DEMONSTRATIO MATHEMATICA
Vol. XVII No 2 1984

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SOME RESULT ON THE EXISTENCE OF SOLUTIONS OF OPERATOR EQUATIONS IN CONTINUOUS FUNCTION SPACES

1. Introduction

The existence of solutions of initial and boundary value problems for ordinary differential and differential-functional equations, the existence problems for integro-functional and functional equations lead us to a fixed point problem for an appropriate operator in some function space, usually in $C(I, R^n)$ space. Here $C(I, R^n)$ is the Banach space of all continuous functions defined on the interval $I = [a,b]$, $0 < a < b$, with values in a real-dimensional space R^n, $\|\cdot\|$ will denote a norm in R^n.

In the literature we can find many particular results concerned with the problem, including those obtained by the clasical Banach contraction mapping principle, the comparison method, the Schauder fixed point theorem and by the more recent results employing measure of noncompactness or degree of mapping notions.

In the present paper we intend to describe a wide class of operators in $C(I, R^n)$ for which the fixed point result can be established by the direct use of the Schauder theorem. The main idea of the paper is: for a given operator defined in $C(I, R^n)$ one has to find such a common modulus of continuity of all functions of some bounded, closed convex subset of $C(I, R^n)$ which remains the same for all images by this operator of the elements of the subset mentioned.

We will find the mentioned modulus of continuity as a solution of some functional inequality related to the operator considered. It seems that the result of the present paper could not be establised by the direct use of the measure of noncompactness method (see [1]).

2. The main result

Let a continuous operator $F: C(I, R^n) \longrightarrow C(I, R^n)$ be given.

Note that in applications sometimes we have to deal with an operator $F_1 : C(I_1, R^n) \longrightarrow C(I, R^n)$ where the interval $I \subset I_1 = [a_1, b_1]$. However this case can be easily reduced to the case where $I_1 = I$ if some functions $\varphi \in C(I_a, R^n)$, $\psi \in C(I_b, R^n)$, $I_a = [a_1, a]$, $I_b = [b, b_1]$, are given. To do this it is enough to define the operator $F : C(I, R^n) \longrightarrow C(I, R^n)$ by the relation $F = F_1 \circ E$, where the operator $E : C(I, R^n) \longrightarrow C(I_1, R^n)$ has the form $Ez = w$, where

$$w(t) = \begin{cases} \varphi(t), & t \in I_a \\ z(t) - l(z(a), z(b))(t) + l(\varphi(a), \psi(b))(t), & t \in I \\ \psi(t), & t \in I_b \end{cases}$$

and

$$l(u,v)(t) = v + \frac{u - v}{a - b}(t-b), \qquad u, v \in R^n.$$

In view of this we can confine our considerations to the functional equation

$$(1) \qquad x(t) = (Fx)(t), \quad t \in I.$$

We introduce the following assumption.

Assumption H_1. Assume that

1^o there exists a nondecreasing operator $\Gamma : C(I, R_+) \longrightarrow C(I, R_+)$, $R_+ = [0, +\infty)$, and an element $x_0 \in C(I, R^n)$ such that for any $x \in C(I, R^n)$,

$$(2) \qquad \|(Fx)(t) - x_0(t)\| \leqslant \Gamma(\|x - x_0\|)(t), \quad t \in I,$$

2^o there exists $g \in C(I, R_+)$ such that

$$(\Gamma g)(t) \leqslant g(t), \quad t \in I. \tag{3}$$

Put

$$K(x_0,g) = \{x \mid x \in C(I, R^n),\ \|x(t)-x_0(t)\| \leqslant g(t),\ t \in I\}.$$

Clearly $K(x_0,g)$ is a bounded closed and convex subset of $C(I, R^n)$. Moreover according to (2) and (3) we get $F(K(x_0,g)) \subset \subset K(x_0,g)$. Unfortunately we cannot apply the Schauder theorem because in the general case the operator F is not compact. But under some additional assumptions we will find a compact subset of $K(x_0,g)$ which is invariant with respect to the operator F.

Let A be a fixed non-empty set, let $F(A, R_+)$ denote the set of all mappings from A to R_+ and $0 \in V \subset F(A, R_+)$. We take the following assumption.

A s s u m p t i o n H_2. Assume that

1^o there exist functions: $\alpha: A \times I \longrightarrow I$, $\omega: A \times I_0 \longrightarrow I_0$, $I_0 = [0, b-a]$, such that

$$\|x(\alpha(\cdot, t)) - x(\alpha(\cdot,s))\| \in V,$$

$$|\alpha(\tau,t) - \alpha(\tau,s)| \leqslant \omega(\tau, |t-s|) \tag{4}$$

for any $t,s \in I$, $\tau \in A$, $x \in K(x_0,g)$,

2^o there exists a nondecreasing and continuous operator $\Omega: I_0 \times V \longrightarrow R_+$ such that $\Omega(0,0) = 0$ and

$$\|(Fx)(t)-(Fx)(s)\| \leqslant \Omega(|t-s|, \|x(\alpha(\cdot,t)) - x(\alpha(\cdot,s))\|), \tag{5}$$

for any $t,s \in I$ and $x \in K(x_0,g)$,

3^o there exists a nondecreasing and continuous function $\gamma: I_0 \longrightarrow R_+$ such that $\gamma(0) = 0$ and

$$\gamma(u) \geqslant \Omega(u, \gamma(\omega(\cdot,u))), \quad u \in I_0. \tag{6}$$

Note that for our further consideration it is enough if the function γ is defined only on some interval $[0,\delta]$, $\delta>0$.

Now we can formulate

T h e o r e m 1. If the assumptions H_1 and H_2 are satisfied then there exists in $K(x_0,g)$ at least one solution of the equation (1).

P r o o f . In order to prove the theorem we take the function determined by condition 3^0 of Assumption H_2 and we define the set

$$S(\gamma)=\left\{x \mid x\in K(x_0,g), \|x(t)-x(s)\|\leqslant\gamma(|t-s|),\quad t,s\in I\right\}.$$

It is obvious that the set $S(\gamma)$ is a convex and compact subset of $C(I,R^n)$. We easily prove that $S(\gamma)$ is invariant with respect to the operator F, i.e. $F(S(\gamma))\subset S(\gamma)$. Because of the relation $F(K(x_0,g))\subset K(x_0,g)$, it is enough to observe that according to the evaluations (4)-(6) for any $x\in S(\gamma)$ and $t,s\in I$ we get

$$\|(Fx)(t)-(Fx)(s)\|\leqslant\Omega(|t-s|,\|x(\alpha(\cdot,t))-x(\alpha(\cdot,s))\|)\leqslant$$
$$\leqslant\Omega(|t-s|,\ \gamma(\omega(\cdot,|t-s|)))\leqslant\gamma(|t-s|).$$

This means that $Fx\in S(\gamma)$. Now the assertion of the theorem is implied by the Schauder fixed point theorem.

3. Discussion of the assumptions

Let us discuss some special cases and possible applications of Theorem 1. First we observe that Assumption H_1 is fulfilled if the operator F is bounded, i.e. if there exists a continuous function $M: I\longrightarrow R_+$ such that $\|(Fx)(t)\|\leqslant M(t)$, $t\in I$.

Now the operator Γ does not depend on x and we can assume that

$$\Gamma(\|x-x_0\|)(t)=\|x_0(t)\|+M(t),$$

so for $g(t)$ we can get the right-hand side of the last equation. This is the simplest possible case. Otherwise the operator Γ should preserve the specific features of the operator F, i.e. it should be an integral or integro-functional if F is of this kind. Having the operator Γ fixed we need to find a solution of inequality (3).

The discussion of possible special cases of the operator Ω and the function ω appearing in Assumption H_2 seems to be more difficult and interesting.

1) The simplest case appears if the operator Ω does not depend on the second variable, i.e.

$$\Omega(u,z) = m(u), \quad u \in I_0, \quad z \in V.$$

Now we can take $\gamma(u) = m(u)$. This is the case which usually occur in discussions of the existence problem for integral equations. Clearly this includes the existence problems for initial and boundary value problems for ordinary differential and differential-delay equations. Evidently the discussion of properties of the set A and the function ω has no meaning in this case.

2) Another rather simple case occurs if the set A is a one-point set, say $A = \{1\}$. Now we can assume that $F(A, R_+) = R_+$, $V = R_+$ and

$$\Omega(u,z) = m(u,z), \quad u \in I_0, \quad z \in R_+.$$

The functions α and ω appearing in the condition 1^0 of Assumption H_2 we can consider as functions of one variable. The evaluations (4) and (5) now take the form

$$|\alpha(t) - \alpha(s)| \leqslant \omega(|t - s|)$$

$$\|(Fx)(t)-(Fx)(s)\| \leqslant m(|t-s|, \|x(\alpha(t))-x(\alpha(s))\|), \quad t,s \in I.$$

For γ we take a solution of the functional inequality

$$\gamma(u) \geqslant m(u,\gamma(\omega(u))), \quad u \in I_0.$$

Now the discussion of the existence of the continuous and nondecreasing solution of this inequality is required. We note that in view of the Banach contraction principle there exists such a solution of the last inequality (equation) if

$$m(u,z) = q(u) + k\,z, \quad k \in [0,1)$$

and the functions q, ω are continuous and nondecreasing. However, if $\omega(u) \leqslant u$ we can take

$$\gamma(u) = \frac{q(u)}{1-k} .$$

Another case, when the existence problem for the mentioned inequality for γ is easy to solve, we obtain if

$$m(u,z) = Q\,u^p + k\,z, \quad \omega(u) \leqslant c \cdot u,$$

for some Q, p, k, $c \in R_+$, $0 < p \leqslant 1$, $0 \leqslant c \leqslant 1$ and $k\,c^p < 1$. Now we get $\gamma(u) = Q\,u^p/1-kc^p$ as a solution to the mentioned inequality.

The case described we meet usually in discussions of the existence problem for the integro-functional equations of the form

$$x(t) = f\left(t, \int_{p(t)}^{r(t)} g(t,s,x(s))ds,\ x(\alpha(t))\right).$$

3) The more general case we have if the set $A = \{1,2,\dots,r\}$. Now we assume that $F(A, R_+) = R_+^r = V$,

$$\alpha(\cdot,t) = (\alpha_1(t),\dots,\alpha_r(t)), \quad t \in I,$$

$$\omega(\cdot,u) = (\omega_1(u),\dots,\omega_r(u)), \quad u \in I_0,$$

$$\Omega(u,z) = m(u,z_1,\dots,z_r), \quad z = (z_1,\dots,z_r) \in R_+^r.$$

In this case the evaluations (4) and (5) take the form

$$|\alpha_i(t)-\alpha_i(s)| \leqslant \omega_i(|t-s|), \quad i=1,2,\ldots,r,$$

$$\|(Fx)(t)-(Fx)(s)\| \leqslant$$

$$\leqslant m(|t-s|, \|x(\alpha_1(t))-x(\alpha_1(s))\|, \ldots, \|x(\alpha_r(t))-x(\alpha_r(s))\|), \quad t,s \in I.$$

Now γ should be a solution of the inequality

$$\gamma(u) \geqslant m(u, \gamma(\omega_1(u)), \ldots, \gamma(\omega_r(u))), \quad u \in I_0.$$

It is not difficult to discuss the conditions which guarantee the existence of the solution γ of this inequality (equation) having the properties required by the condition 3^o of Assumption H_2 if

$$m(u, z_1, \ldots, z_r) = q(u) + \sum_{i=1}^{r} k_i(u) z_i, \quad u \in I_0.$$

Clearly, the problem can be easily solved if k_i are constant and their sum is less than one. If $\omega_i(u) \leqslant u$, $i=1,2,\ldots,r$, $k_i(u) = k_i^*$ and

$$k = \sum_{i=1}^{r} k_i^* < 1$$

we can take $\gamma(u) = q(u)/1-k$. Otherwise, if $q(u) = Q\, u^p$, $\omega_i(u) \leqslant c_i \cdot u$, $k_i(u) = k_i^*$, $i = 1,2,\ldots,r$, $u \in I_0$, $0 < p \leqslant 1$, $0 \leqslant c_i \leqslant 1$, $Q \geqslant 0$ and

$$k^* = \sum_{i=1}^{r} k_i^* c_i^{\,p} < 1,$$

then we get $\gamma(u) = Q\, u^p/1-k^*$, $u \in I_0$. If some c_i are greater than one then the same formula for γ is valid for $u \in [0,\delta]$, where $\delta > 0$ is such that $c_i \cdot \delta \leqslant b-a$, $i = 1,2,\ldots,r$.

This considerations are useful for discussion of the existence problems for suitable generalization of the integro-functional equation mentioned in 2).

There is a variety of other possible cases involving infinite sets A (intervals for instance). They lead us to very complicated functional inequalities for the function γ, for instance of the form

$$\gamma(u) \geqslant m\left(u, \int_0^d \gamma(\omega(\tau,u))d_\tau K(\tau,u)\right),$$

where the integral is the Stieltjes one.

4. <u>A generalization of the main result</u>

Sometimes the operator F has more complicated form and Assumption H_2 does not hold. It takes place for instance if we have to consider the existence problem for differential delay equations of neutral type with transformed argument depending on the derivative of unknown function. In such situation in order to establish some existence result for equation (1) we modify Assumption H_2 as follows

A s s u m p t i o n H_3. Assume that

1^0 there exist functions $\alpha: A\times I\times K(x_0,g) \longrightarrow I$, $\alpha_1: A\times I \longrightarrow I$, $\omega: A\times I_0\times V \longrightarrow I_0$, $\omega_1: A\times I_0 \longrightarrow I_0$, such that for any $t,s \in I$, $x \in K(x_0,g)$,

$$\|x(\alpha(\cdot,t,x)) - x(\alpha(\cdot,s,x))\| \in V,$$

$$\|x(\alpha_1(\cdot,t)) - x(\alpha_1(\cdot,s))\| \in V$$

and

$$|\alpha(\tau,t,x) - \alpha(\tau,s,x)| \leqslant$$

$$\leqslant \omega(\tau,|t-s|,\|x(\alpha_1(\cdot,t))-x(\alpha_1(\cdot,s))\|),$$

$$|\alpha_1(\tau,t)-\alpha_1(\tau,s)| \leqslant \omega_1(\tau,|t-s|), \ \tau \in A,$$

2^0 there exists a nondecreasing and continuous operator $\Omega : I_0 \times V \longrightarrow R_+$ such that $\Omega(0,0) = 0$ and for any $t,s \in I$, $x \in K(x_0,g)$

$$\|(Fx)(t)-(Fx)(s)\| \leq \Omega(|t-s|,\|x(\alpha(\cdot,t,x))-x(\alpha(\cdot,s,x))\|),$$

3^0 there exists a nondecreasing and continuous function $\gamma : I_0 \longrightarrow R_+$ such that $\gamma(0) = 0$ and

$$\gamma(u) \geq \Omega\left(u,\gamma\left(\omega(\cdot,u,\gamma(\omega_1(\cdot,u)))\right)\right), \quad u \in I_0.$$

Now we can state

T h e o r e m 2. If the assumptions H_1 and H_3 are fulfilled, then there exists in $K(x_0,g)$ at least one solution of the equation (1).

The proof of this theorem runs by the same way as for the previous one.

It is clear that the last inequality for the function γ is more complicated than the inequality (6) but there exist situations when it is not difficult to find the solution of this inequality. It is certainly the case when $A = \{1\}$ and

$$\Omega(u,z) = Q\,u + k\,z,\ \omega(u,z) = r \cdot u + l \cdot z, \quad \omega_1(u) = c \cdot u,$$

Q, k, l, c, $r \in R_+$. Now $\gamma(u) = G \cdot u$, where $G > 0$ is a solution to the inequality

$$G \geq Q + k\,r\,G + k\,l\,c\,G^2.$$

Such a solution obviously exists if $k\,r < 1$ and $(1-kr)^2 - 4Qklc > 0$.

There are possible further generalizations of Assumption H_2. For instance, if in Assumption H_3 we assume that α_1 depends on x, then we need to assume that there exist some additional functions α_2 and ω_2 which play with respect to the function α_1 the same role as functions α_1 and ω_1 play in

Assumption H_2 with respect to the function α. This process can be iterated many times.

5. Remarks

The method described in this paper can be easily extended to the equations of type (1) with the unknown function of multidimensional variable. Also an extension of the method for operators F in $L_p(a,b)$ is possible.

The present paper essentially developes an idea which in particular cases can be found in [2]-[7].

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Received December 23, 1982.