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NONLINEAR VOLTERRA INTEGRAL EQUATIONS IN ORLICZ SPACES

Let X be a separable Banach space. In this paper we investigate the integral equation

$$(1) \qquad x(t) = p(t) + \int_0^t f(t,s,x(s))ds,$$

where a solution x is a function from a compact interval $J = [0,a]$ into X. We give sufficient conditions for the existence of solutions of (1) belonging to the generalized Orlicz space $L_\varphi(J,X)$. Moreover, we prove that the set S of all solutions $x \in L_\varphi(J,X)$ of (1) is a compact R_δ, i.e. S is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts. Throughout this paper we assume that $D = [0,d]$, $R_+ = [0,\infty)$ and μ is the Lebesgue measure in R; the symbol $\int$ denotes the Bochner integral.

1. Orlicz spaces

A function $\varphi: R_+ \times D \to R_+$ is called a (generalized) N-function if

(i) $\varphi(0,t) = 0$ for almost all $t \in D$;

(ii) for almost every $t \in D$ the function $u \to \varphi(u,t)$ is convex and nondecreasing on R_+;

(iii) for any $u \in R_+$ the function $t \to \varphi(u,t)$ is L-measurable on D;

(iv) for almost every $t \in D$

$$\lim_{u\to 0} \frac{\varphi(u,t)}{u} = 0 \text{ and } \lim_{u\to\infty} \frac{\varphi(u,t)}{u} = \infty .$$

For any N-function φ we may define an N-function φ^* by

$$\varphi^*(u,t) = \sup_{v\geq 0} (uv - \varphi(v,t)) \quad (u \geq 0,\ t \in D);$$

it is called the complementary function to φ.

For a given subinterval J of D we denote by $L_\varphi(J,R)$ the set of all L-measurable functions $x:J \to R$ for which the number

$$\|x\|_\varphi = \inf\left\{ r>0: \int_J \varphi(|x(t)|/r,t)dt \leq 1\right\}$$

is finite. $L_\varphi(J,R)$ is called the (generalized) Orlicz space. It is well known (cf. [3], [5]) that $\langle L_\varphi(J,R), \|\cdot\|_\varphi\rangle$ is a Banach space, and the convergence in $L_\varphi(J,R)$ implies the convergence in measure. Moreover, for any functions $u \in L_\varphi(J,R)$ and $v \in L_{\varphi^*}(J,R)$, the function uv is integrable and

$$\int_J |u(t)v(t)|\, dt \leq 2\|u\|_\varphi \|v\|_{\varphi^*} \quad \text{(Hölder's inequality).}$$

Assume now that φ satisfies Condition A:

$$\int_D \varphi(u,t)dt < \infty \quad \text{for all } u>0.$$

Denote by $E_\varphi(J,R)$ the closure in $L_\varphi(J,R)$ of the set of all simple functions. Obviously, $E_\varphi(J,R)$ is a Banach subspace of $L_\varphi(J,R)$.

Lemma 1. The following statements are equivalent:
(a) $x \in E_\varphi(J,R)$;
(b) $x \in L_\varphi(J,R)$ and x has absolutely continuous norm, i.e. for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x\chi_T\|_\varphi < \varepsilon$ for every measurable subset T of J with $\mu(T) < \delta$;

(c) $\int_J \varphi(\lambda|x(t)|,t)dt<\infty$ for all $\lambda>0$.

P r o o f . We prove only (b) $\Longrightarrow$ (c), because (a) $\Longleftrightarrow$ (c) and (a) $\Longrightarrow$ (b) have been shown by A. Kozek (cf. [5], Prop.3.3 and Prop.3.4). Let x be a function from $L_\varphi(J,R)$ with absolutely continuous norm. For a given $\lambda>0$ we choose $\delta>0$ such that $\|\lambda x\chi_T\|_\varphi\leqslant 1$ for every measurable subset T of J with $\mu(T)<\delta$. Since $J=\bigcup_{i=1}^{n} T_i$, where $(T_i)_{i=1,\dots,n}$ is a family of disjoint subintervals of J such that $\mu(T_i)<\delta$, we have

$$\int_J \varphi(\lambda|x(t)|,t)dt = \sum_{i=1}^{n}\int_{T_i}\varphi(\lambda|x(t)|,t)dt \leqslant \sum_{i=1}^{n}\|\lambda x\chi_{T_i}\|_\varphi\leqslant n.$$

L e m m a 2. If a sequence $(x_n)\subset E_\varphi(J,R)$ has equi-absolutely continuous norms and converges in measure, then (x_n) converges in $E_\varphi(J,R)$.

P r o o f . We repeat the proof of Lemma 11.2 from [6]. For a given $\varepsilon>0$ put $G_{mn}=\{t\in J: |x_n(t)-x_m(t)|>\eta\}$, where $\eta=\varepsilon/3\|\chi_J\|_\varphi$. Choose $\delta>0$ in such a way that $\|x_n\chi_T\|_\varphi<\varepsilon/3$ for $n=1,2,\dots$ and any measurable subset T of J such that $\mu(T)<\delta$. Since the sequence (x_n) converges in measure, there exists a positive integer n_0 such that $\mu(G_{mn})<\delta$ for $m,n>n_0$. Hence

$$\|x_n-x_m\|_\varphi\leqslant\|(x_n-x_m)\chi_{G_{mn}}\|_\varphi+\|(x_n-x_m)\chi_{J\setminus G_{mn}}\|_\varphi\leqslant$$

$$\leqslant\|x_n\chi_{G_{mn}}\|_\varphi+\|x_m\chi_{G_{mn}}\|_\varphi+\eta\|\chi_J\|_\varphi\leqslant\varepsilon$$

for $m,n>n_0$, so that (x_n) satisfies the Cauchy condition for the convergence in $E_\varphi(J,R)$. As the space $E_\varphi(J,R)$ is complete, this implies the convergence of (x_n) in $E_\varphi(J,R)$.

2. Measures of noncompactness

For any bounded subset A of X the ball measure of noncompactness of A, denoted $\beta(A)$, is defined to be the infimum of positive numbers ε such that A can be covered by a finite number of balls of radius smaller than ε. The fundamental properties of β are given in [7] and [4]. Further, for a given subinterval J of D, denote by $L^1(J,X)$ the Lebesgue space of all (Boochner) integrable functions $x: J \to X$, provided with the norm $\|x\|_1 = \int_J \|x(t)\| dt$. We shall always assume that all functions from $L^1(J,X)$ are extended to R by putting $x(t)=0$ outside J. Let β_1 be the ball measure of noncompactness in $L^1(J,X)$. For any set V of functions belonging to $L^1(J,X)$ denote by v the function defined by $v(t) = \beta(V(t))$ for $t \in J$ (under the convention that $\beta(A) = \infty$ if A is unbounded), where $V(t) = \{x(t): x \in V\}$.

Lemma 3. (cf. [10], Th.1). Let V be a countable subset of $L^1(J,X)$ such that there exists $\psi \in L^1(J,R)$ such that $\|x(t)\| \leqslant \psi(t)$ for all $x \in V$ and $t \in J$. Then the function v is integrable on J and for any measurable subset T of J

$$\beta\left(\left\{\int_T x(t)dt:\ x \in V\right\}\right) \leqslant \int_T v(t)dt. \tag{2}$$

Moreover, if

$$\lim_{h \to 0} \sup_{x \in V} \int_J \|x(t+h) - x(t)\| dt = 0,$$

then

$$\beta_1(V) \leqslant \int_J v(t)dt. \tag{3}$$

3. An existence theorem

In this section we assume that

1° $M,N : R_+ \times D \to R_+$ are complementary N-functions and M satisfies Condition A;

2° $\varphi: R_+ \times D \to R_+$ is an N-function satisfying Condition A and such that

$$u \leqslant \lambda\varphi(u,t) + h(t) \quad \text{for all } u \geqslant 0 \text{ and a.a. } t \in D, \tag{4}$$

where λ is a positive number and $h \in L^1(D,R)$.

3^o $(t,s,x) \to f(t,s,x)$ is a function from $D^2 \times X$ into X which is continuous in x for a.e. $t,s \in D$, and strongly measurable in (t,s) for every $x \in X$.

4^o $\|f(t,s,x)\| \leqslant K(t,s)g(s,\|x\|)$ for $t,s \in D$ and $x \in X$, where

(i) $(s,u) \to g(s,u)$ is a function from $D \times R_+$ into R_+, measurable in s and continuous in u, and there exist $\alpha, \gamma > 0$ and $b \in L^1(D,R)$, $b \geqslant 0$, such that $N(\alpha g(s,u),s) \leqslant \gamma\varphi(u,s) + b(s)$ for all $u \geqslant 0$ and a.a. $s \in D$.

(ii) $(t,s) \to K(t,s)$ is a function from D^2 into R_+ such that $K(t,\cdot) \in E_M(D,R)$ for a.e. $t \in D$ and the function $t \to \|K(t,\cdot)\|_M$ belongs to $E_\varphi(D,R)$.

For any subinterval J of D denote by $L_\varphi(J,X)$ the set of all strongly measurable functions $x: J \to X$ such that $\|x\| \in L_\varphi(J,R)$. Analogously we define $E_\varphi(J,X)$. Then $L_\varphi(J,X)$ is a Banach space with the norm $\|x\|_\varphi = \|\|x\|\|_\varphi$. Owing to (4) it is clear that $L_\varphi(J,X) \subset L^1(J,X)$.

We introduce an operator F defined by

$$F(x)(t) = \int_0^t f(t,s,x(s))ds \quad (t \in D,\ x \in E_\varphi(D,X)).$$

From 4^o it follows that for any $x \in E_\varphi(D,X)$ and $t \in D$

$$\|g(\cdot,\|x\|)\chi_{[0,t]}\|_N = \frac{1}{\alpha}\|\alpha g(\cdot,\|x\|)\chi_{[0,t]}\|_N \leqslant$$

$$\leqslant \frac{1}{\alpha}\left(1 + \int_0^t N(\alpha g(s,\|x(s)\|),s)ds\right) \leqslant \frac{1}{\alpha}\left(1 + \int_0^t b(s)ds + \gamma\int_0^t \varphi(\|x(s)\|,s)ds\right)$$

and, by the Hölder inequality,

$$\|F(x)(t)\| \leqslant \int_0^t K(t,s)g(s,\|x(s)\|)ds \leqslant k(t)\|g(\cdot,\|x\|)\chi_{[0,t]}\|_N,$$

where $k(t) = 2\|K(t,\cdot)\chi_{[0,t]}\|_M$. Hence

(5) $$\|F(x)(t)\| \leqslant \frac{1}{\alpha} k(t)\left(1 + \int_0^t b(s)ds + \gamma \int_0^t \varphi(\|x(s)\|,s)ds\right)$$

and

(6) $$\|F(x)\chi_T\|_\varphi \leqslant \frac{1}{\alpha} \|k\chi_T\|_\varphi\left(1 + \int_0^d b(s)ds + \gamma \int_0^d \varphi(\|x(s)\|,s)ds\right)$$

for $t \in D$, $x \in E_\varphi(D,X)$ and any measurable subset T of D. Let us remark that, in view of 4^0(ii) and (4), $k \in E_\varphi(D,R)$ and $k \in L^1(D,R)$. Similarly it can be shown that for any $t \in D$ such that $K(t,\cdot) \in E_M(D,R)$

(7) $$\int_P \|f(t,s,x(s))\| ds \leqslant$$
$$\leqslant \frac{2}{\alpha} \|K(t,\cdot)\chi_P\|_M \left(1 + \int_0^t b(s)ds + \gamma \int_0^t \varphi(\|x(s)\|,s)ds\right)$$

for any measurable subset P of $[0,t]$ and $x \in E_\varphi(D,X)$.

By Lemma 1 from (6) we conclude that F maps $E_\varphi(D,X)$ into itself. We shall show that F is continuous. Let x_n, $x_0 \in E_\varphi(D,X)$ and $\lim_{n\to\infty} \|x_n - x_0\|_\varphi = 0$. Suppose that $\|F(x_n) - F(x_0)\|_\varphi$ does not converge to 0 as $n \to \infty$. Thus there are $\varepsilon > 0$ and a subsequence (x_{n_j}) such that

(8) $$\|F(x_{n_j}) - F(x_0)\|_\varphi > \varepsilon \quad \text{for} \quad j=1,2,\ldots$$

and $\lim_{j\to\infty} x_{n_j}(t) = x_0(t)$ for a.e. $t \in D$.

From Lemma 1 and from the inequality

$$\int_0^d \varphi(\|x_n(s)\|,s)ds \leqslant \frac{1}{2} \int_0^d \varphi(2\|x_n(s)-x_0(s)\|,s)ds +$$
$$+ \frac{1}{2} \int_0^d \varphi(2\|x_0(s)\|,s)ds$$

it follows the boundedness of the sequence $\left(\int_0^d \varphi(\|x_n(s)\|,s)ds\right)$. Consequently, by (7), for a.e. $t \in D$ the sequence $(\|f(t,s,x_n(s))\|)$ is equi-integrable on $[0,t]$. As for a.e. $t \in D$

$$\lim_{j\to\infty} f(t,s,x_{n_j}(s)) = f(t,s,x_0(s)) \text{ for a.e. } s \in [0,t],$$

the Vitali convergence theorem proves that

$$\lim_{j\to\infty} F(x_{n_j})(t) = F(x_0)(t) \text{ for a.e. } t \in D.$$

Moreover, in view of (6), the sequence $(F(x_{n_j}))$ has equi-absolutely continuous norms in $L_\varphi(D,X)$. Hence, by Lemma 2,

$$\lim_{j\to\infty} \|F(x_{n_j}) - F(x_0)\|_\varphi = 0$$

which contradicts (8).

Now we are going to establish our existence theorem for (1). Let $(t,s,u) \longrightarrow h(t,s,u)$ be a nonnegative function defined for $0 \leqslant s \leqslant t \leqslant d$, $u \geqslant 0$, satisfying the following conditions:

(i) for any nonnegative $u \in E_\varphi(D,R)$ there exists the integral $\int_0^t h(t,s,u(s))ds$ for almost every $t \in D$;

(ii) for any a, $0 < a \leqslant d$, $u = 0$ a.e. is the only nonnegative function on $[0,a]$ which belongs to $E_\varphi([0,a],R)$ and satisfies

$$u(t) \leqslant \int_0^t h(t,s,u(s))ds \quad \text{almost everywhere on } [0,a].$$

Moreover, let B_φ^r denote the closed ball in $E_\varphi(D,X)$ with center 0 and radius r.

T h e o r e m 1. Suppose that

$$(9) \quad \lim_{\tau\to 0} \sup_{x \in B_\varphi^r} \int_0^d \|F(x)(t+\tau) - F(x)(t)\|dt = 0 \quad \text{for any } r > 0$$

and

(10) $$\beta(f(t,s,Z)) \leqslant h(t,s,\beta(Z))$$

for almost every $t,s \in D$ and for each bounded subset Z of X.

Then for any $p \in E_\varphi(D,X)$ there exist an interval $J = [0,a]$ and a function $x \in L_\varphi(J,X)$ which satisfies (1) almost everywhere on J.

P r o o f . Fix a function $p \in E_\varphi(D,X)$. We choose a positive number $a < \min(d,\omega_+)$, where $[0,\omega_+)$ is the maximal interval of existence of the maximal continuous solution z of the integral equation

(11) $$z(t) = \frac{1}{2}\int_0^t \varphi\left(2\|p(s)\| + \frac{2}{\alpha}\, k(s)(1 + \|b\|_1 + \gamma z(s)), s\right)ds.$$

Let $J = [0,a]$. For simplicity we put $L^1 = L^1(J,X)$, $L_\varphi = L_\varphi(J,X)$ and $E_\varphi = E_\varphi(J,X)$.

For any positive integer n we define a function $u_n : J \longrightarrow X$ by

$$u_n(t) = \begin{cases} p(t) & \text{for } 0 \leqslant t \leqslant a_n \\ p(t) + \int_0^{t-a_n} f(t,s,u_n(s))ds & \text{for } a_n \leqslant t \leqslant a, \end{cases}$$

where $a_n = a/n$. By repeating the argument from the proof of (5), it can be shown that

$$\|u_n(t)\| \leqslant \|p(t)\| \quad \text{for} \quad 0 \leqslant t \leqslant a_n$$

and

$$\|u_n(t)\| \leqslant \|p(t)\| + \frac{2}{\alpha}\|K(t,\cdot)\chi_{[0,t-a_n]}\|_M \times$$

$$\times \left(1 + \int_0^{t-a_n} b(s)ds + \gamma \int_0^{t-a_n} \varphi(\|u_n(s)\|, s)\, ds\right)$$

for $a_n \leqslant t \leqslant a$.

As $2\|K(t,\cdot)\chi_{[0,t-a_n]}\|_M \leq k(t)$ for $a_n \leq t \leq a$, from this we deduce that $u_n \in E_\varphi$ and

$$\|u_n(t)\| \leq \|p(t)\| + \frac{1}{\alpha} k(t)\left(1 + \|b\|_1 + \gamma \int_0^t \varphi(\|u_n(s)\|,s)ds\right) \tag{12}$$

for $t \in J$.

Consequently

$$\varphi(\|u_n(t)\|,t) \leq \frac{1}{2}\varphi(2\|p(t)\| +$$

$$+ \frac{2}{\alpha} k(t)\left(1 + \|b\|_1 + \gamma \int_0^t \varphi(\|u_n(s)\|,s)ds\right),t\Big).$$

As $k \in E_\varphi(D,R)$ and $p \in E_\varphi(D,X)$, putting

$$z_n(t) = \int_0^t \varphi(\|u_n(s)\|,s)ds$$

and integrating the last inequality between 0 and t, we get

$$z_n(t) \leq \frac{1}{2}\int_0^t \varphi(2\|p(s)\| + \frac{2}{\alpha} k(s)(1 + \|b\|_1 + \gamma z_n(s)),s)ds \text{ for } t \in J.$$

Applying now Th. 2 of [1], we infer that $z_n(t) \leq z(t)$ for $t \in J$, where z is the maximal continuous solution of (11). Hence

$$\int_0^a \varphi(\|u_n(s)\|,s)ds \leq r \quad \text{for} \quad n = 1,2,\ldots, \tag{13}$$

where $r = \max_{t\in J} z(t)$, so that

$$\|u_n\|_\varphi \leq r+1 \quad \text{for} \quad n = 1,2,\ldots . \tag{14}$$

From (13), (5) and (7) it follows that

$$\|u_n(t) - p(t) - F(u_n)(t)\| = \|F(u_n)(t)\| \leqslant \frac{1}{\alpha} k(t)(1+\|b\|_1 + \gamma r)$$

for $t \in [0, a_n]$, and

$$\|u_n(t) - p(t) - F(u_n)(t)\| = \|\int_{t-a_n}^{t} f(t,s,u_n(s))ds\| \leqslant$$

$$\leqslant \frac{2}{\alpha} \|K(t,\cdot)\chi_{[t-a_n,t]}\|_M (1 + \|b\|_1 + \gamma r) \quad \text{for} \quad t \in [a_n, a],$$

so that

$$\|u_n(t) - p(t) - F(u_n)(t)\| \leqslant \frac{1}{\alpha} k_n(t)(1 + \|b\|_1 + \gamma r) \quad \text{for} \quad t \in J,$$

where

$$k_n(t) = \begin{cases} k(t) & \text{if} \quad 0 \leqslant t \leqslant a_n \\ 2\|K(t,\cdot)\chi_{[t-a_n,t]}\|_M & \text{if} \quad a_n \leqslant t \leqslant a. \end{cases}$$

By 4^0 (ii) we have $\lim_{n\to\infty} k_n(t) = 0$ and $k_n(t) \leqslant k(t)$ for a.e. $t \in J$. As $k \in E_\varphi(D,R)$ and $k \in L^1(D,R)$, this implies that

(15) $$\lim_{n\to\infty} (u_n(t) - p(t) - F(u_n)(t)) = 0 \quad \text{for a.e. } t \in J,$$

(16) $$\lim_{n\to\infty} \|u_n - p - F(u_n)\|_1 = 0$$

and

(17) $$\lim_{n\to\infty} \|u_n - p - F(u_n)\|_\varphi = 0.$$

Let $V = \{u_n : n = 1,2,\dots\}$ and $W = F(V)$. In view of (16) and (15) we have

(18) $\beta_1(V) = \beta_1(W)$ and $\beta(V(t)) = \beta(W(t))$ for a.e. $t \in J$.

Moreover, by (5) and (13),

(19) $\|F(u_n)(t)\| \leqslant Ak(t)$ for $t \in J$ and $n = 1,2,\ldots,$

where $A = \frac{1}{\alpha}(1 + \|b\|_1 + \gamma r)$.

On the other hand, from (9) and (14) it follows that

$$\lim_{\tau \to 0} \sup_n \int_0^a \|F(u_n)(t+\tau) - F(u_n)(t)\| dt = 0.$$

As $k \in L^1(D,R)$, by Lemma 3 from this we deduce that the function $t \longrightarrow v(t) = \beta(W(t))$ is integrable on J,

(20) $$\beta_1(W) \leqslant \int_0^a v(t)dt$$

and

(21) $$v(t) \leqslant A\, k(t) \quad \text{for a.e. } t \in J.$$

Fix $t \in J$ for which (10) holds and $K(t,\cdot) \in E_M(D,R)$. Then, by (7) and (13), we have

(22) $$\int_P \|f(t,s,u_n(s))\| ds \leqslant 2A \|K(t,\cdot)\chi_P\|_M$$

for any measurable subset P of $[0,t]$ and $n = 1,2,\ldots$. Furthermore, by the Egoroff theorem and (15), for any $\varepsilon > 0$ there exists a closed subset J_ε of J such that $\mu(J \setminus J_\varepsilon) < \varepsilon$ and

$$\lim_{n\to\infty} (u_n(s) - p(s) - F(u_n)(s)) = 0 \text{ uniformly on } J_\varepsilon.$$

Hence, in virtue of the Luzin theorem, from (19) and (22) we infer that for a given $\varepsilon > 0$ there exist a closed subset T of $[0,t]$ and a positive number ϱ such that

(23) $\|u_n(s)\| \leqslant \varrho$ for $s \in T$ and $n = 1,2,\ldots$

and

(24) $\int_P \|f(t,s,u_n(s))\| ds \leqslant \varepsilon$ for $n = 1,2,\ldots,$

where $P = [0,t] \setminus T$. Since

$$\|f(t,s,u_n(s))\| \leqslant K(t,s)g(s,\|u_n(s)\|),$$

from (23) it follows that

$$\|f(t,s,u_n(s))\| \leqslant \psi(s) \quad \text{for} \quad s \in T \text{ and } n = 1,2,\ldots,$$

where $\psi(s) = K(t,s)g(s,\varrho)$. As $K(t,\cdot) \in E_M(D,R)$ and $g(\cdot,\varrho) \in$ $\in L_N(D,R)$, the Hölder inequality proves that $\psi \in L^1(T,R)$.

Put

$$Z = \{ f(t,\cdot,u_n(\cdot)) : n = 1,2,\ldots \}$$

and

$$\int_T Z(s)ds = \left\{ \int_T f(t,s,u_n(s))ds : n = 1,2,\ldots \right\}.$$

Then, by (2), we have

$$\beta\left(\int_T Z(s)ds\right) \leqslant \int_T \beta(Z(s))ds.$$

Moreover, (24) implies that

$$\beta\left(\int_P Z(s)ds\right) \leqslant \varepsilon.$$

Since $F(V) \subset \int_T Z(s)ds + \int_P Z(s)ds$, we obtain

$$\beta(F(V)(t)) \leqslant \int_T \beta(Z(s))ds + \varepsilon.$$

On the other hand, from (10) it follows that

$$\beta(Z(s)) \leqslant h(t,s,\beta(V(s))) \quad \text{for a.e.} \quad s \in [0,t].$$

Thus, by (18),

$$v(t) \leqslant \int_T h(t,s,v(s))ds + \varepsilon \leqslant \int_0^t h(t,s,v(s))ds + \varepsilon.$$

As ε is arbitrary, this proves that

$$v(t) \leqslant \int_0^t h(t,s,v(s))ds.$$

Since this inequality holds for a.e. $t \in J$ and, by (21), $v \in E_\varphi(J,R)$, we deduce that $v(t) = 0$ for a.e. $t \in J$. Consequently, by (20) and (18), $\beta_1(V) = 0$, so that the set V is relatively compact in L^1. Thus we can find a subsequence (u_{n_j}) of (u_n) which is convergent in L^1. On the other hand, from (12) and (13) it follows that the sequence (u_n) has equi-absolutely continuous norms in L_φ. Hence the sequence (u_{n_j}) converges in E_φ to a function u. By (17) and the continuity of F, this implies that $\|u - p - F(u)\|_\varphi = 0$, so that

$$u(t) = p(t) + \int_0^t f(t,s,u(s))ds \text{ for a.e. } t \in J.$$

4. Solution funnels

T h e o r e m 2. Under the assumptions of Theorem 1, for any $p \in E_\varphi(D,X)$ there exists an interval $J = [0,a]$ such that the set S of all solutions of (1) belonging to $E_\varphi(J,X)$ is a compact R_δ.

P r o o f . Fix $p \in E_\varphi(D,X)$ and choose numbers a and r in the same way as in the proof of Th. 1. Let

$$U = \left\{ x \in E_\varphi : \int_0^a \varphi(\|x(s)\|, s)ds \leqslant r + 1 \right\}.$$

Obviously, $U \subset B_\varphi^{r+2}$.

For any positive integer n and $x \in E_\varphi$ put

$$F_n(x)(t) = \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant a_n \\ \int_0^{t-a_n} f(t,s,x(s))ds & \text{if } a_n \leqslant t \leqslant a, \end{cases}$$

where $a_n = a/n$. Similarly as for F in the proof of Th. 1, it can be shown that F_n is a continuous mapping of E_φ into itself, and

$$(25) \qquad \|F_n(x)(t)\| \leqslant \frac{1}{\alpha} k(t) \left(1 + \|b\|_1 + \gamma \int_0^t \varphi(\|x(s)\|, s)ds\right)$$

for $x \in E_\varphi$ and $t \in J$. Moreover, arguing similarly as in the proof of (17), we obtain

$$\lim_{n\to\infty} \|F(x) - F_n(x)\|_\varphi = 0 \quad \text{uniformly in } x \in U.$$

Put $G(x) = p + F(x)$ and $G_n(x) = p + F_n(x)$ $(x \in U)$. Then G and G_n are continuous mappings of U into E_φ and

$$(26) \qquad \lim_{n\to\infty} \|G(x) - G_n(x)\|_\varphi = 0 \quad \text{uniformly in } x \in U.$$

Fix n. It can be easily verified that for any $x, y \in U$

$$(27) \qquad x - G_n(x) = y - G_n(y) \Longrightarrow x = y.$$

Suppose that x_j, $x_o \in U$ and

(28) $$\lim_{j\to\infty} \| x_j - G_n(x_j) - x_o + G_n(x_o) \|_\varphi = 0.$$

Since $G_n(x_j)(t) = G_n(x_o)(t) = p(t)$ for $0 \leqslant t \leqslant a_n$, (28) implies that $\lim_{j\to\infty} \|(x_j - x_o)\chi_{[0,a_n]}\|_\varphi = 0$. Further,

$$x_j(t) - x_o(t) = (x_j(t) - G_n(x_j)(t) - x_o(t) + G_n(x_o)(t)) +$$

$$+ (F_n(x_j\chi_{[0,a_n]})(t) - F_n(x_o\chi_{[0,a_n]})(t))$$

for $a_n \leqslant t \leqslant 2a_n$ and $j = 1,2,\ldots$. By (28) and the continuity of F_n this proves that $\lim_{j\to\infty} \|(x_j - x_o)\chi_{[a_n,2a_n]}\|_\varphi = 0$. By repeating this argument we get

$$\lim_{j\to\infty} \|(x_j - x_o)\chi_{[0,ia_n]}\|_\varphi = 0$$

for $i = 1,2,\ldots,n$, so that $\lim_{j\to\infty} \|x_j - x_o\|_\varphi = 0$. From this and (27) it follows that the mapping $I - G_n : U \longrightarrow E_\varphi$ is a homeomorphism into (I - the identity mapping).

We choose a number q, $0 < q \leqslant 1/2$, such that the maximal continuous solution z_q of the integral equation

$$z(t) = q + \frac{1}{2}\int_0^t \varphi(2\|p(s)\| + \frac{2}{\alpha}k(s)(1 + \|b\|_1 + \gamma z(s)), s)ds$$

is defined on J and $z_q(t) \leqslant 1 + z(t)$ for $t \in J$, where z is the maximal solution of (11). Let $B_\varphi^q = \{ x \in E_\varphi : \|x\|_\varphi \leqslant q \}$. For a given n and $y \in B_\varphi^q$ we define a sequence of functions x_i, $i = 1,2,\ldots,n$, by

$$x_1(t) = y(t) + p(t) \qquad \text{for } 0 \leqslant t \leqslant a_n$$

$$\tilde{x}_i(t) = \begin{cases} x_i(t) & \text{for } 0 \leqslant t \leqslant ia_n \\ 0 & \text{for } ia_n < t \leqslant a \end{cases}$$

$$x_{i+1}(t) = x_i(t) \qquad \text{for } 0 \leqslant t \leqslant ia_n$$

$$x_{i+1}(t) = y(t) + p(t) + F_n(\tilde{x}_i)(t) \quad \text{for } ia_n \leqslant t \leqslant (i+1)a_n.$$

Then $x_n \in E_\varphi$ and $x_n(t) = y(t) + p(t) + F_n(x_n)(t)$ for $t \in J$, and consequently, by (25),

$$\|x_n(t)\| \leqslant \|y(t)\| + \|p(t)\| + \frac{1}{\alpha} k(t)\left(1 + \|b\|_1 + \gamma \int_0^t \varphi(\|x_n(s)\|, s)ds\right).$$

Hence

$$\varphi(\|x_n(t)\|, t) \leqslant \frac{1}{2}\varphi(2\|y(t)\|, t) + \frac{1}{2}\varphi(2\|p(t)\| +$$

$$+ \frac{2}{\alpha} k(t)\left(1 + \|b\|_1 + \gamma \int_0^t \varphi(\|x_n(s)\|, s)ds\right), t\Big)$$

for $t \in J$. Putting $w_n(t) = \int_0^t \varphi(\|x_n(s)\|, s)ds$ and integrating the above inequality between 0 and t, we get

$$w_n(t) \leqslant q + \frac{1}{2}\int_0^t \varphi(2\|p(s)\| + \frac{2}{\alpha} k(s)(1 + \|b\|_1 + \gamma w_n(s)), s)ds$$

for $t \in J$. By Th. 2 of [1] this implies that $w_n(t) \leqslant z_q(t) \leqslant r+1$ for $t \in J$, and hence $x_n \in U$. This proves that

(29) $\qquad B_\varphi^q \subset (I - G_n)(U)$ for all n.

Now we shall show that

(30) $(I - G)^{-1}(Y)$ is compact for any compact subset Y of E_φ.

Let Y be a given compact subset of E, and let (u_n) be an infinite sequence in $(I - G)^{-1}(Y)$. Since $u_n - p - F(u_n) \in Y$ for $n = 1,2,\dots$, we can find a subsequence (u_{n_j}) of (u_n) and $y \in Y$ such that

$$\lim_{j\to\infty} \| u_{n_j} - p - F(u_{n_j}) - y \|_\varphi = 0.$$

As, by (4), the convergence in L_φ implies the convergence in L^1, we have $\lim_{j\to\infty} \| u_{n_j} - p - F(u_{n_j}) - y\|_1 = 0$. By passing to a subsequence if necessary, we may assume that

$$\lim_{j\to\infty} (u_{n_j}(t) - p(t) - F(u_{n_j})(t)) = y(t) \text{ for a.e. } t \in J.$$

Putting $V = \{u_{n_j} : j = 1,2,\dots\}$ and repeating the argument from the proof of Th. 1, we conclude that the set V is relatively compact in E_φ. As U is a complete metric subspace of E_φ, this proves (30). From (26), (29) and (30) it follows that the mapping $G : U \to E_\varphi$ satisfies all assumptions of Th. 7 of [2], and therefore the set $(I - G)^{-1}(0)$ is a compact R_δ. On the other hand, if $x \in S$, then analogously as for u_n in the proof of (13), it can be shown that $\int_0^a \varphi(\|x(s)\|, s)ds \leqslant r$, i.e. $x \in U$. Thus $S = (I - G)^{-1}(0)$, which ends the proof of Th. 2.

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