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FINSLER DIFFERENTIAL SPACE

In this paper we suggest a study of the theory of Finslers spaces on the differential finite dimensional spaces. We define a function $F: T_p(M,C) \to R$ fulfiling the conditions (1) and (2). Further we define functions ${}_eF$, F_e and $F_{e|ij}$ on the R^m and functions $\hat{F}$, $\hat{\hat{F}}$ and $\widetilde{F}$ on the $T_p^0(M,C) = T_p(M,C)-\{0\}$ (definitions (3), (20), (22), (7), (11) and (29) respectively). The function $\widetilde{F}$ additionaly satisfies condition (33). The generalization of the Finslers space is defined by this function as the pair $((M,C),F)$. Further we define the generalized notion of indicatrix and the notion of metric.

Let (M,C) be a Hausdorff differential space of finite dimension. (The general theory of differential spaces can be found in [6]). We consider the tangent bundle to (M,C) ([3]): $T(M,C) = \bigcup_{p\in M} (T_p(M,C),C')$, where C' is of small differential structure on $T(M,C)$ containing set $C_0 = \{\alpha \circ \pi ; \alpha \in C\} \cup \{\alpha_* ; \alpha \in C\}$, $\pi(v) = p$ for $v \in T_p(M,C)$, $\alpha_*(v) = v(\alpha)$.

Let $T^0(M,C) = (T(M,C)-\{0_p ; p \in M\},\ C'_{T(M,C)-\{0_p;p\in M\}})$ where $T(M,C) = \bigcup_{p\in M} T_p(M,C)$.

Let D be a set such that $T^0(M,C) \subset D \subset T(M,C)$.

We shall consider a function $F \in C'_D$ such that

$$(1) \qquad F(av) = a\,F(v) \quad \text{for} \quad a>0,\ v\in D \text{ and } v \neq 0,$$

$$(2) \qquad F(v) > 0 \quad \text{for} \quad v \in D \text{ and } v \neq 0.$$

By $e = (e_1,\dots,e_m)$ we denote a basis of the linear space $T_p(M,C)$ for $p \in M$. In order to investigate the properties of function F we define a new function ${}_eF: R^m \supset D_e \to R$ by the equation

$$(3) \qquad {}_eF(v^1,\dots,v^m) := F(v^h e_h) \quad \text{for} \quad (v^1,\dots,v^m) \in {}_eD,$$

where ${}_eD = \{(v^1,\dots,v^m); v^h e_h \in D\}$.

It is known that the function (3) is smooth ([7]).

We denote by $F_{|i}$ the derivative of the function $F: R^m \to R$ with respect to the i-th variable. From (1) it follows that

$$(4) \quad {}_eF_{|i}(av^1,\dots,av^m) = {}_eF_{|i}(v^1,\dots,v^m) \text{ for } a>0 \text{ and } (v^1,\dots,v^m) \in {}_eD.$$

We consider a different basis $e' = (e_{1'},\dots,e_{m'})$ of $T_p(M,C)$ such that $e_{i'} = A^i_{i'} e_i$. Then from (4) it follows that

$${}_{e'}F(v^{1'},\dots,v^{m'}) = F(v^{h'} e_{h'}) = F(v^{h'} A^h_{h'} e_h) = {}_eF(v^{h'} A^1_{h'},\dots,v^{h'} A^m_{h'}).$$

Hence it follows

$$(5) \qquad {}_{e'}F_{|i'}(v^{1'},\dots,v^{m'}) = {}_eF_{|i}(v^{h'} A^1_{h'},\dots,v^{h'} A^m_{h'}) \cdot A^i_{i'}$$

and

$$(6) \quad {}_{e'}F_{|i'}(v^{1'},\dots,v^{m'}) x^{i'} = {}_eF_{|i}(v^1,\dots,v^m) x^i, \text{ where } x^i = A^i_{i'} x^{i'}.$$

Let $T^o_p(M,C) = T_p(M,C) - \{0\}$. Using (6) we may associate with each arbitrary $v \in T^o_p(M,C)$, $p \in M$, the mapping $\hat{F}(v): T_p(M,C) \to R$ defined by relation

$$(7) \qquad \hat{F}(v)(x) := {}_eF_{|i}(v^1,\dots,v^m) x^i \quad \text{for} \quad x \in T_p(M,C),$$

where $x = x^i e_i$, $v = v^h e_h$. From (4) we have

$$\hat{F}(av)(x) = \hat{F}(v)(x) \quad \text{for each} \quad x \in T_p(M,C).$$

Hence it follows

(8) $$\hat{F}(av) = \hat{F}(v) \quad \text{for } a>0,\ v \in T_p^o(M,C).$$

Now it is evident that the function $\hat{F}$ is positively homogeneous of degree 0.

We can easily verify the

(9) $$\hat{F}(v)(ax+by) = a\hat{F}(v)(x)+b\hat{F}(v)(y)$$

$$\text{for } a,b \in R,\ v \in T_p^o(M,C)\ x,y \in T_p(M,C).$$

From (4) it follows that

(10) $${}_eF_{|ij}(av^1,\dots,av^m)a = {}_eF_{|ij}(v^1,\dots,v^m)$$

$$\text{for } a>0,\ (v^1,\dots,v^m) \in {}_eD.$$

We note that in view of (5) for $e' = (e_{1'},\dots,e_{m'})$

$${}_{e'}F_{|i'j'}(v^{1'},\dots,v^{m'})\, x^{i'} y^{j'} = {}_eF_{|ij}(v^1,\dots,v^m)x^iy^j.$$

Hence we can define a function $\hat{\hat{F}}$, which for each $v \in T_p^o(M,C)$ associates the mapping $\hat{\hat{F}}(v): T_p(M,C) \times T_p(M,C) \longrightarrow R$ defined by the relation

(11) $$\hat{\hat{F}}(v)(x,y) := {}_eF_{|ij}(v^1,\dots,v^m)x^iy^j \quad \text{for } x,y \in T_p(M,C),$$

where $x = x^h e_h$ and $y = y^h e_h$.

The function $\hat{\hat{F}}$ is well defined. Moreover from (10) we have

(12) $$\hat{\hat{F}}(av)(x,y) = \frac{1}{a} F(v)(x,y) \quad \text{for } x,y \in T_p(M,C),\ v \in T_p^o(M,C) \text{ and } a>0.$$

Hence

$$\hat{\hat{F}}(av) = \frac{1}{a}\hat{\hat{F}}(v) \quad \text{for} \quad v \in T_p^o(M,C) \text{ and } a>0.$$

It is evident that the function $\hat{\hat{F}}$ is positively homogeneous of degree -1. By (1) and (3) we have

(13) ${}_eF(av^1,\dots,av^m)=aF(v^he_h)$ for $v\in T_p^o(M,C)$ and $a>0$, where $v=v^he_h$.

From (13) by Euler's theorem on homogeneous functions we have

$$(14)\qquad {}_eF_{|i}(av^1,\dots,av^m)v^i = F(v^he_h)$$

and when a = 1 we have

$$(15)\qquad {}_eF_{|i}(v^1,\dots,v^m)v^i = F(v^he_h) \text{ for } (v^1,\dots,v^m)\in {}_eD.$$

From (7) and (15) it follows that

$$(16)\qquad \hat{F}(v)(v) = F(v) \quad \text{for} \quad v\in T_p^o(M,C).$$

When we differentiate (4) with respect to a, we obtain

$$(17)\qquad {}_eF_{|ij}(av^1,\dots,av^m)v^j = 0.$$

Moreover, if we set a = 1, then

$$(18)\qquad {}_eF_{|ij}(v^1,\dots,v^m)v^jx^i = 0 \quad \text{for} \quad (v^1,\dots,v^m)\in {}_eD.$$

From (11) and (12) it follows that

$$(19)\qquad \hat{\hat{F}}(v)(x,v) = 0 \quad \text{for} \quad v\in T_p^o(M,C),\ x\in T_p(M,C),\ p\in M.$$

Now we shall consider a new smooth function $F_e : {}_eD \longrightarrow R$ given by

$$(20)\qquad F_e(v^1,\dots,v^m) := \frac{1}{2}F^2(v^he_h) \text{ for } (v^1,\dots,v^m)\in {}_eD,$$

where ${}_eD = \{(v^1,\dots,v^m);\ v^he_h\in D\}$. Hence

$$(21)\qquad F_{e|i}(v^1,\dots,v^m) = F(v^he_h)\,{}_eF_{|i}(v^1,\dots,v^m)$$

and

(22) $$F_{e|ij}(v^1,\dots,v^m) = {}_eF_{|j}(v^1,\dots,v^m)\cdot{}_eF_{|i}(v^1,\dots,v^m) + {}_eF(v^1,\dots,v^m)\cdot{}_eF_{|ij}(v^1,\dots,v^m).$$

From (15) and (18) we have

(23) $$F_{e|ij}(v^1,\dots,v^m)v^i = {}_eF_{|j}(v^1,\dots,v^m)\cdot F(v^he_h)$$

and

(24) $$F_{e|ij}(v^1,\dots,v^m)v^iv^j = F^2(v^he_h).$$

We may also deduce the following equalities

(25a) $$F_{e'|i'j'}(v^{1'},\dots,v^{m'}) = F_{e|ij}(v^{h'}A^1_{h'},\dots,v^{h'}A^m_{h'})\cdot A^i_{j'}\,A^i_{i'}$$

and

(25) $$F_{e'|i'j'}(v^{1'},\dots,v^{m'})\,w^{i'}z^{j'} = F_{e|ij}(v^1,\dots,v^m)w^iz^j.$$

Using the equalities (25) we may associate with each arbitrary $v\in T^o_p(M,C)$, $p\in M$, a mapping $\tilde{F}(v):T_p(M,C)\times T_p(M,C)\longrightarrow R$ defined by the relation

(26) $$\tilde{F}(v)(w,z) := F_{e|ij}(v^1,\dots,v^m)w^iz^j \quad \text{for } w,z\in T_p(M,C),$$
$$v=v^he_h,\ w=w^ie_i,\ z=z^je_j.$$

The mapping $\tilde{F}$ is well defined. From (24) and (26) it follows that

(27) $$\tilde{F}(v)(v,v) = F^2(v) \quad \text{for } v\in T^o_p(M,C).$$

Now we shall quote a few properties of the functions $\tilde{F}$, $\hat{F}$ and $\hat{\hat{F}}$. From (23) and (22) we have

$$(28)\qquad \hat{\hat{F}}(v)(w,z) = \frac{\tilde{F}(v)(w,z)}{F(v)} - \frac{\tilde{F}(v)(v,z)\cdot\tilde{F}(v)(v,w)}{F^3(v)},$$

where $v \in T_p^o(M,C)$, $w,z \in T_p(M,C)$. From (22), (7), (11) and (26) we deduce that

$$(29)\qquad \tilde{F}(v)(w,z) = \hat{F}(v)(z)\cdot\hat{F}(v)(w) + \hat{\hat{F}}(v)(w,z)\cdot F(v).$$

As a corollary from the definition $\tilde{F}$ we obtain the identities

$$(30)\qquad \tilde{F}(v)(w,z) = \tilde{F}(v)(z,w),$$

and

$$(31)\quad \tilde{F}(v)(a_1w_1+a_2w_2, b_1z_1+b_2z_2) = a_1b_1\,\tilde{F}(v)(w_1,z_1) + $$
$$+ a_1b_2\tilde{F}(v)(w_1,z_2) + a_2b_1\tilde{F}(v)(w_2,z_1) + a_2b_2\tilde{F}(v)(w_2,z_2)$$

for $v \in T_p^o(M,C)$, $w,z \in T_p(M,C)$, $a_1,a_2,b_1,b_2 \in R$.

Thus $\tilde{F}(v)$ is a bilinear symmetric mapping of $T_p(M,C)\times T_p(M,C)$ into R.

Now we shall prove the following facts.

S t a t e m e n t 1. If $v \in T_p^o(M,C)$, $w,z \in T_p(M,C)$, $p \in M$ and $a > 0$, then

$$(32)\qquad \tilde{F}(av) = \tilde{F}(v).$$

P r o o f . Using (29) we have

$$\tilde{F}(av)(w,z) = \hat{F}(av)(z)\cdot\hat{F}(av)(w) + \hat{\hat{F}}(av)(w) + \hat{\hat{F}}(av)(w,z)\cdot F(av) =$$

$$= \hat{F}(v)(z)\cdot\hat{F}(v)(w) + \frac{1}{a}\,\hat{\hat{F}}(v)(w,z)\cdot aF(v) = \tilde{F}(v)(w,z).$$

Thus (32) is true.

Two basic assumptions (1) amd (2), which are analogous to the known assumptions in Finsler spaces ([5]), have been used in the former considerations. Following the theories of these spaces a condition corresponding to the assumption of positiveness of a quadratic form is formulated.

Because of this we now additionally suppose that

$$(33)\qquad \tilde{F}(v)(w,w) > 0 \quad\text{for}\quad v,w \in T_p^o(M,C),\ p \in M.$$

L e m m a 1. If $v,w \in T_p^o(M,C)$, $p \in M$ and $w = \lambda v$ for $\lambda \in R$, then

(34) $$\hat{\hat{F}}(v)(w,w) > 0.$$

P r o o f . It should be noticed that from the Schwarz inequality we have

(35) $$\tilde{F}(v)(w,w)\cdot\tilde{F}(v)(v,v) > (\tilde{F}(v)(w,v))^2.$$

From (28) it follows that $\hat{\hat{F}}(v)(w,w) = \frac{1}{F(v)}\left(\tilde{F}(v)(w,w) - \frac{(\tilde{F}(v)(v,w))^2}{\tilde{F}(v)(v,v)}\right)$. In view of (35) and (36) $\hat{\hat{F}}(v)(w,w) > 0$.

We may now prove

T h e o r e m 1. If $v,\bar{v} \in T_p^o(M,C)$ and $v+\bar{v} \in T_p^o(M,C)$ then

(36) $$F(v + \bar{v}) \leqslant F(v) + F(\bar{v}).$$

P r o o f . Let $v,z \in T_p^o(M,C)$. From the Taylor theorem we have

$$_eF(z^1,\dots,z^m) = {_eF}(v^1,\dots,v^m) + {_eF}_{|i}(v^1,\dots,v^m)(z^i-v^i) +$$

$$+ \frac{1}{2}\, {_eF}_{ij}(\Theta^1v^1+(1-\Theta^1)z^1,\dots,\Theta^mv^m+(1-\Theta^m)z^m)(z^i-v^i)(z^j-v^j),$$

where $z = z^he_h$, $v = v^ke_k$, $0 < \Theta^i < 1$.

Hence $F(z) = F(v)+\hat{F}(v)(z-v) + \frac{1}{2}\hat{\hat{F}}(u)(z-v,z-v)$ if $(z-f) \in T_p^o(M,C)$ and $u = (\Theta^hv^h+(1-\Theta^h)z^h)e_h \in T_p^o(M,C)$. Taking into account Lemma 1 we obtain $\hat{\hat{F}}(u)(z-v,z-v) > 0$. Because $\hat{F}(v)(v) = F(v)$, therefore $F(z) > \hat{F}(v)(z)$ for $v,z \in T_p^o(M,C)$. Hence $F(v) > \hat{F}(v+\bar{v})(v)$, $F(\bar{v}) > \hat{F}(v+\bar{v})(\bar{v})$ and $F(v)+F(\bar{v}) > \hat{F}(v+\bar{v})(v+\bar{v}) = F(v+\bar{v})$. $F(v)+F(\bar{v}) = F(v+\bar{v})$ if and only if $v = \lambda v$ for $\lambda > 0$.

Let us introduce the definition

(37) $$\text{Indicatrix}_pF := \{v\in T_p^o(M,C);F^2(v)=1\} = \{v\in T_p^o(M,C);\tilde{F}(v)(v,v)=1\}.$$

We may now define a set

$$(38) \qquad A_{F,p} := \left\{ v : v \in T_p(M,C) \cap D \text{ and } F(v) \leqslant 1 \right\}.$$

T h e o r e m 2. The set $A_{f,p} \cup \{0\}$ is convex.

P r o o f . 1°. Let $v, \bar{v} \in T_p^0(M,C)$, $F(v) \leqslant 1$ and $F(\bar{v}) \leqslant 1$. We consider $w \in T_p^0(M,C)$ such that $w = (1-\lambda)v + \lambda\bar{v}$ for $\lambda \in \langle 0,1 \rangle$. In view of the inequality $F(v+\bar{v}) \leqslant F(v) + F(\bar{v})$ we have $F(w) = F((1-\lambda)v + \lambda\bar{v}) \leqslant (1-\lambda)F(v) + \lambda F(\bar{v}) \leqslant 1$.

2°. If $w = 0$, then in view of definition (38) $w \in A_{F,p} \cup \{0\}$.

Now we shall prove that $A_{F,p} \neq \emptyset$. Let $v \in T_p^0(M,C)$. Hence $v_0 = \frac{v}{F(v)} \in A_{F,p}$, since $F(v_0) = F\left(\frac{v}{F(v)}\right) = \frac{F(v)}{F(v)} = 1$. Hence $A_{F,p} \neq \emptyset$.

We may now define the length of an arbitrary vector $v \in T_p(M,C)$. The length of v is given by

$$(39) \qquad |v|_F = \begin{cases} F(v) & \text{when } v \neq 0 \\ 0 & \text{when } v = 0. \end{cases}$$

Thus according to (1) and (2) it follows that

1° $\qquad |v+w|_F \leqslant |v|_F + |w|_F,$

2° $\qquad |av|_F = a|v|_F \quad \text{for} \quad a \geqslant 0.$

If we additionally assume that $F(-v) = F(v)$ for $-v, v \in T_p^0(M,C)$, then the indicatrix$_p F$ is symmetric and the additional condition

3° $\qquad |av|_F = |a||v|_F$

is to be imposed, where $a \in R$.

The formula 3° follows immediately from (30) and (1). Namely for $a < 0$ we have $|av|_F = F(av) = F((-a)(-v)) = -aF(-v) = |a|F(v) = |a||v|_F$.

The indicatrix$_p F$ plays the role of the unit sphere in the geometry of the vector space $T_p(M,C)$. The pair $(T_p(M,C), |\cdot|_F)$, where $|\cdot|_F$ satisfy conditions 1° and 2° will be called a Minkowskian space of function F.

Now we turn to the expression $F_{e|j'i'}(v^{1'},\dots,v^{m'}) =$
$= F_{e|ij}(v^{h'}A^1_{h'},\dots,v^{h'}A^m_{h'})A^j_{j'}A^i_{i'}$. Hence
$F_{e'|i'j'k'}(v^{1'},\dots,v^{m'})x^{i'}y^{j'}z^{k'} = F_{e|ijk}(v^1,\dots,v^m)x^iy^jz^k$.

This suggest that we can easily (irrespectively of any basis) introduce a function F given by

$$\tilde{\tilde{F}}(v)(x,y,z) := \frac{1}{2}F_{e|ijk}(v^1,\dots,v^m)x^iy^jz^k \tag{40}$$

for $v \in T^o_p(M,C), x,y,z \in T_p(M,C)$, where $v = v^he_h$, $x = x^ie_i$, $y = y^je_j$, $z = z^ke_k$.

As a corollary to (40) we obtain

Assertion 2.

(i) $\tilde{\tilde{F}}(av)(x,y,z) = \frac{1}{a}\tilde{\tilde{F}}(v)(x,y,z)$ for $a>0$,
$v \in T^o_p(M,C), x,y,z \in T_p(M,C)$, $p \in M$,

(ii) $\tilde{\tilde{F}}(v)(v,y,z) = \tilde{\tilde{F}}(v)(x,v,z) = \tilde{\tilde{F}}(v)(x,y,v) = 0$ for
$v \in T^o_p(M,C), x,y,z \in T_p(M,C)$, $p \in M$.

The pair $((M,C),F)$, where (M,C) is a Hausdorff differential finite dimension space, $F \in C'_{T^o(M,C)}$ satisfies conditions (1) and (2), and the function $\tilde{F}$ is defined by the relation (26), satisfying condition (33), will be called a Finsler differential space in the sense of Sikorski.

BIBLIOGRAPHY

[1] H. Akbar-Zadeh: Les espaces de Finsler et certaines de leurs generalizations, Ann. Sci. Ecole Norm. Sup (3) 80 (1963) 1-79.

[2] K. Cegiełka: The differential space of a differential module, Bull. Soc. Sci. Letters Łódź 28 (1978), 14.

[3] M. Matsumoto: Foundations of Finsler Geometry and special Finsler spaces, Berlin 1977.

[4] H. M a t u s z c z y k : Ślebodzińskis formula for Lie derivative of tensor fields in a differential space, Czechoslovakian-GDR-Polish Scientific School on Differential Geometry, February 1978.

[5] H. R u n d : The differential geometry of Finsler spaces, Berlin 1959.

[6] R. S i k o r s k i : Wstęp do geometrii różniczkowej, Warszawa 1972.

[7] W. W a l i s z e w s k i : Finsler differential spaces, Coll. Math. (to appear).

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