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ON CERTAIN CHARACTERISTICS OF THE FAMILY OF PRIME FILTERS OF DISTRIBUTIVE LATTICE

Many papers appeared on the subject of prime filters family in a distributive lattice (see [1], [2], [3], [5]). Among others there are algebraic characteristics given in [3].

In this paper we give a set theoretical characteristics of the family of prime filters of a distributive lattice. For any non-empty set K by $\Phi(K)$ we denote the set of all families $\mathfrak{X}\subseteq 2^K$ such that following conditions are satisfied:

(C1) $$a = b \Longleftrightarrow \bigvee_{H\in\mathfrak{X}} (a\in H \Longleftrightarrow b\in H),$$

(C2) $$\exists_{c\in K} \bigvee_{H\in\mathfrak{X}} (c\in H \Longleftrightarrow a\in H \wedge b\in H),$$

(C3) $$\exists_{c\in K} \bigvee_{H\in\mathfrak{X}} (c\in H \Longleftrightarrow a\in H \vee b\in H),$$

(C4) $$\emptyset\notin\mathfrak{X},\quad K\notin\mathfrak{X},\quad \text{for every } a,b\in K.$$

In formulas (C1)-(C4) the symbols: $\wedge$, $\vee$, $\Longleftrightarrow$ stand for conjunction, alternative and equivalence respectively.

Let $\mathfrak{X}\in\Phi(K)$. We define the relation $\leqslant_{\mathfrak{X}}$ in K as follows

$$a\leqslant_{\mathfrak{X}} b \Longleftrightarrow \bigvee_{H\in\mathfrak{X}} (a\in H \Rightarrow b\in H),$$

for every $a,b\in K$.

It is easy to verify that the relation $\leqslant_{\mathfrak{X}}$ is an order on the set K and $\langle K, \leqslant_{\mathfrak{X}} \rangle$ is a distributive lattice.

Let $\underline{K} = \langle K, \leqslant \rangle$ be a lattice. Then we denote the family of all prime filters of this lattice by $\mathcal{P}(\underline{K})$.

T h e o r e m 1. If $\underline{K} = \langle K, \leqslant \rangle$ is a non-trivial distributive lattice, then $\mathcal{P}(\underline{K})$ is a maximal element in the set $\Phi(K)$ ordered by inclusion.

P r o o f . Let $\underline{K} = \langle K, \leqslant \rangle$ be a distributive lattice including at least two different elements. Therefore the family $\mathcal{P}(\underline{K})$ satisfies conditions (C1)-(C4). Hence $\mathcal{P}(\underline{K}) \in \Phi(K)$.

Let now $\mathfrak{X} \in \Phi(K)$ and

(1) $$\mathcal{P}(\underline{K}) \subseteq \mathfrak{X}.$$

We will prove that $\mathcal{P}(\underline{K}) = \mathfrak{X}$.

In view of the assumption (1) it is sufficient to prove the equation

(E) $$\leqslant_{\mathfrak{X}} = \leqslant_{\mathcal{P}(\underline{K})}.$$

The inclusion $\leqslant_{\mathfrak{X}} \subseteq \leqslant_{\mathcal{P}(\underline{K})}$ is obvious. The inverse inclusion will be proved by contradiction. Let us have for certain $a, b \in K$ the following assumptions:

(2) $$a \leqslant_{\mathcal{P}(\underline{K})} b,$$

(3) $$a \nleqslant_{\mathfrak{X}} b.$$

Let $a \cap_{\mathfrak{X}} b$ denote the infimum of elements a, b in the lattice $\langle K, \leqslant_{\mathfrak{X}} \rangle$. Then, according to (3) we have: $a \nleqslant_{\mathcal{P}(\underline{K})} a \cap_{\mathfrak{X}} b$. Thus for some $H \in \mathcal{P}(\underline{K})$ we infer

(4) $$a \in H,$$

(5) $$a \cap_{\mathfrak{X}} b \in H.$$

Hence, in view of (1) we have $b \notin H$. At the same time from (2) and (4) we obtain $b \in H$. We receive the contradiction, which ends the proof of equation (E). Thus Theorem 1 is proved.

T h e o r e m 2. If $K \neq \emptyset$ and $\mathfrak{X}$ is a maximal element of the poset $\langle \Phi(K), \subseteq \rangle$ then $\mathfrak{X}$ is the family of all prime filters of the lattice $\langle K, \leqslant_{\mathfrak{X}} \rangle$.

We omit an easy proof of Theorem 2.

REFERENCES

[1] M.E. A d a m s : The poset of prime ideals of a distributive lattice, Algebra Universalis, 5 (1975) 141-142.

[2] R. B a l b e s : On the partially ordered set of prime ideals of a distributive lattice, Canadian J. Math. 23 (1971) 866-874.

[3] R. B a l b e s : On the partially ordered set of prime ideals of a distributive lattice II, Bull. Acad. Polon. Sci., 26 (1978) 771-773.

[4] G. G r ä t z e r : Lattice theory. San Francisco, 1971.

[5] T.D. S p e e d : On the order of prime ideals, Algebra Universalis 2 (1972) 85-87.

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