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Adam Buraczewski

A GENERALIZATION OF DETERMINANT THEORY OF FREDHOLM OPERATORS IN BANACH SPACES

1. Introduction

It is shown how to obtain effectively analytic formulae for the determinant systems of linear continuous mappings $A = I + T$ in Banach spaces, where T^k is a quasi-nuclear (or nuclear) operator for some natural k. The obtained result is a generalization of the determinant theory of operators of the form $I + T$, T being quasi-nuclear, i.e. when $k = 1$.

R. Sikorski [2] has shown how to construct effectively a determinant system for any linear and continuous Fredholm operator of the form $I + T$ in a Banach space X, where T is a quasi-nuclear operator.

The purpose of this paper is to show how to construct effectively a determinant system for Fredholm operators of the form $I + T$ in X under the assumption that T^k is quasi-nuclear for some positive integer k. Thus the obtained result is a generalization of Sikorski's theory.

2. Preliminaries

In what follows Ξ and X denote two fixed Banach spaces over the same real or complex field $\mathcal{F}$. The letters ξ, η and x, y (with indices if necessary) always denote elements of Ξ and X, respectively. Every mapping into $\mathcal{F}$ is called a functional. We recall (see [4]) that Ξ and X are conjugate in the sense that there exists a continuous bilinear functional ξx on $\Xi \times X$ such that

(a) $\xi x = 0$ for every $x \in X$ implies $\xi = 0$;

(a') $\xi x = 0$ for every $\xi \in \Xi$ implies $x = 0$.

We assume that the norms in Ξ and X satisfy the conditions

$$\|\xi\| = \sup_{\|x\| \leqslant 1} |\xi x|, \quad \|x\| = \sup_{\|\xi\| \leqslant 1} |\xi x|.$$

Let $op(\Xi, X)$ be the class of all continuous bilinear functionals A defined on $\Xi \times X$, ξAx being the value of A at (ξ, x) such that each $A \in op(\Xi, X)$ can simultaneously be interpreted as an endomorphism $\eta = \xi A$ in Ξ and as endomorphism $y = Ax$ in X, defined by the relationship

$$\xi Ax = (\xi A)x = \xi(Ax).$$

Clearly, a bilinear functional $A \in op(\Xi, X)$ interpreted as the endomorphism $\eta = \xi A$ in Ξ is the adjoint of the endomorphism $y = Ax$ in X, and the elements of $op(\Xi, X)$ will be called operators. The bilinear functional K defined by the formula

$$\xi Kx = \xi x_0 \cdot \xi_0 x,$$

where ξ_0 and x_0 are fixed non-zero elements, will be called a one-dimensional operator, denoted by $x_0 \cdot \xi_0$.

3. Main theorem

Let $s \geqslant 1$ be an integer, let

(1) $$\alpha_0 = 1, \quad \alpha_1, \ldots, \alpha_{s-1}$$

be all solutions of the equation $\alpha^s = 1$, and let $T \in op(\xi, X)$. Then we can factorise the operator $I - T^s$ as follows

(2) $$I - T^s = (I - T)(I - \alpha_1 T) \ldots (I - \alpha_{s-1} T).$$

We now prove the following theorem which will be needed for our purpose.

T h e o r e m 1. Let $T \in \mathrm{op}(\Xi, X)$ be an operator such that T^k is quasi-nuclear for some integer $k \geq 1$. Then for every integer $l \geq k$ there exists an integer $s \geq l$ such that

$$(3) \qquad I - T^s = A_0(I - T) = (I - T)A_0,$$

where

$$(4) \qquad A_0 = (I - \alpha_1 T) \dots (I - \alpha_{s-1} T)$$

is invertible and the α_j $(j = 1,\dots,s-1)$ are the numbers given in (1).

P r o o f . Suppose that the contrary holds, i.e. there exists an integer $l \geq k$ such that for all $s \geq l$ the operator A_0 defined by (4) is not invertible. Let us choose an integer $s_1 \geq l$ and let

$$I - T^{s_1} = (I - T)(I - \alpha_{s_1,1}T) \dots (I - \alpha_{s_1,s_1-1}T),$$

where $\alpha_{s_1,1},\dots,\alpha_{s_1,s_1-1}$ are all roots of the equation $\alpha^{s_1} = 1$ which differ from 1. Then there must exist at least one factor $I - \alpha_{s_1,1}T$, say, which is not invertible. Consequently there exists a point $x_1 \in X$ such that

$$x_1 = \alpha_{s_1}Tx_1 \quad \text{and} \quad \|x_1\| = 1,$$

where we denote $\alpha_{s_1,1}$ by α_{s_1} for simplicity. Let us now choose an integer $s_2 > s_1$ satisfying the following condition. There exists a root $\alpha_{s_2} \neq 1$ of the equation $\alpha^{s_2} = 1$ such that $\alpha_{s_2} \neq \alpha_{s_1}$ and the operator $I - \alpha_{s_2}T$ is not invertible. Consequently there exists a point $x_2 \in X$ such that

$$x_2 = \alpha_{s_2}Tx_2 \quad \text{and} \quad \|x_2\| = 1.$$

Continuing this process we find inductively that there exist sequences

$$x_1 < x_2 < \dots < s_j < \dots,$$

$$\alpha_{s_1}, \alpha_{s_2}, \dots, \alpha_{s_j}, \dots$$

and

$$x_1, x_2, \dots, x_j, \dots$$

such that

(5) $$\alpha_{s_j}^{s_j} = 1, \quad \alpha_{s_j} \neq 1 \ (j=1,2,\dots), \quad \alpha_{s_i} \neq \alpha_{s_j} \text{ for } i \neq j,$$

and

(6) $$x_j = \alpha_{s_j} T x_j, \quad \|x_j\| = 1 \text{ for } j = 1,2,\dots .$$

Formula (6) can be re-written in the form

(7) $$\beta_{s_j} x_j = T x_j,$$

there $\beta_{s_j} = \frac{1}{\alpha_{s_j}}$ for $j = 1,2,\dots$.

We now show that the set $X_0 = \{x_1, x_2, \dots\}$ is linearly independent. To this end suppose that

(8) $$r_1 x_{j_1} + r_2 x_{j_2} + \dots + r_p x_{j_p} = 0.$$

Appyling successively formula (7) to (8) we obtain after i steps $(i = 0,1,\dots,p-1)$ the system of equations

$$r_1 \beta_{s_{j_1}}^{i} x_{j_1} + r_2 \beta_{s_{j_2}}^{i} x_{j_2} + \dots + r_p \beta_{s_{j_p}}^{i} x_{j_p} = 0$$

$$(i = 0,1,\dots,p-1).$$

Since all $\beta_{s_{j_1}},\dots,\beta_{s_{j_2}}$ are different, we conclude that

$$\begin{vmatrix} 1, & 1, & \dots, 1 \\ \beta_{s_{j_1}}, & \beta_{s_{j_2}}, & \dots, \beta_{s_{j_p}} \\ \beta^2_{s_{j_1}}, & \beta^2_{s_{j_2}}, & \dots, \beta^2_{s_{j_p}} \\ \dots & \dots & \dots \\ \beta^{p-1}_{s_{j_1}}, & \beta^{p-1}_{s_{j_2}}, & \dots, \beta^{p-1}_{s_{j_p}} \end{vmatrix} \neq 0$$

as a Vandermonde determinant. This proves that the vectors $x_{j_1},\dots,x_{j_p}$ are linearly independent and so the set X_0 is linearly independent. Consequently, the linear set $\mathrm{lin}(X_0)$ spanned by the vectors $x_1, x_2, \dots$ is an infinite dimensional vector subspace of X.

Since the product of two quasi-nuclear operators is nuclear (see [2]) and therefore compact, it follows that T^s is compact for $s \geqslant 2k$. Thus we now assume that $s \geqslant 2k$, so that T^s is compact.

Now let $U \subset \mathrm{Lin}(X_0)$ be any bounded set, and let

$$y = r_1 x_{j_1} + \dots + r_p x_{j_p} \in U.$$

Then, by virtue of (6), we have

$$\text{(9)} \qquad r_1 x_{j_1} + \dots + r_p x_{j_p} = T^s\left(r_1 \alpha^s_{s_j} x_{j_1} + \dots + r_p \alpha^s_{n_{j_p}}\right).$$

It follows from (6) that

$$\left\| \alpha^s_{n_{j_i}} \; T^s_{x_{j_i}} \right\| = 1 \quad (i = 1,\dots,p)$$

and, in view of (9), the subset U of $\mathrm{Lin}(X_0)$ is the image of the compact operator T^s. This means that every bounded subset U

of $\mathrm{Lin}(X_0)$ is relatively compact. Consequently $\mathrm{Lin}(X_0)$ must be finite dimensional, which is a contradiction. This completes the proof of Theorem 1.

Applying the same Theorem 1 to the operator $-T$ instead of T, we obtain

$$I + (-1)^{s+1}T^s = A_0(I + T) = (I + T)A_0, \tag{10}$$

where $s \geqslant 1$, and

$$A_0 = (I + \alpha_1 T) \dots (I + \alpha_{s-1}T) \tag{11}$$

is invertible.

4. Structure of analytic formulae for the determinant system

Let $T \in \mathrm{op}(\Xi, X)$ be such that T^k is a quasi-nuclear operator for some positive integer k. It follows from Theorem 1, in view of formula (4), that there exists a positive integer $s \geqslant k$ such that

$$I + T = A_0^{-1}(I + (-1)^{s+1}T^s) = (I + (-1)^{s+1}T^s)A_0^{-1}, \tag{12}$$

where A_0 is given by (11).

Let F be a quasi-nucleus, (see [4]), which determines the quasi-nuclear operator T^k, i.e.

$$\xi T x^k = F(x, \xi), \tag{13}$$

which can also be written in the form

$$\xi T x^k = F_{\eta y}(\eta x \cdot \xi y), \tag{14}$$

η and y being dummy variables. For a fixed operator $C \in \mathrm{op}(\Xi, X)$ let us define nuclei CF and FC on $\mathrm{op}(\Xi, X)$ by the formulae

$$(CF)(A) = F(AC), \quad (FC)(A) = F(CA). \tag{15}$$

Then in view of (13) we obtain

(16) $$(CF)(x.\xi) = F(x.\xi C) = \xi CTx,$$

(17) $$(FC)(x.\xi) = F(Cx.\xi) = \xi TCx.$$

It follows from the above formulae that the nucle

(18) $$((-1)^{s+1}T^{s-k})F \text{ and } F((-1)^{s+1}T^{s-k})$$

determine the same quasi-nuclear operator $(-1)^{s+1}T^{s}$.

We shall use the following properties of determinant systems (see [3]).

If $D_n\begin{pmatrix}\xi_1,\dots,\xi_n\\ x_1,\dots,x_n\end{pmatrix}$ $(n = 0,1,2,\dots)$ is a determinant system for $A\in op(\Xi, X)$ and $A_o\in op(\Xi, X)$ is invertible, then

(19) $$D_n\begin{pmatrix}\xi_1,\dots,\xi_n\\ A_o^{-1}x_1,\dots,A_o^{-1}x_n\end{pmatrix} \quad (n = 0,1,2,\dots),$$

is a determinant system for A_oA, and similarly,

(20) $$D_n\begin{pmatrix}\xi_1A_o^{-1},\dots,\xi_nA_o^{-1}\\ x_1,\dots,x_n\end{pmatrix} \quad (n = 0,1,2,\dots)$$

is a determinant system for AA_o.

To prove the structure of the determinant system for $I + T$ and bearing in mind that F is a given quasi-nucleus of the quasi-nuclear operator T^k we first obtain, on the basis of Theorem 7 in [4], the determinant system for $I + T^k$ as follows:

Let F be a quasi-nucleus of T^k, and let (Θ_n) be a determinant system for I, i.e.

(21) $$\Theta_o = 1, \quad \Theta_n\begin{pmatrix}\xi_1,\dots,\xi_n\\ x_1,\dots,x_n\end{pmatrix} = \begin{vmatrix}\xi_1x_1 & \dots & \xi_1x_n\\ \dots & \dots & \dots\\ \xi_nx_1 & \dots & \xi_nx_n\end{vmatrix} \quad \text{for } n=1,2,\dots,$$

and let, for m,n = 0,1,2,...,

(22) $$D_{n,m}(F)\begin{pmatrix}\xi_1,\dots,\xi_n\\ x_1,\dots,x_n\end{pmatrix} = \frac{1}{m!}\, F_{\eta_1 y_1}\dots F_{\eta_m y_m}\Theta_{n+m}\begin{pmatrix}\eta_1,\dots,\eta_m,\xi_1,\dots,\xi_n\\ y_1,\dots,y_m,x_1,\dots,x_n\end{pmatrix}.$$

Then $(D_n(F))$, defined by the formula

(23) $$D_n(F)\begin{pmatrix}\xi_1,\dots,\xi_n\\ x_1,\dots,x_n\end{pmatrix} = \sum_{m=0}^{\infty} D_{n,m}(F)\begin{pmatrix}\xi_1,\dots,\xi_n\\ y_1,\dots,y_m\end{pmatrix} \text{ for } n=0,1,2,\dots,$$

is a determinant system for $I + T^k$.

Suppose now that formulae (22), (23) hold and consider the nuclei in (18), which determine the same quasi-nuclear operator T^s, F being a quasi-nucleus for T^k. In view of (18) and ([1] page 297), we obtain for m = 0,1,2,...

(24) $$D_{n,m}((-1)^{s+1}T^{s-k}F)\begin{pmatrix}\xi_1,\dots,\xi_n\\ x_1,\dots,x_n\end{pmatrix} =$$

$$= \frac{(-1)^{m(s+1)}}{m!}\,(T^{s-k}F)_{\eta_1 y_1}\dots(T^{s-k}F)_{\eta_m y_m}\Theta_{n+m}\begin{pmatrix}\eta_1,\dots,\eta_m,\xi_1,\dots,\xi_n\\ y_1,\dots,y_m,x_1,\dots,x_n\end{pmatrix} =$$

$$= \frac{(-1)^{m(s+1)}}{m!}\,F_{\eta_1 y_1}\dots F_{\eta_m y_m}\Theta_{n+m}\begin{pmatrix}\eta_1,\dots,\eta_m,\xi_1,\dots,\xi_n\\ T^{s-k}_{y_1},\dots,T^{s-k}_{y_m},x_1,\dots,x_n\end{pmatrix},$$

and similarly

(25) $$D_{n,m}(F(-1)^{s+1}T^{s-k})\begin{pmatrix}\xi_1,\dots,\xi_n\\ x_1,\dots,x_n\end{pmatrix} =$$

$$= \frac{(-1)^{m(s+1)}}{m!}\,(FT^{s-k})_{\eta_1 y_1}\dots FT^{s-k})_{\eta_m y_m}\Theta_{n+m}\begin{pmatrix}\eta_1,\dots,\eta_m,\xi_1,\dots,\xi_n\\ y_1,\dots,y_m,x_1,\dots,x_n\end{pmatrix} =$$

$$= \frac{(-1)^{m(s+1)}}{m!}\,F_{\eta_1 y_1}\dots F_{\eta_n y_m}\Theta_{n+m}\begin{pmatrix}\eta_1 T^{s-k},\dots,\eta_m T^{s-k},\xi_1,\dots,\xi_n\\ y_1,\dots,y_m, \quad x_1,\dots,x_n\end{pmatrix}.$$

Consequently, bearing in mind formula (12), and in view of (19) and (20), we can obtain four formulae for the determinant system of the operator I + T. For example, by virtue of (24) and (19) we obtain a determinant system (D_m) for I + T defined by

$$D_n\begin{pmatrix}\xi_1,\dots,\xi_n\\ x_1,\dots,x_n\end{pmatrix} = \sum_{m=0}^{\infty} D_{n,m}\begin{pmatrix}\xi_1,\dots,\xi_n\\ x_1,\dots,x_n\end{pmatrix},$$

where

$$D_{n,m}\begin{pmatrix}\xi_1,\dots,\xi_n\\ x_1,\dots,x_n\end{pmatrix} =$$

$$= \frac{(-1)^{m(s+1)}}{m!} F_{\eta_1 y_1}\dots F_{\eta_m y_m}\Theta_{n+m}\begin{pmatrix}\eta_1,\dots,\eta_m, & \xi_1,\dots,\xi_n\\ T^{s-k}_{y_1},\dots,T^{s-k}_{y_m}, & A_0x_1,\dots,Ax_n\end{pmatrix}.$$

$$(n,m = 0,1,2,\dots).$$

Similarly, by virtue of (24) and (20), we obtain a determinant system (D_n) for I + T defined by

$$D_n\begin{pmatrix}\xi_1,\dots,\xi_n\\ x_1,\dots,x_n\end{pmatrix} = \sum_{m=0}^{\infty} D_{n,m}\begin{pmatrix}\xi_1,\dots,\xi_n\\ x_1,\dots,x_n\end{pmatrix} \qquad (n = 0,1,2,\dots),$$

where

$$D_{n,m}\begin{pmatrix}\xi_1,\dots,\xi_n\\ x_1,\dots,x_n\end{pmatrix} =$$

$$= \frac{(-1)^{m(s+1)}}{m!} F_{\eta_1 y_1}\dots F_{\eta_m y_m}\Theta_{n+m}\begin{pmatrix}\eta_1T^{s-k},\dots,\eta_mT^{s-k},\xi_1A_0\\ y_1,\dots,y_m,x_1,\dots,x_n\end{pmatrix}$$

$$(n,m = 0,1,\dots).$$

Now if k = 1, i.e. when T is quasi-nuclear, then s = k = 1, A_0 = I, so that we obtain Sikorski's formula for the determinant system of I + T given in [4] (see Theorem 7). This solves the problem.

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DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF PAPUA NEW GUINEA
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