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MODULAR APPROXIMATION BY A FILTERED FAMILY OF THE HAMMERSTEIN OPERATORS

1. Introduction

In this paper we introduce the notion of boundedness of a filtered family (T_v) of Hammerstein operators in a modular space. This notion is used to get a theorem on modular convergence of $T_v x$.

Let X be a real vector space. A function $\varrho : X \to [0, \infty]$ is called a modular on X, if $\varrho(x) = 0$ iff $x = 0$, $\varrho(-x) = \varrho(x)$ and $\varrho(ax+by) \leqslant \varrho(x) + \varrho(y)$ for $a,b \geqslant 0$, $a+b = 1$, $x,y \in X$. If $\varrho(ax+by) \leqslant a\varrho(x) + b\varrho(y)$ for $a,b \geqslant 0$, $a+b = 1$, then ϱ is called convex modular on X. The modular space X_ϱ generated by ϱ is defined as

$$X_\varrho = \left\{ x \in X : \varrho(ax) \to 0 \quad \text{as} \quad a \to 0 \right\}.$$

We define in X a modular convergence (ϱ-convergence) $x_n \xrightarrow{\varrho} 0$ by the condition: there exists an $a>0$ such that $\varrho(ax_n) \to 0$ as $n \to \infty$. The ϱ-closure of a set $S \subset X_\varrho$ is defined as the set of all elements $x \in X_\varrho$ such that $x_n - x \xrightarrow{\varrho} 0$ for a sequence of $x_n \in S$. In the case of convex ϱ, $\|x\|_\varrho = \inf \left\{ u>0 : \varrho\left(\frac{x}{u}\right) \leqslant 1 \right\}$ defines a norm in X_ϱ. Convergence $x_n \to 0$ in norm in X_ϱ is equivalent to the condition $\varrho(ax_n) \to 0$ as $n \to \infty$ for every $a>0$. Obviously, norm convergence implies ϱ-convergence but not conversely.

Let (Ω, Σ, μ) be a measure space, such that $\Omega = [0,b)$, $0 < b < \infty$, μ = Lebesgue measure in the σ-algebra Σ of all Lebesgue measurable subsets of $[0,b)$. Let X be the space of all extended real-valued, Σ-measurable functions $x = x(\cdot)$ over $[0,b)$, finite μ-almost everywhere, two functions equal μ - a.e. will be treated as the same element of X.

Let U be a nonempty set.

Let $\varrho_u : [0,b) \times R \to [0,\infty]$, $u \in U$, be a family of functions such that for every $u \in U$:

(1) $\varrho_u(t,r)$ is a continuous function of r, equal to zero iff $r = 0$, $\varrho_u(t,-r) = \varrho_u(t,r)$ for every $r \in R$, and nondecreasing for $r > 0$, for every $t \in [0,b)$,

(2) $\varrho_u(t,r)$ is a measurable function of $t \in [0,b)$ for every $r \in R$.

Now, taking

$$\varrho_s(x) = \sup_{u \in U} \int_0^b \varrho_u(t,x(t))dt, \tag{3}$$

we see that ϱ_s is a modular on X. Let X_{ϱ_s} be the respective modular space. Throughout this paper we assume (1)-(3).

The results of this paper extends the results of [1] from linear operators to the Hammerstein operators.

2. A General Theorem

Let V be a nonempty set and let $\mathcal{V}$ be a filter of subsets of V.

D e f i n i t i o n 1. A function $g: V \to R$ tends to zero with respect to $\mathcal{V}$, $g(v) \xrightarrow{\mathcal{V}} 0$, if for every $\varepsilon > 0$ there is a set $V_0 \in \mathcal{V}$ such that $|g(v)| < \varepsilon$ for all $v \in V_0$.

D e f i n i t i o n 2. A family $T = (T_v)_{v \in V}$ of operators $T_v : X_{\varrho_s} \to X_{\varrho_s}$ will be called $\mathcal{V}$-bounded if there exists positive numbers k_1, k_2 and a function $g: V \to R_+$ such that $g(v) \xrightarrow{\mathcal{V}} 0$ and for all $x,y \in X_{\varrho_s}$ there is a set $V_{x,y} \in \mathcal{V}$ for which

$$(4) \qquad \varrho_s(a(T_v x - T_v y)) \leqslant k_1 \varrho_s(ak_2(x-y)) + g(v)$$

for every $v \in V_{x,y}$ and every $a > 0$.

Let us remark that if ϱ_s is convex, then the constant k_1 may be always taken equal 1.

D e f i n i t i o n 3. An operator $\tau_v : X \longrightarrow X$ such that $\tau_v x(t) = x(t+v)$, where x is extended to the whole R b-periodically will be called the translation operator.

Let us extend the functions $\varrho_u(t,r)$ b-periodically with respect to the variable $t \in [0,b)$ to the whole R, i.e., $\varrho_u(t,r) = \varrho_u(t+b,r)$ for $t,r \in R$, $u \in U$.

D e f i n i t i o n 4. We shall say that the family of functions $(\varrho_u)_{u \in U}$ is τ-bounded, if there exist positive constants n_1, n_2 such that

$$(5) \quad \varrho_u(t-v,r) \leqslant n_1 \varrho_u(t,n_2 r) + f_u(t,v) \quad \text{for } r,t,v \in R,\ u \in U,$$

where the functions $f_u : R \times R \longrightarrow R_+$ are measurable and b-periodic with respect to the first variable and such that if

$$h(v) = \sup_{u \in U} \int_0^b f_u(t,v)\,dt \quad \text{for every } v \in R,$$

then

$$H = \sup_{v \in R} h(v) < \infty \quad \text{and} \quad h(v) \longrightarrow 0 \text{ as } v \longrightarrow 0 \text{ or } v \longrightarrow b.$$

Let us remark that if $\varrho_u(t,r)$ are convex as functions of r for every $u \in U$, then we may take in the above definition $n_1 = 1$.

Now let $V = R$ and let $\mathcal{V}$ be the filter of all neighbourhoods of zero in R.

T h e o r e m 1. If the family $(\varrho_u)_{u \in U}$ is τ-bounded and if for every $c > 0$ $c \in X_{\varrho_s}$ and

$$(6) \qquad \int_0^b \sup_{u \in U} \varrho_u(t,c)\,dt < \infty$$

then the family τ of translation operators is $\mathcal{V}$-bounded and for every $x \in X_{\rho_s}$ there is $c>0$ such that

$$\omega_{\tau}(cx,\delta) =$$

$$= \sup_{|v|\leq\delta} \sup_{u\in U} \int_0^b \rho_u(t,c(x(t+v) - x(t)))dt \longrightarrow 0 \quad \text{as} \quad \delta \longrightarrow 0^+.$$

Theorem 1 is a slight modification of Theorem 3 from [1], so the proof of that theorem will be omited.

3. On convergence of the integral Hammerstein operators

Let W be a nonempty set and let $\mathcal{W}$ be a filter in W.

Let $K_w:[0,b) \longrightarrow R_+$ for $w \in W$ be integrable in $[0,b)$ and let

$$(7) \qquad \sigma(w) = \int_0^b K_w(t)dt \xrightarrow{\mathcal{W}} 1, \qquad \sigma = \sup_{w\in W} \int_0^b K_w(t)dt < \infty,$$

$$\sigma_\delta(w) = \int_\delta^{b-\delta} K_w(t)\,dt \xrightarrow{\mathcal{W}} 0 \quad \text{for every} \quad 0<\delta<\frac{b}{2}.$$

Let us extend K_w b-periodically to the whole R.

Let $F:[0,b) \times R \longrightarrow R$ be measurable and let us extend F b-periodically to the whole R. Let

$$(8) \qquad T_w\, x(s) = \int_0^b K_w(t-s)F(t,x(t))dt \quad \text{for every} \quad w \in W.$$

We prove first

P r o p o s i t i o n 1. Let the family $(\rho_u)_{u\in U}$ be τ-bounded and for every $t \in [0,b)$ let $\rho_u(t,r)$ be a convex function of r for every $u \in U$. Let the assumptions (7) hold and let $F(t,0) = 0$ for every $t \in [0,b)$. If there exists $L>0$ such that $|F(t,r) - F(t,v)| \leq L|r-v|$ for every $t \in [0,b)$ and $r,v \in R$, then $T_w:X_{\rho_s} \longrightarrow X_{\rho_s}$ for every $w \in W$ and $T = (T_w)_{w\in W}$ is $\mathcal{W}$-bounded, where T_w are given by (8).

P r o o f . It is sufficient to prove that T is $\mathcal{W}$-bounded and $W_{x,y} = W$ for all $x,y \in X_{\varrho_s}$. In fact, if T is $\mathcal{W}$-bounded and $W_{x,y} = W$ for all $x,y \in X_{\varrho_s}$, then (see (4)) there exist positive numbers k_1, k_2 and a function $g: W \longrightarrow R_+$, $g(w) \xrightarrow{\mathcal{W}} 0$, such that for all $x,y \in X_{\varrho_s}$ and every $a > 0$

$$\varrho_s(a(T_w x - T_w y)) \leqslant k_1 \varrho_s(ak_2(x-y)) + g(w)$$

for every $w \in W$. In particular, if $y = 0$ we have

$$\varrho_s(aT_w x) \leqslant k_1 \varrho_s(ak_2 x) + g(w).$$

From this we obtain that $T_w: X_{\varrho_s} \longrightarrow X_{\varrho_s}$. By the b-periodicity of $\varrho_u(\cdot,r)$, $x(\cdot)$, $F(\cdot,r)$, Jensen's inequality and τ-boundedness of $(\varrho_u)_{u\in U}$ with $n_1 = 1$, $n_2 = n > 1$ we obtain for $x,y \in X_{\varrho_s}$, $a > 0$, $w \in W$:

$$\varrho_s(a(T_w x - T_w y)) = \sup_{u\in U} \int_0^b \varrho_u\Big(s, \frac{a}{\sigma(w)} \int_0^b K_w(t)\sigma(w)(F(t+s,x(t+s)) -$$

$$- F(t+s,y(t+s)))dt\Big)ds \leqslant$$

$$\leqslant \frac{1}{\sigma(w)} \sup_{u\in U} \int_0^b \int_0^b K_w(t)\varrho_u\Big(s, a\sigma\big(F(t+s,x(t+s))-F(t+s,y(t+s))\big)\Big)\, dt\, ds =$$

$$= \frac{1}{\sigma(w)} \sup_{u\in U} \int_0^b K_w(t) \int_0^b \varrho_u\Big(r-t, a\sigma\big(F(r,x(r))-F(r,y)r)\big)\Big)\, dr\, dt \leqslant$$

$$\leqslant \varrho_s(na\sigma L(x-y)) + \frac{1}{\sigma(w)} \int_0^b K_w(t)h(t)dt = \varrho_s(na\sigma L(x-y)) + g(w),$$

where $g(w) = \frac{1}{\sigma(w)} \int_0^b K_w(t)h(t)dt.$

It is easy to prove that $g(w) \xrightarrow{\mathcal{W}} 0$ (see [1]). From this we obtain that $T = (T_w)_{w \in W}$ is $\mathcal{W}$-bounded.

Now, we are able to prove the following theorem.

Theorem 2. Let the assumptions of Proposition 1 and Theorem 1 hold. If there exists $f \in X$ such that

(9) $$|f(t)| \leqslant L \quad \text{for every} \quad t \in R,$$

(10) $$|F(t,r) - F(s,r)| \leqslant |f(t-s)\, r| \quad \text{for all} \quad r,t,s \in R,$$

(11) $$\int_0^b K_w(t)|f(t)|\,dt \xrightarrow{\mathcal{W}} 0,$$

then for every $x \in X_{\varrho_s}$ there exists $a > 0$ such that

$$\varrho_s(a(T_w x - \mathbf{F}x)) \xrightarrow{\mathcal{W}} 0,$$

where $\mathbf{F}$ is given by the formula

(12) $$\mathbf{F}x(s) = F(s,x(s)) \quad \text{for} \quad x \in X_{\varrho_s} \quad \text{and} \quad s \in [0,b).$$

Proof. By Proposition 1, we have $T_w x \in X_{\varrho_s}$. From Theorem 1 we infer that $\omega_\tau(cx,\delta) \longrightarrow 0$ as $\delta \to 0^+$ for sufficiently small $c > 0$. For $x \in X_{\varrho_s}$ and $a > 0$, we have

(13) $$\varrho_s(a(T_w x - \mathbf{F}x)) =$$

$$= \sup_{u \in U} \int_0^b \varrho_u\left(s, \frac{1}{\sigma(w)} \int_0^b K_w(t) a\sigma(w) F(s+t, x(s+t))dt - aF(s,x(s))\right)ds \leqslant$$

$$\leqslant \frac{1}{2} \sup_{u \in U} \int_0^b \varrho_u\left(s, \frac{1}{\sigma(w)} \int_0^b K_w(t) 2a\sigma(w) \Big(F(s+t,x(s+t)) -\right.$$

$$\left. - F(s,x(s))\Big)\, dt\right) ds + \frac{1}{2} \sup_{u \in U} \int_0^b \varrho_u\Big(s, 2a(\sigma(w)-1)F(s,x(s))\Big)\, ds \leqslant$$

$$\leq \frac{1}{2\sigma(w)} \sup_{u\in U} \int_0^b K_w(t) \int_0^b \varrho_u\Big(2a\sigma\big(F(s+t,x(s+t))-F(s,x(s))\big)\Big)\, ds\, dt +$$

$$+ \frac{1}{2} \sup_{u\in U} \int_0^b \varrho_u(s, 2aL(\sigma(w)-1)x(s))ds.$$

From (10) and (13) we have

$$\varrho_s(a(T_w x - Fx)) \leq \tag{14}$$

$$\leq \frac{1}{4\sigma(w)} \sup_{u\in U} \int_0^b K_w(t) \int_0^b \varrho_u(s, 4a\sigma f(t)x(s+t))ds\, dt +$$

$$+ \frac{1}{4\sigma(w)} \sup_{u\in U} \int_0^b K_w(t) \int_0^b \varrho_u(s, 4aL\sigma(x(s+t)-x(s)))ds\, dt +$$

$$+ \frac{1}{2} \sup_{u\in U} \int_0^b \varrho_u(s, 2aL(\sigma(w)-1)x(s))ds.$$

Now, we split the second integral on the right-hand side of this inequality into three integrals over intervals $[0,\delta)$, $[\delta, b-\delta)$, $[b-\delta, b)$, where $0<\delta<\frac{b}{2}$ is arbitrary. The first integral is estimated as follows

$$\sup_{u\in U} \int_0^\delta K_w(t) \int_0^b \varrho_u\big(s, 4aL\sigma(x(s+t)-x(s))\big)\, ds\, dt \leq \tag{15}$$

$$\leq \int_0^\delta K_w(t)\, \varrho_s(4aL\sigma(\tau_t x - x))dt \leq \sigma(w)\,\omega_\tau(4aL\sigma x, \delta)$$

and the third one, by substitution $t = b-u$,

$$\sup_{u\in U} \int_{b-\delta}^b K_w(t) \int_0^b \varrho_u\big(s, 4aL\sigma(x(s+t) - x(s))\big)\, ds\, dt \leq \tag{16}$$

$$\leq \sigma(w)\omega_\tau(4aL\sigma x, \delta).$$

Finally, for the second integral (see [1]) we have

$$(17)\quad \sup_{u\in U}\int_{\delta}^{b-\delta} K_w(t)\int_0^b \varrho_u\Big(s,4aL\big(x(s+t)-x(s)\big)\Big)\,ds\,dt \leqslant$$

$$\leqslant \frac{1}{2}\int_{\delta}^{b-\delta} K_w(t)\big(\varrho_s(8anL\sigma x) + h(t) + \varrho_s(8anL\sigma x)\big)dt \leqslant$$

$$\leqslant \Big(\varrho_s(8anL\sigma x) + \frac{1}{2}H\Big)\int_{\delta}^{b-\delta} K_w(t)\,dt.$$

The first integral on the right-hand side of the inequality (14) is estimated as follows

$$(18)\qquad \sup_{u\in U}\int_0^b K_w(t)\int_0^b \varrho_u(s,4a\sigma f(t)x(s+t))\,ds\,dt \leqslant$$

$$\leqslant \frac{1}{L}\sup_{u\in U}\int_0^b K_w(t)|f(t)|\int_0^b \varrho_u(s,4aL\sigma x(s+t))\,ds\,dt \leqslant$$

$$\leqslant \frac{1}{L}\big(\varrho_s(4anL\sigma x) + H\big)\int_0^b K_w(t)|f(t)|\,dt.$$

From (7), (14), (15), (16), (17), (18) for sufficiently small $a>0$ we obtain that

$$\varrho_s(a(T_w x - Fx)) \xrightarrow{W^{\nu}} 0.$$

Now let g_1 and F_1 be such that: $g_1:[0,b)\longrightarrow R$, $F_1:[0,b)\times R\longrightarrow R$ are measurable, and there is $\underline{L}>0$ for which

$$(19)\qquad |g_1(t)| < \underline{L} \quad \text{for} \quad t\in[0,b),$$

there is $\varepsilon_0\in(0,b)$ such that

$$(20)\qquad F_1(t,r) = \begin{cases} 0 & \text{for} \quad t\in[0,\varepsilon_0]\cup[b-\varepsilon_0,b) \\ g_2(t,r) & \text{for} \quad t\in(\varepsilon_0,b-\varepsilon_0) \end{cases}$$

for $r \in R$, and there is $L_2 > 0$ such that

(21) $|g_2(t,r)-g_2(t,u)| \leqslant L_2|r-u|$ for $t \in [0,b)$, $r,u \in R$,

(22) $F_1(t,0) = 0$ for every $t \in [0,b)$.

Let us extend g_1 and F_1 b-periodically to the whole R. Let

(23) $\underline{T}_w\, x(s) = \int\limits_0^b K_w(t-s)\Big(g_1(t)x(t)+F_1(t-s,x(t))\Big)\,dt, \quad w \in W.$

T h e o r e m 3. Let the assumptions of Theorem 1 hold. Let the family $(\varrho_u)_{u\in U}$ be τ-bounded and for every $t \in [0,b)$ let $\varrho_u(t,r)$ be a convex function with respect to r, for every $u \in U$. If the assumptions (7), (19)-(22) hold and there is $\delta_0 > 0$ such that for every $x \in X_{\varrho_s}$ there is $c > 0$ for which

$$\sup_{u\in U} \int\limits_0^{\delta_0} K_w(t) \int\limits_0^b \varrho_u(s,c(g_1(t+s) - g_1(s))x(s))\,ds\,dt \xrightarrow{\mathcal{W}} 0$$

$$\sup_{u\in U} \int\limits_{b-\delta_0}^{b} K_w(t) \int\limits_0^b \varrho_u(s,c(g_1(t+s) - g_1(s))x(s))\,ds\,dt \xrightarrow{\mathcal{W}} 0,$$

then for every $x \in X_{\varrho_s}$ there exists $a > 0$ such that

$$\varrho_s(a(\underline{T}_w x - Gx)) \xrightarrow{\mathcal{W}} 0,$$

where $\underline{T}_w$ is given by (23) and G is given by formula

$$Gx(t) = g_1(t)x(t) \quad \text{for} \quad x \in X_{\varrho_x}, \quad t \in [0,b).$$

P r o o f . It is easy to see that the assumptions of Proposition 1 hold. Hence $\underline{T}_w x \in X_{\varrho_s}$ for $x \in X_{\varrho_s}$ and $\underline{T} = (\underline{T}_w)_{w\in W}$

is $\mathcal{W}$-bounded. The rest of the proof is analogous to that of Theorem 2 and we omit it.

Let $\underline{K}_w : [0,b) \times [0,b) \longrightarrow R_+$ for $w \in W$ be integrable in $[0,b) \times [0,b)$ and

(24) if $s_1 < s_2$, then $\underline{K}_w(t,s_1) \leqslant \underline{K}_w(t,s_2)$ for $t \in [0,b)$.

We let $\underline{K}_w(t,b) = \lim\limits_{s \to b^-} \underline{K}_w(t,s)$.

$$(25) \qquad \sigma(w) = \sup_{s \in [0,b)} \left| \int_0^b \underline{K}_w(t,s)dt - 1 \right| \xrightarrow{\mathcal{W}} 0,$$

$$(26) \quad \overline{\sigma}(w) = \int_0^b \underline{K}_{w^-}(t,b)dt, \quad 0 < \overline{\sigma}(w) < \infty, \quad \sigma = \sup_{w \in W} \overline{\sigma}(w) < \infty,$$

$$(27) \qquad \sigma_\delta(w) = \int_\delta^{b-\delta} \underline{K}_{w^-}(t,b)dt \longrightarrow 0 \quad \text{for} \quad 0 < \delta < \frac{b}{2} .$$

Now we extend $\underline{K}_w(t,r)$ periodically with respect to the variables $t,r \in [0,b)$ to the whole $R \times R$ i.e. $\underline{K}_w(t+b,r) = \underline{K}_w(t,r)$, $\underline{K}_w(t,r+b) = \underline{K}_w(t,r)$ for $t,r \in R$. Let

$$(28) \qquad \underline{T}_w x(s) = \int_0^b \underline{K}_w(t-s,s)F(t,x(t))dt.$$

P r o p o s i t i o n 2. Let the family $(\varrho_u)_{u \in U}$ be τ-bounded and let for every $t \in [0,b)$, $\varrho_u(t,r)$ be convex as a function of r for every $u \in U$. Let the assumptions (24)-(27) hold. Let $F(t,0) = 0$ for $t \in [0,b)$. If there exists $L > 0$ such that $|F(t,r) - F(t,v)| \leqslant L|r-v|$ for every $t \in [0,b)$ and all $r,v \in R$, then $\underline{T}_w : X_{\varrho_s} \longrightarrow X_{\varrho_s}$ for every $w \in W$ and $\underline{\underline{T}} = (\underline{T}_w)_{w \in W}$ is $\mathcal{W}$-bounded, where $\underline{T}_w$ are given by (28).

P r o o f . It suffices to prove that $\underline{\underline{T}}$ is $\mathcal{W}$-bounded.

Let $x,y \in X_{\varrho_s}$ and $a > 0$. From the assumptions we have

(29)
$$\varrho_s(a(\underline{\underline{T}}_w x - \underline{\underline{T}}_w y)) =$$

$$= \sup_{u\in U} \int_0^b \varrho_u\Big(s, a\int_0^b K_w(t,s)(F(t+s,x(t+s)-F(t+s,y(t+s))) dt\Big)ds \leqslant$$

$$\leqslant \sup_{u\in U} \int_0^b \varrho_u\Big(s, a\int_0^b \underline{K}_{\overline{w}}(t,b)L(x(t+s)-y(t+s))dt\Big)ds \leqslant$$

$$\leqslant \frac{1}{\overline{\sigma}(w)} \sup_{u\in U} \int_0^b\int_0^b \underline{K}_{\overline{w}}(t,b)\varrho_u\Big(s, a\sigma L(x(t+s)-y(t+s))\Big) ds\ dt =$$

$$= \frac{1}{\overline{\sigma}(w)} \sup_{u\in U} \int_0^b\int_0^b \underline{K}_{\overline{w}}(t,b)\varrho_u\Big(r,t, a\sigma L(x(r)-y(r))\Big) dr\ dt \leqslant$$

$$\leqslant \frac{1}{\overline{\sigma}(w)} \sup_{u\in U} \int_0^b\int_0^b \underline{K}_{\overline{w}}(t,b)\varrho_u\Big(r, an\sigma L(x(r)-y(r))\Big) dt\ dr +$$

$$+ \frac{1}{\overline{\sigma}(w)} \sup_{u\in U} \int_0^b\int_0^b \underline{K}_{\overline{w}}(t,b)f_u(r,t)dr\ dt = \varrho_s(an\sigma L(x-y))+g(w),$$

where

(30)
$$g(w) = \frac{1}{\overline{\sigma}(w)} \int_0^b \underline{K}_{\overline{w}}(t,b)h(t)dt.$$

It is easy to prove that $g(w) \xrightarrow{\mathcal{W}} 0$ (see [1]). From this and (29), (30) we obtain the $\mathcal{W}$-boundedness of $\underline{\underline{T}}$.

T h e o r e m 4. Let the assumptions of Proposition 2 and Theorem 1 hold. If there exist $N>0$ and $f\in X$ such that $|F(t,r)-F(s,r)| \leqslant |f(t-s)\ r| \leqslant N|r|$ for $r\in R$, $t,s\in[0,b)$ and $\int_0^b \underline{K}_{\overline{w}}(t,b)|f(t)|dt \xrightarrow{\mathcal{W}} 0$, then for every $x\in X_{\varrho_s}$ there exists $a>0$ such that $\varrho_s(a(\underline{\underline{T}}_w x - \mathbf{F}x)) \xrightarrow{\mathcal{W}} 0$, where $\underline{\underline{T}}_w$ are given by (28) and $\mathbf{F}$ is given by formula $\mathbf{F}x(s) = F(s,x(s))$ for $x\in X_{\varrho_s}$, $s\in[0,b)$.

P r o o f . Since the proof is analogous to that of Theorem 2 we give only its outline. Let $x \in X_{\varrho_s}$, $a > 0$.

We have from the assumptions

(31)
$$\varrho_s(a(\underline{\underline{T}}_w x - Fx)) \leqslant$$
$$\leqslant \frac{1}{2} \sup_{u\in U} \int_0^b \varrho_u\left(s, 2a \int_0^b \underline{K}_w(t,s)(F(t+s,x(t+s)-F(s,x(s))) dt\right) ds +$$
$$+ \frac{1}{2} \sup_{u\in U} \int_0^b \varrho_u\left(s, 2a \left(\int_0^b \underline{K}_w(t,s)dt-1\right) F(s,x(s))dt\right) ds \leqslant$$
$$\leqslant \frac{1}{4} \sup_{u\in U} \int_0^b \varrho_u\left(s, 4a \int_0^b \underline{K}_{\overline{w}}(t,b)f(t)x(s+t)dt\right) ds +$$
$$+ \frac{1}{4} \sup_{u\in U} \int_0^b \varrho_u\left(s, 4a \int_0^b \underline{K}_{\overline{w}}(t,b)L(x(t+s)-x(s))dt\right) ds +$$
$$+ \frac{1}{2}\varrho_s(2aL\sigma(w)x).$$

It is easy to see that

(32) $\frac{1}{2} \varrho_s(2aL\sigma(w)x) \xrightarrow{W} 0$ for sufficiently small $a > 0$,

(33) $$\sup_{u\in U} \int_0^b \varrho_u\left(s, 4a \int_0^b \underline{K}_{\overline{w}}(t,b)L(x(t+s)-x(s))dt\right) ds \xrightarrow{W} 0$$

for sufficiently small $a > 0$ (see the proof of Theorem 2).

To end the proof we must show that

(34) $$\sup_{u\in U} \int_0^b \varrho_u\left(s, 4a \int_0^b \underline{K}_{\overline{w}}(t,b)f(t)x(s+t)dt\right) ds \xrightarrow{W} 0$$

for sufficiently small $a > 0$.

From the assumptions we obtain

$$(35)\quad \sup_{u\in U}\int_0^b \varrho_u\left(s, 4a\int_0^b \underline{K}_{\bar{w}}(t,b)f(t)x(s+t)dt\right)ds \leqslant$$

$$\leqslant \frac{1}{\bar{\sigma}(w)} \sup_{u\in U}\int_0^b\int_0^b \underline{K}_{\bar{w}}(t,b)\varrho_u(s, 4a\sigma f(t)x(s+t))dt\; ds \leqslant$$

$$\leqslant \frac{1}{\bar{\sigma}(w)} \sup_{u\in U}\int_0^b\int_0^b \underline{K}_{\bar{w}}(t,b)\varrho_u(r, 4an\sigma f(t)x(r))dr\; dt + g(w),$$

$$(36)\quad \sup_{u\in U}\int_0^b\int_0^b \underline{K}_{\bar{w}}(t,b)\varrho_u(r, 4an\sigma f(t)x(r))dr\; dt \leqslant$$

$$\leqslant \frac{1}{N} \sup_{u\in U}\int_0^b\int_0^b \underline{K}_{\bar{w}}(t,b)|f(t)|\varrho_u(r, 4an\sigma Nx(r))dr\; dt =$$

$$= \frac{1}{N}\varrho_s(4an\sigma Nx)\int_0^b \underline{K}_{\bar{w}}(t,b)|f(t)|dt \xrightarrow{\mathcal{W}} 0$$

for sufficiently small $a>0$.

From (31)-(36) we obtain Theorem 4.

4. On convergence in Generalized Orlicz Sequence Spaces

We are now going to the case of the space X of all sequences $x = (x_j)$ and to a modular ϱ of the form

$$(37)\qquad \varrho(x) = \sum_{i=0}^{\infty} \varphi_i(x_i),$$

where $\varphi = (\varphi_i)$ is a sequence of φ-functions, i.e. $\varphi : N\times R \longrightarrow R_+$. We shall investigate a family of Hammerstein operators in the generalized Orlicz sequence space ℓ^{φ}. Here V will be the set N of all nonnegative integers and the filter $\mathcal{V}$ will consists of all sets $V_0 \subset V$ which are complements of finite sets. The set W and the filter $\mathcal{W}$ of its subsets will be as previously.

Let us introduce the family $r = (r_m)_{m\in N}$ by the formula

(38) $$r_m x(i) = \begin{cases} 0 & \text{for } i \leqslant m, \\ x(i) & \text{for } i > m. \end{cases}$$

Definition 1. We shall say that $\varphi = (\varphi_i)_{i=0}^{\infty}$ is τ-bounded, if there exist constants $k_1, k_2 \geqslant 1$ and a double sequence $(\eta_{n,j})$ such that

(39) $$\varphi_n(u) \leqslant k_1 \varphi_{n+j}(k_2 u) + \eta_{n,j} \quad \text{for} \quad u \in R, \quad n > j \geqslant 0,$$

where $\eta_{n,j} \geqslant 0$, $\eta_{n,o} = 0$, $\sum_{n=0}^{\infty} \eta_{n,j} < \infty$ uniformly with respect to j. We shall say that φ is τ_+-bounded, if there are constants $k_1, k_2 \geqslant 1$ and a double sequence $\varepsilon_{n,j}$ such that

$$\varphi_{n+j}(u) \leqslant k_1 \varphi_n(k_2 u) + \varepsilon_{n,j} \quad \text{for} \quad u \in R, \quad n,j = 0,1,2,\ldots,$$

where $\varepsilon_{n,j} \geqslant 0$, $\varepsilon_{n,o} = 0$, $\varepsilon_j = \sum_{n=0}^{\infty} \varepsilon_{n,j} \to 0$ as $j \to \infty$, $s = \sup_{j\in N} \varepsilon_j < \infty$.

Let us write $e_\ell = (\delta_{i,\ell})_{i=o}^{\infty}$, where $\delta_{i,\ell}$ is the Kronecker symbol.

It is easy to prove that

Proposition 3.

(a) The family $r = (r_m)_{m=o}^{\infty}$ is $\mathcal{V}$-bounded.

(b) The set of linear combinations of sequences $e_0, e_1, e_2, \ldots$ is ϱ-dense in ℓ^φ.

(c) $\varrho(a r_m e_\ell) \to 0$ as $m \to \infty$ for every ℓ and every $a > 0$.

From Theorem 1 of [1] and from Proposition 3 we obtain

Theorem '1'. For every $x \in \ell^\varphi$ there is an $a > 0$ for which $\varrho(a r_m x) \to 0$ as $m \to \infty$.

Let $\mathcal{W}$ be a filter of subsets of a set W and let $K_W : N \to R_+$ be such that

$$(40)\qquad \sigma(w) = \sum_{j=0}^{\infty} K_{w,j} \leqslant \sigma < \infty,\ K_{w,o} \xrightarrow{\mathcal{W}} 1,\ \frac{K_{w,j}}{\sigma(w)} \xrightarrow{\mathcal{W}} 0$$

for $j=1,2,\ldots$ $w \in W$.

Let $T_w x = ((T_w x)_i)_{i=0}^{\infty}$, where $(T_w x)_i = \sum_{j=0}^{\infty} K_w(i-j)F(j,x(j))$, where $F: N\times R \to R$, $x(j) = x_j$.

Proposition 4. Let $\varphi = (\varphi_i)_{i=0}^{\infty}$ be τ_+-bounded and let φ_i be convex for $i = 0,1,2,\ldots$. Let the assumptions (40) hold. If there exists $L>0$ such that

$$|F(u,v) - F(u,r)| \leqslant L|v-r| \quad \text{for } u \in N,\ v,r \in R$$

and $F(n,0) = 0$ for $n \in N$, then $T_w: \ell^{\varphi} \to \ell^{\varphi}$ for every $w \in W$ and $T = (T_w)_{w\in W}$ is $\mathcal{W}$-bounded.

Proof. It is enough to show that T is $\mathcal{W}$-bounded. We have for every $a>0$ and all $x,y \in \ell^{\varphi}$, $w \in W$

$$\varrho(a(T_w x - T_w y)) =$$

$$= \sum_{i=0}^{\infty} \varphi_i \left(a \sum_{j=0}^{i} K_w(j)(F(i-j,x(i-j)) - F(i-j,y(i-j))\right) \leqslant$$

$$\leqslant \frac{1}{\sigma(w)} \sum_{j=0}^{\infty} K_w(j) \sum_{i=j}^{\infty} \varphi_i\left(\sigma(w)L(x(i-j) - y(i-j))\right) \leqslant$$

$$\leqslant k_1\varrho(ak_2\sigma L(x-y)) + c(w),$$

where $c(w) = \frac{1}{\sigma(w)} \sum_{j=1}^{\infty} K_w(j)\varepsilon_j \xrightarrow{\mathcal{W}} 0$ (see [1]).

Theorem 5. Let the assumptions of Proposition 4 hold. If there exists $a>0$ such that $\sum_{i=1}^{\infty} \varphi_i(aK_w(i)) \xrightarrow{\mathcal{W}} 0$, then for every $x \in \ell^{\varphi}$, there exists $a>0$ such that $\varrho(a(T_w x - \mathbf{F}x)) \xrightarrow{W} 0$ where $\mathbf{F}$ is given by the formula

$$\mathbf{F}x(i) = F(i,x(i)) \text{ for all } x \in \ell^{\varphi},\ i \in N.$$

P r o o f . Let $x \in \ell^{\varphi}$, $a>0$. From the assumptions we obtain

$$(41)\quad \varrho(a(T_w x - \mathbf{F}x)) = \sum_{i=0}^{\infty} \varphi_i\left(a\left(\sum_{j=1}^{i} K_w(j)F(i-j,x(i-j)) - F(i,x(i))\right)\right) =$$

$$= \varphi_0\left(a\left(K_w(0)F\left(0,x(0)\right) - F\left(0,x(0)\right)\right)\right) +$$

$$+ \sum_{i=1}^{\infty} \varphi_i\left(a\left(K_w(0)F\left(i,x(i)\right) - F\left(i,x(i)\right)\right) +\right.$$

$$\left. + \sum_{j=1}^{i} K_w(j)F\left(i-j,x(i-j)\right)\right) \leqslant$$

$$\leqslant \varphi_0\left(a\left(K_w(0)F\left(0,x(0)\right) - F\left(0,x(0)\right)\right)\right) +$$

$$+ \frac{1}{2}\varphi_1\left(2a\left(K_w(0)F\left(1,x(1)\right)\right) - F\left(1,x(1)\right)\right) +$$

$$+ \frac{1}{2}\varphi_1\left(2a\left(K_w(1)F\left(0,x(0)\right)\right)\right) + \dots +$$

$$+ \frac{1}{2(m-1)}\varphi_{m-1}\left(2(m-1)a\left(K_w(0)F\left(m-1,x(m-1)\right) - F\left(m-1,x(m-1)\right)\right)\right) +$$

$$+ \frac{1}{2(m-1)}\varphi_{m-1}\left(2(m-1)a\ K_w(m-1)F\left(0,x(0)\right)\right) + \dots +$$

$$+ \frac{1}{2(m-1)}\varphi_{m-1}\left(2(m-1)a\ K_2(2)F\left(m-2,x(m-3)\right)\right) +$$

$$+\frac{1}{2}\varphi_{m-1}\Big(2a\ K_w(1)F\big(m-2,x(m-2)\big)\Big)+$$

$$+\frac{1}{2}\sum_{i=m}^{\infty}\varphi_i\Big(2a\sum_{j=1}^{i-m+2}\Big(K_w(j)F\big(i-j,x(i-j)\big)\Big)\Big)+$$

$$+\frac{1}{2(m-1)}\sum_{i=m}^{\infty}\varphi_i\Big(2(m-1)a\Big(K_w(0)F\big(i,x(i)\big)-F\big(i,x(i)\big)\Big)\Big)+$$

$$+\frac{1}{2(m-1)}\sum_{i=m}^{\infty}\varphi_i\Big(2(m-1)a\ K_w(i)F\big(0,x(0)\big)\Big)+\dots+$$

$$+\frac{1}{2(m-1)}\sum_{i=m}^{\infty}\varphi_i\Big(2(m-1)a\ K_w(i-m+2)F\big(m-2,x(m-2)\big)\Big)\leqslant$$

$$\leqslant\sum_{i=0}^{\infty}\varphi_i\Big(2am\Big(K_w(0)F\big(i,x(i)\big)-F\big(i,x(i)\big)\Big)\Big)+$$

$$+\sum_{i=1}^{\infty}\varphi_i(2amLx(0)K_w(i))+\dots+\sum_{i=m-1}^{\infty}\varphi_i(2amLx(m-2)K_w(i))+$$

$$+\sum_{i=m-1}^{\infty}\varphi_i\Big(2a\sum_{j=1}^{i-m+2}K_w(j)F\big(i-j,x(i-j)\big)\Big).$$

It is easy to see that for sufficiently small $a>0$:

$$\sum_{i=1}^{\infty}\varphi_i(2amLx(0)K_w(i))\xrightarrow{\mathcal{W}}0 \tag{42}$$

$$\sum_{i=m-1}^{\infty}\varphi_i(2amLx(m-l-1)K_w(i))\xrightarrow{\mathcal{W}}0\quad\text{for}\quad l=1,2,\dots,m-2, \tag{43}$$

$$(44)\quad \sum_{i=0}^{\infty} \varphi_i\Big(2am(K_w(0)-1)F(i,x(i))\Big) \leqslant \varrho\Big(2amLx(K_w(0)-1)\Big) \xrightarrow{W} 0.$$

On the other hand, from the assumptions and from Proposition 3 we obtain

$$(45)\quad \sum_{i=m-1}^{\infty} \varphi_i\Big(2a \sum_{j=1}^{i-m+2} K_w(j)F(i-j,x(i-j))\Big) \leqslant$$

$$\leqslant \sum_{j=1}^{\infty} K_w(j) \sum_{i=j+m-2}^{\infty} \varphi_i(2a\sigma Lx(i-j)) \xrightarrow{W} 0$$

as $m \to \infty$ for sufficiently small $a>0$.

From (41)-(45) we have:

for every $x \in \ell^{\varphi}$ there exists $a>0$ such that $\varrho(a(T_w x - Fx)) \xrightarrow{W} 0$.

Let $K_w : N \times N \to R_+$ for $w \in W$ be such that

(46) there exists $M \in N$ such that $K_w(j,l) \leqslant K_w(j,M)$

for all $j,l \in N$ and $0 < \bar{\sigma}(w) = \sum_{j=0}^{\infty} K_w(j,M) \leqslant \sigma < \infty$.

$$(47)\quad \sigma(w) = \sup_{i \in N} \big| K_w(0,i) - 1 \big| \xrightarrow{W} 0.$$

Let $\underline{T}_w x = ((T_w x)_i)_{i=0}^{\infty}$, where $(\underline{T}_w x)_i = \sum_{j=0}^{i} K_w(i-j,i)F(j,x(j))$, where $F : N \times R \to R$.

P r o p o s i t i o n 5. Let $\varphi = (\varphi_i)_{i=0}^{\infty}$ be τ_+-bounded and let φ_i be convex for $i=0,1,2,\ldots$. Let the assumption (46) hold and $\dfrac{K_w(j,M)}{\bar{\sigma}(w)} \xrightarrow{W} 0$ for $j=1,2,\ldots$.

If there exists $L>0$ such that

$$|F(n,u) - F(n,v)| \leqslant L|u-v| \quad \text{for } n \in N,\ u,v \in R$$

and $F(n,0) = 0$ for every $n \in N$, then $\underline{T}_w: \ell^{\varphi} \longrightarrow \ell^{\varphi}$ for every $w \in W$ and $\underline{T} = (\underline{T}_w)_{w \in W}$ is $\mathcal{W}$-bounded.

The proof is analogous to that of Proposition 2 and Proposition 4 and we omit it.

T h e o r e m 6. Let the assumptions of Proposition 5 hold. If the assumption (47) holds and there exists $a > 0$ such that $\sum_{i=0}^{\infty} \varphi_i(a(K_w(i,M)) \xrightarrow{\mathcal{W}} 0$, then for every $x \in \ell^{\varphi}$ there is $a > 0$ for which $\varrho(a(\underline{T}_w x - Fx)) \longrightarrow 0$, where F is given by formula

$$Fx(i) = F(i,x(i)) \quad \text{for all} \quad x \in \ell^{\varphi},\ i \in N.$$

The proof is analogous to that of Theorem 4 and Theorem 5 and we omit it.

We give some final remarks:

1. If we take $F(t,x(t)) = x(t)$ in Theorem 2, we obtain Theorem 4 of [1].

2. If we take $F(t,x(t)) = x(t)$ in Theorem 5 we obtain Theorem 5 of [1].

3. If in Chapter 3 we restrict X to the set of all essentially-bounded functions, then in Theorems 2, 3, 4 ϱ-convergence can be replaced by norm convergence in X_{ϱ_s}.

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