cot\frac{\sigma-s}{2}\,d\sigma$$

in $L_r(S)$, $r = \frac{2p\varrho}{\varrho+p}$, i.e.

$$(16)\qquad A_r = \begin{cases} \tan\left(\frac{\pi}{2r}\right) & \text{if } 1<r\leq 2\\ \cot\left(\frac{\pi}{2r}\right) & \text{if } 2\leq r<\infty. \end{cases}$$

C o r o l l a r y . In particular, there exists a solution $\Phi(z)\in C_\lambda(\overline{D^+})\cap C_1(\overline{D^-})$, $\lambda = \frac{1}{q}$, to Problem E if g = const,

$$(17)\qquad 2\gamma = \sup_{\substack{s\in[-\pi,\pi]\\ \varphi\in\mathbb{R}}} [m_0(s,\varphi)] - \inf_{\substack{s\in[-\pi,\pi]\\ \varphi\in\mathbb{R}}} [m_0(s,\varphi)] < \frac{\pi}{K},$$

where $m_0(s,\varphi) = \arctan F_\varphi(s,\varphi)$, and

$$(18)\qquad \sup_{\substack{s\in[-\pi,\pi]\\ \varphi\in\mathbb{R}}} \|F_s(s,\varphi)\|_\varrho < \infty.$$

In case of $\varrho = \infty$ one has $K = 2p$ and (17) is fulfilled for p sufficiently near to 1 if the oscillation 2γ of the function $\arctan[F_\varphi(s,\varphi)]$ is smaller than $\frac{\pi}{2}$; cf. [8] for further examples of Theorem 1, especially with $2\gamma_R \sim \arctan[\lambda_0 R^\alpha]$, $0 \leqslant \alpha < 1$ as $R \to \infty$.

5. Problem E_1

We distinguish three cases: G is real, G is purely imaginary, and G is complex.

In the first case the additional condition (9a) reads

$$v(0) = c\,G - v_0(0) \quad \text{in} \quad z = 0 \tag{19}$$

and the problem (7), (19) has the form of Problem Q_1 of [8]. We pose the following Assumption B_0 on the function F:

(i) There hold the estimations

$$\left| F_s(s,\varphi) + F_\varphi(s,\varphi)\frac{du_0}{ds} \right| \leqslant E_0(s) \in L_\varrho(\Gamma), \quad \varrho > 1, \tag{20}$$

for almost all $s \in [-\pi,\pi]$ and all $\varphi \in \mathbb{R}$, and

$$2\gamma = \sup_{\substack{s\in[-\pi,\pi]\\ \varphi\in\mathbb{R}}} [m_0(s,\varphi)] - \inf_{\substack{s\in[-\pi,\pi]\\ \varphi\in\mathbb{R}}} [m_0(s,\varphi)] < \frac{\pi}{2}\,\frac{\varrho-1}{\varrho}, \tag{21}$$

where as above $m_0(s,\varphi) = \arctan[F_\varphi(s,\varphi)]$.

(ii) $F(s,\varphi)$ is strictly monotone in φ for almost all $s \in [-\pi,\pi]$ and possesses the (finite or identically infinite) limit functions $F^{\pm}(s) = \lim_{\varphi\to\pm\infty} F(s,\varphi)$ uniformly in $s \in [-\pi,\pi]$.

From Theorem 4 of [8] then follows

T h e o r e m 2. Under Assumption B_0 the Problem E_1 with real constant G has a unique solution $\Phi(z) \in C_\lambda(\overline{D^+}) \cap C_\mu(\overline{D^-})$, $\lambda = \min\left(1 - \frac{1}{p}, \mu\right)$, with $\Phi'(t) \in L_p(S)$ for some $1 < p < \varrho$, if the constant

$$C = \int_{-\pi}^{\pi} f(s)ds - \int_{-\pi}^{\pi} v_0(t)ds - 2\pi\,[cG - v_0(0)]$$

lies between the limits

$$F^{\pm} = \int_{-\pi}^{\pi} F^{\pm}(s)ds.$$

In the second case $G = iG_2$ we have the additional condition

$$(22) \qquad u(0) = -[cG_2 + u_0(0)]$$

and the Problem (7), (22) has the form of Problem Q_2 of [8]. From Theorem 5 of [8] one obtains

Theorem 3. If for some p with $1 < p < \varrho$ there exists

$$R \geq |cG_2 + u_0(0)| + 2\,(2\pi)^{\frac{1}{q}}\,C_R,$$

where q is the conjugate exponent to p and C_R is given by (10) with (11) - (14), then Problem E_1 with purely imaginary constant $G = iG_2$ has a unique solution $\Phi(z) \in C_\lambda(\overline{D^+}) \cap C_\mu(\overline{D^-})$, $\lambda = \min\left(\frac{1}{q}, \mu\right)$, with $\Phi'(t) \in L_p(S)$.

Corollary. In particular, there exists a unique solution $\Phi(z) \in C_\lambda(\overline{D^+}) \cap C_1(\overline{D^-})$, $\lambda = \frac{1}{q}$, to Problem E_1 with purely imaginary constant $G = iG_2$, if g = const and the assumptions (17), (18) are fulfilled.

6. Problem E_1 (continuation)

In the case of a complex constant $G = G_1 + iG_2$ Problem E_1 leads to a Riemann-Hilbert problem with boundary condition of the form

$$(23) \qquad v(t) + \Psi(s, u(t)) = h(s) \quad \text{on } S$$

and additional condition

$$(24) \qquad A\,u(0) + B\,v(0) = l \quad \text{in } z = 0,$$

where A, B, l are non-vanishing real constants.

Problems of this type have not been considered explicitely in [8] but can be dealt with in analogous way as Problems Q_1, Q_2 there. So the problem (23), (24) is equivalent to the problem with the boundary condition

$$(25) \qquad \frac{\partial u}{\partial r} + \Psi_u(s,u)\frac{\partial u}{\partial s} + \Psi_s(s,u) = h'(s) \quad \text{on} \quad S$$

and the additional condition

$$(26) \qquad A\int_{-\pi}^{\pi} u(t)ds - B\int_{-\pi}^{\pi}\Psi(s,u(t))ds = 2\pi l - B\int_{-\pi}^{\pi} h(s)ds$$

for the harmonic function u. The problem (25), (26) in its turn is equivalent to the fixed point equation

$$(27) \qquad u(s) = k + \int_0^s L(\sigma,u)d\sigma,$$

where the real parameter $k = k[u]$ is a solution of the equation

$$(28) \qquad k - \frac{B}{A}\frac{1}{2\pi}\int_{-\pi}^{\pi}\Psi\left(s,k+\int_0^s L(\sigma,u)d\sigma\right)ds =$$

$$= \frac{1}{A} - \frac{B}{A}\frac{1}{2\pi}\int_{-\pi}^{\pi} h(s)ds - \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^s L(\sigma,u)d\sigma\, ds.$$

The kernel $L(s,u)$ is given by the same expression as for Problems Q_1, Q_2 in [8].

We now assume that there hold the inequalities

$$(29) \qquad |\Psi(s,u)| \leq \Psi_0,$$

$$(30) \qquad \frac{B}{A}\Psi_u(s,u) < 1$$

for almost all $s \in [-\pi,\pi]$ and all $u \in \mathbb{R}$. Then the equation (28) has a unique solution $k = k[u]$ which depends continuously upon u in the maximum norm topology for any Hölder continuous function u. If $|u| \leq R$, the solution k satisfies the estimation

$$(31) \qquad |k| \leq K_0 + (2\pi)^{\frac{1}{q}} C_R,$$

where

$$(32) \qquad K_0 = \frac{1}{|A|} \left| 1 - \frac{B}{2\pi} \int_{-\pi}^{\pi} h(s)ds \right| + \left| \frac{B}{A} \right| \Psi_0$$

and C_R is given by (10) with

$$(33) \qquad M = \| h' \|_\varrho, \quad r_R = \sup_{\substack{s \in [-\pi,\pi] \\ |u| \leq R}} \| \Psi_s(s,u) \|_\varrho$$

and

$$(34) \qquad m(s,u) = \arctan \left[\Psi_u(s,u) \right].$$

Therefore, Schauder's fixed point theorem yields the existence of a solution $w(z)$ to problem (23), (24), if for some p with $1 < p < \varrho$ there exists $R \geq K_0 + 2(2\pi)^{\frac{1}{q}} C_R$. Moreover, the solution is unique because the difference $w(z) = w_1(z) - w_2(z) = u(z) + iv(z)$ of two solutions $w_1(z)$, $w_2(z)$ satisfies the boundary condition

$$(35) \qquad v(t) + \chi(s)u(t) = 0 \quad \text{on} \quad S$$

with the continuous function

$$(36) \qquad \chi(s) = \int_0^1 \Psi_u(s, u_1(t) + \tau[u_2(t) - u_1(t)])d\tau$$

and the condition

$$(37) \qquad A\, u(0) + B\, v(0) = 0$$

in $z = 0$. The Riemann-Hilbert problem (35) has one linearly independent solution (over the field of real numbers)

$$W(z) = e^{-i\gamma(z)}, \quad \gamma(z) = S[\arctan \chi(s)], \tag{38}$$

where S means the Schwarz operator. But the condition (37) for $W(z)$ leads to the relation

$$\frac{B}{A} \tan \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \arctan \chi(s) ds \right] = 1 \tag{39}$$

which is impossible because of the assumption (30). Applying these results to the Riemann-Hilbert problem (7) with (9a), we obtain the following theorem.

T h e o r e m 3. Let the function F fulfil the estimations

$$|F(s,\varphi)| \leqslant F_0, \tag{40}$$

$$1 + \frac{G_1}{G_2} \left[F_s(s,\varphi) + F_\varphi(s,\varphi) \frac{du_0}{ds} \right] > 0 \tag{41}$$

for almost all $s \in [-\pi,\pi]$ and all $\varphi \in \mathbb{R}$.

If for some p with $1 < p < \varrho$ there exists

$$R \geqslant \frac{1}{|G_2|} \left| c|G|^2 - G_1 v_0(0) + G_2 u_0(0) - \right.$$

$$\left. - \frac{G_1}{2\pi} \int_{-\pi}^{\pi} [f(s) - v_0(t)] ds \right| + \left| \frac{G_1}{G_2} \right| F_0 + 2\,(2\pi)^{\frac{1}{q}}\, C_R,$$

where C_R is given by (10) with (11) - (14), then the Problem E_1 with complex constant $G = G_1 + iG_2$ has a unique solution $\Phi(z) \in C_\lambda(\overline{D^+}) \cap C_\mu(\overline{D^-})$, $\lambda = \min\left(\frac{1}{q}, \mu\right)$ with $\Phi'(t) \in L_p(S)$.

C o r o l l a r y . In particular, there exists a unique solution $\Phi(z) \in C_\lambda(\overline{D^+}) \cap C_1(\overline{D^-})$, $\lambda = \frac{1}{q}$, to Problem E_1 with complex constant $G = G_1 + iG_2$, if g = const and the assumptions (17), (40) and

$$(42) \qquad 1 + \frac{G_1}{G_2} F_s(s,\varphi) > 0$$

are fulfilled.

R e m a r k . The Corollary also holds, if the boundedness condition (40) is replaced by

$$(40') \qquad |F(s,\varphi)| \leqslant F_0 + F_1|\varphi|^{\delta}, \quad 0 \leqslant \delta < 1.$$

BIBLIOGRAPHY

[1] F a n g A i n o n g : On integral operator and (non-linear mixed) boundary value problem, Scientia Sinica, Ser.A, 25, (1982) 225-236.

[2] L u C h i e n K e : On compound boundary problems, Scientia Sinica, Ser.A, 14 (1965) 1545-1555.

[3] A. M a m o u r i a n : On the mixed boundary value problems for a first order elliptic equation in the plane, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astron. Phys. 23 (1975) 1249-1254.

[4] A. M a m o u r i a n : General transmission and boundary value problems for first order elliptic equations in multiply-connected plane domains, Demonstratio Math. 12 (1979) 785-802.

[5] I.N. M u s k h e l i s h v i l i : Singular integral equations. Gröningen 1953.

[6] I.S. R o g o z h i n a : The Hilbert problem for a piecewise analytic function (Russ.), Kabardino-Balk. Gos. Univ. Ucen. Zap., Ser. Fiz. Mat. 19 (1963) 259-263.

[7] W e n G u o - c h u n : On compound boundary value problem with shift for nonlinear elliptic complex equations of first order, Complex Variables (to appear).

[8] L.v. Wolfersdorf: A class of nonlinear Riemann-Hilbert problems for holomorphic functions, Math. Nachr. (to appear).
[9] J. Wolska-Bochenek: A compound non-linear boundary value problem in the theory of pseudo-analytic functions, Demonstratio Math. 4 (1972) 105-117.

SEKTION MATHEMATIK DER BERGAKADEMIE FREIBERG DDR-9200 FREIBERG;
INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WARSAW,
00-661 WARSZAWA
Received May 16, 1983.