DEMONSTRATIO MATHEMATICA
Vol. XVII No 2 1984

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GRACEFUL UNICYCLIC GRAPHS

1. The terminology used in this paper follows that of Bollobás [3]. In particular, V(G) (resp. E(G)) stands for the set of vertices (resp. edges) of a graph G, $|G|$ denotes its order and e(G) its size.

Let G be a graph. Any one-to-one function f on the set of vertices of G and with values in the set of integers is called a numbering of G. The value of an edge xy of a graph G is defined to be equal to $|f(x)-f(y)|$ and is denoted by f(xy). We say that a numbering f is graceful if it assigns to the vertices of G non-negative integers less than or equal to e(G) in such a way that no two edges of G get the same value. If in addition, for some integer k and for every edge $xy \in E(G)$ either $f(x) \leqslant k < f(y)$ or $f(y) \leqslant k < f(x)$ then we say that f is α-graceful. The vertex of G numbered with 0 is called the base of G (under the numbering f).

Investigations in the area of graceful graphs were originated in 1966 by Rosa [7] who considered various ways of numbering the vertices of a graph, among others α-valuations (in this paper referred to as α-graceful numberings) and β-valuations (only some years later called graceful numberings by Golomb [5]). His investigations arose from the conjecture of Ringel [6] that if T is any tree with n edges then the complete graph K_{2n+1} can be decomposed into 2n+1 subgraphs isomorphic to T. Rosa conjectured that every tree has a β-valuation (is graceful) and proved that this conjecture is stronger than that of Ringel. Since then many classes of trees (e.g. paths, caterpillars, symmetrical trees, olive trees) as well as many classes of other graphs (e.g. wheels, prisms, cycles with a chord) were shown to be graceful, however, the Rosa conjecture still remains a conjecture. For more details and further references the reader is referred to the survey papers by Bermond [1] and Bloom [2].

In this paper we exhibit some classes of graceful unicyclic graphs; among other things all dragons are shown to be graceful.

2. Let us start with the following theorem.

Theorem 1. Let G and H be graphs with disjoint sets of vertices. Assume that G is graceful, $v \in V(G)$ being its base under some graceful numbering g, H is α-graceful, $w \in V(H)$ being its base under some α-graceful numbering h. Then the graph F obtained by identifying v and w in $G \cup H$ is graceful.

Proof. Define $V_1 = \{x \in V(H): h(x) \leqslant k\}$ and $V_2 = \{x \in V(H): h(x) > k\}$, where k is an integer guaranteed by the definition of α-graceful numbering. Clearly V_1 and V_2 are independent sets of vertices of H. Put $r_1 = \max\{h(x): x \in V_1\}$ and $r_2 = \min\{h(x): x \in V_2\}$. We have $r_1+1 = r_2$ since otherwise no edge in H would have 1 as its value. Define a numbering f on V(F) as follows:

$$f(x) = \begin{cases} r_1-h(x) & x \in V_1 \\ e(G)+e(H)+r_2-h(x) & x \in V_2 \\ r_1+g(x) & x \in V(G). \end{cases}$$

First notice that f is well defined since $r_1+g(v) = r_1 = r_1-h(w)$. It is clear that f is a one-to-one function and for every $x \in V(F)$ $0 \leqslant f(x) \leqslant e(G)+e(H) = e(F)$. Moreover, one can easily see that if $e \in E(G)$ then $f(e) = g(e)$ and if $e \in E(H)$ then $f(e) = e(G) + e(H) + 1 - h(e)$.

Figure 1 (a) shows two graceful graphs G and H together with suitable numberings (one of them being an α-graceful numbering) and Figure 1 (b) shows the graph F obtained by identifying the bases of G and H, together with the numbering supplied by Theorem 1.

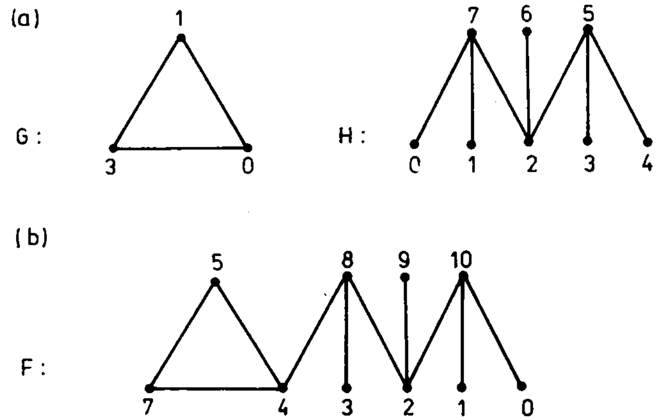

Figure 1

By a <u>dragon</u> $D_k(m)$ we mean the graph obtained by identyfying a vertex of the cycle C_k (i.e. the cycle of k vertices) with a pendant vertex of the path P_{m+1} (i.e. the path of m+1 vertices), and by $S_k(m)$ we mean the graph obtained by identifying a vertex of C_k with the vertex of degree m in the star $K(1,m)$.

C o r o l l a r y 2. Let H be a caterpillar and let $v \in V(H)$ be the base of H under some α-graceful numbering. (Caterpillars were shown to be α-graceful by Rosa [7]). Then the graph obtained by identifying in $C_k \cup H$, $k \equiv 0, 3 (\mathrm{mod}\ 4)$, v and an arbitrary vertex of C_k is graceful.

P r o o f . Rosa [7] proved that if $k \equiv 0, 3 (\mathrm{mod}\ 4)$ then C_k is graceful and clearly every vertex is its base under some graceful numbering. Hence the assertion follows from Theorem 1.

C o r o l l a r y 3. If $k \equiv 0, 3 (\mathrm{mod}\ 4)$ then $D_k(m)$ is graceful.

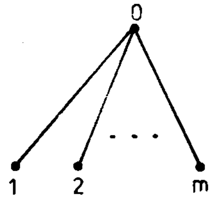

Figure 2

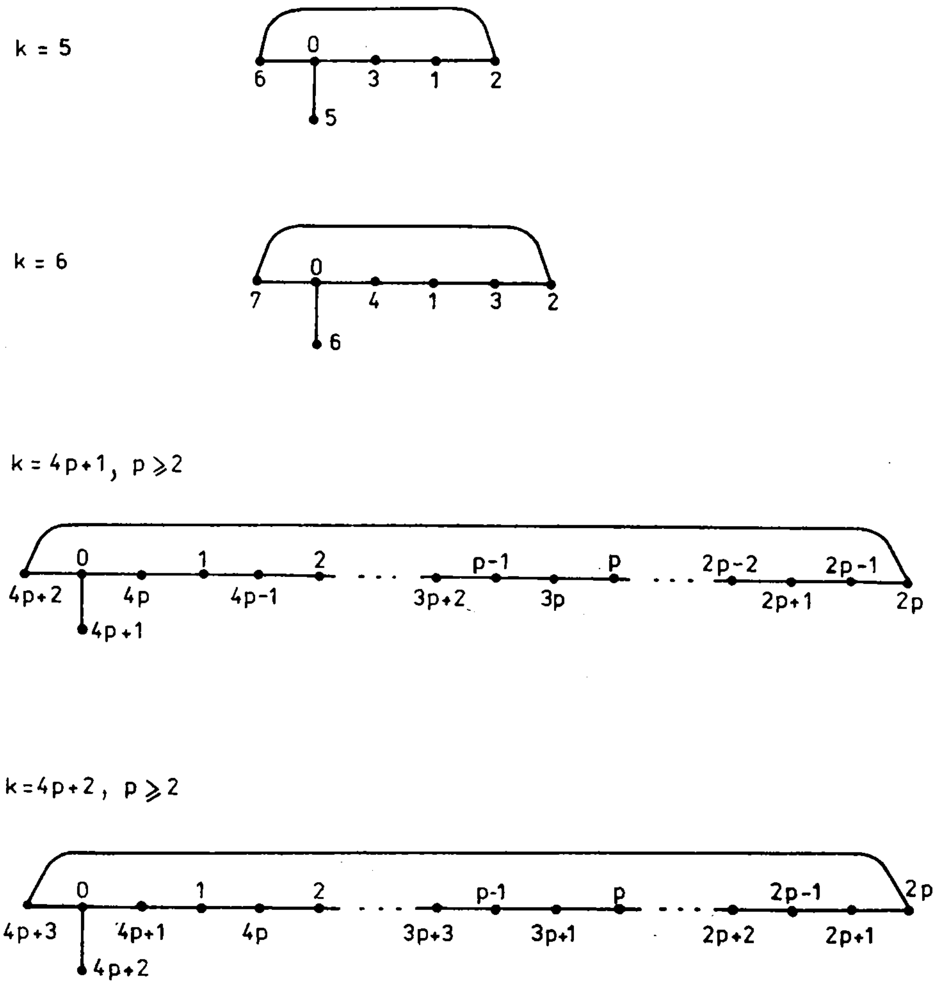

Figure 3

P r o o f . Rosa proved that every vertex of degree 1 in the path P_{m+1} is its base under some α-graceful numbering of P_{m+1}, and so the result follows from Corollary 2.

C o r o l l a r y 4. $S_k(m)$, $k \geqslant 3$, $m \geqslant 1$, is graceful. Moreover, the only vertex of $S_k(m)$ of degree m+2 is a base of it.

P r o o f . For $k \equiv 0,3 (\mathrm{mod}\ 4)$ the assertion follows from Corollary 2 since $K(1,m)$ is a caterpillar and there is an α-graceful numbering of its vertices in which the vertex of degree m receives 0 (see Figure 2).

So assume now that $k \equiv 1,2 (\mathrm{mod}\ 4)$. Suitable numberings of $S_k(1)$ are shown in Figure 3.

If $m > 1$ then $S_k(m)$ is obtained by identifying the vertex of degree 3 in $S_k(1)$, which is a base of $S_k(1)$, and the vertex of degree m-1 in $K(1,m-1)$. Hence $S_k(m)$ is graceful by Corollary 2.

C o r o l l a r y 5. Let H be a caterpillar and let v be its base under some α-graceful numbering. Then the graph F obtained by identifying in $S_k(m) \cup H$ v and the vertex of $S_k(m)$ of degree m+2 is graceful.

An illustration of this Corollary is given in Figure 4.

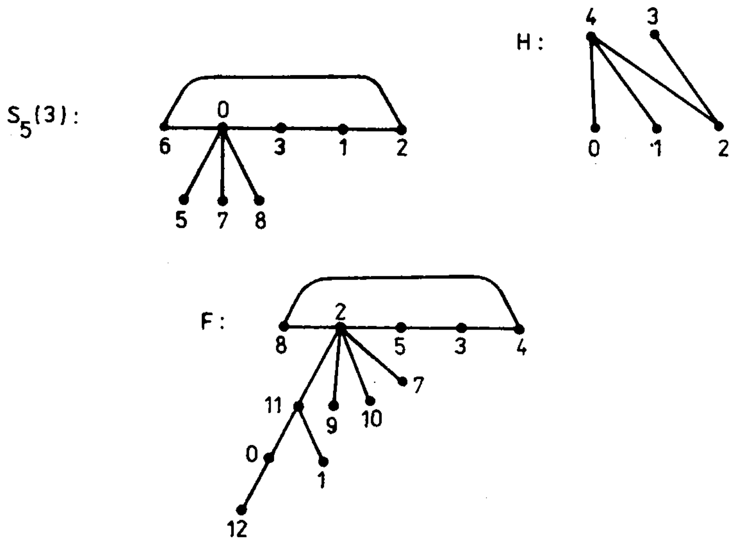

Figure 4

3. In this section we shall prove that all dragons are graceful.

Theorem 6. $D_k(m)$, $k \geqslant 3$, $m \geqslant 1$, is graceful.

Proof. Consider a dragon $D_k(m)$. By Corollary 3 we can restrict ourselves to the case of $k \equiv 1,2 \pmod 4$, we can also assume that $m \geqslant 2$, since $D_k(1) = S_k(1)$. We shall need the following lemma.

Lemma. For every $n \geqslant 2$ there is a function $f_n : \{1,2,\ldots,n+1\} \to \{0,2,3,\ldots,n+1\}$ such that $f_n(1) = 0$ and $\{|f_n(i)-f_n(i+1)| : i=1,2,\ldots,n\} = \{1,2,\ldots,n\}$.

Proof. For $2 \leqslant n \leqslant 11$ the functions given in Figure 5 have the required properties.

i	1	2	3	4	5	6	7	8	9	10	11	12
$f_2(i)$	0	2	3									
$f_3(i)$	0	3	4	2								
$f_4(i)$	0	4	3	5	2							
$f_5(i)$	0	5	2	6	4	3						
$f_6(i)$	0	6	2	7	4	5	3					
$f_7(i)$	0	7	2	8	4	5	3	6				
$f_8(i)$	0	8	2	9	4	7	3	5	6			
$f_9(i)$	0	9	2	10	4	8	3	6	5	7		
$f_{10}(i)$	0	10	2	11	4	9	3	7	6	8	5	
$f_{11}(i)$	0	11	2	12	4	10	3	8	5	9	7	6

Figure 5

For $n \geqslant 12$ define

$$f_n(i) = \begin{cases} f_{n-2}(i) & i \text{ is odd, } i \leqslant n-1 \\ f_{n-2}(i)+2 & i \text{ is even, } i \leqslant n-1 \\ f_{n-2}(n-1) & i = n \\ f_{n-2}(n-3) & i = n+1, \end{cases}$$

if n is odd and

$$f_n(i) = \begin{cases} f_{n-2}(i) & i \text{ is odd, } i \leqslant n-1 \\ f_{n-2}(i)+2 & i \text{ is even, } i \leqslant n-1 \\ f_{n-2}(n-4) & i = n \\ f_{n-2}(n-2) & i = n+1 \end{cases}$$

if n is even we leave to the reader the simple, but rather lengthy proof that the functions defined fulfil the requirements of the assertion.

We are ready to continue the proof of the theorem. In the sequel V and E will stand for the set of vertices and the set of edges of $D_k(m)$, respectively. We split our reasoning into five cases:

1. $k = 4p+1$, $m \geqslant 2p$ or m is odd.
2. $k = 4p+1$, $p > 1$, $2 < m \leqslant 2p-2$, m is even.
3. $k = 4p+1$, $p > 1$, $m = 2$.
4. $k = 4p+2$, $m > 2$.
5. $k = 4p+2$, $m = 2$.

We shall present only suitable numberings of the vertices of $D_k(m)$. Proofs that these numberings are actually graceful are left to the reader.

Case 1. Put $V = \{v_0, v_1, \ldots, v_{4p+m}\}$ and $E = \{v_s v_{s+1} : s=0,1,\ldots,4p+m-1\} \cup \{v_0 v_{4p}\}$ and let $c_1, c_2, \ldots, c_{4p+m}$ be the sequence of all integers from 1 to $4p+m+1$ except $2p$ in descending order. Number v_0 with $a_0 = 0$ and v_{s+1}, $0 \leqslant s \leqslant 4p+m-1$ with $a_{s+1} = a_s + (-1)^s c_{s+1}$.

In the remaining cases $c_1, c_2, \ldots, c_{k-1}$ stands for the sequence of all integers from m to $m+k$ in descending order, except exactly two of them, say b_1 and b_2, which will be defined in each of the cases independently.

Case 2. Put $V = \{v_0, v_1, \ldots, v_{4p+m}\}$, $E = \{v_i v_{i+1} : i=0,1,\ldots,4p-1,\ 4p+1,\ldots,4p+m-1\} \cup \cup \{v_0 v_{4p}, v_{4p-1} v_{4p+1}\}$, $b_1 = m+1$ and $b_2 = 2p+1$. Number v_0 with

$a_0 = 0$, v_i, $1 \leqslant i \leqslant 4p$, with $a_i = a_{i-1} + (-1)^{i-1} c_i$, and v_i, $4p+1 \leqslant i \leqslant 4p+m$, with $a_i = 2p+f_{m-1}(i-4p)$.

Case 3. Put $V = \{v_0, v_1, \ldots, v_{4p+2}\}$, $E = \{v_i v_{i+1} : i=0,1,\ldots,4p-1\} \cup \{v_{2p}v_{4p+1}, v_{4p+1}v_{4p+2}, v_0 v_{4p}\}$, $b_1 = 2p$ and $b_2 = 2p+1$. Number v_0 with $a_0 = 0$, v_i, $1 \leqslant i \leqslant 4p$, with $a_i = a_{i-1} + (-1)^{i-1} c_i$, v_{4p+1} with $a_{4p+1} = 3p+1$ and v_{4p+2} with $a_{4p+2} = 3p+2$.

Case 4. Put $V = \{v_0, v_1, \ldots, v_{4p+m+1}\}$, $E = \{v_i v_{i+1} : i=0,1,\ldots,4p,4p+2,\ldots,4p+m\} \cup \{v_0 v_{4p+1}, v_{4p} v_{4p+2}\}$, $b_1 = m+1$ and $b_2 = 2p+m$. Number v_0 with $a_0 = 0$, $1 \leqslant i \leqslant 4p+1$, with $a_i = a_{i-1} + (-1)^{i-1} c_i$, and v_i, $4p+2 \leqslant i \leqslant 4p+m+1$ with $a_i = 2p+m+1 - f_{m-1}(i-4p-1)$.

Case 5. Put $V = \{v_0, v_1, \ldots, v_{4p+3}\}$, $E = \{v_i v_{i+1} : i=0,1,\ldots,4p\} \cup \{v_0 v_{4p+1}, v_{2p-1} v_{4p+2}, v_{4p+2} v_{4p+3}\}$, $b_1 = 2p+4$ and $b_2 = 2p+5$. Number v_0 with $a_0 = 0$, v_i, $1 \leqslant i \leqslant 4p+1$, with $a_i = a_{i-1} + (-1)^{i-1} c_i$, v_{4p+2} with $a_{4p+2} = p$ and v_{4p+3} with $a_{4p+3} = p+1$.

Examples 1 to 5 illustrate all the five cases considered.

Example 1. $p = 1$, $k = 5$, $m = 3$, $c_i = 9-i$, $1 \leqslant i \leqslant 6$, $c_7 = 1$.

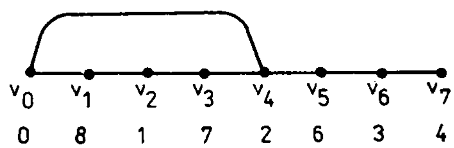

Figure 6

Example 2. $p = 3$, $k = 13$, $m = 4$, $b_1 = 5$, $b_2 = 7$, $c_i = 18-i$, $1 \leqslant i \leqslant 10$, $c_{11} = 6$, $c_{12} = 4$.

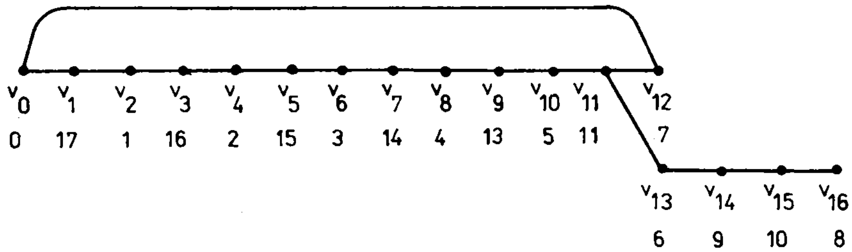

Figure 7

E x a m p l e 3. $p = 2$, $k = 9$, $m = 2$, $b_1 = 4$, $b_2 = 5$, $c_i = 12-i$, $1 \leqslant i \leqslant 6$, $c_7 = 3$, $c_8 = 2$.

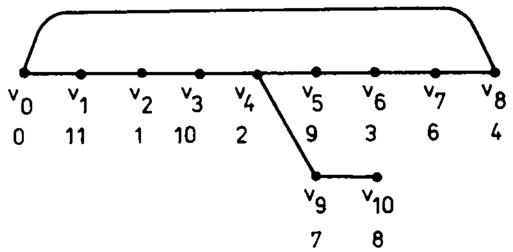

Figure 8

E x a m p l e 4. $p = 2$, $k = 10$, $m = 3$, $b_1 = 4$, $b_2 = 7$, $c_i = 14-i$, $1 \leqslant i \leqslant 6$, $c_7 = 6$, $c_8 = 5$, $c_9 = 3$.

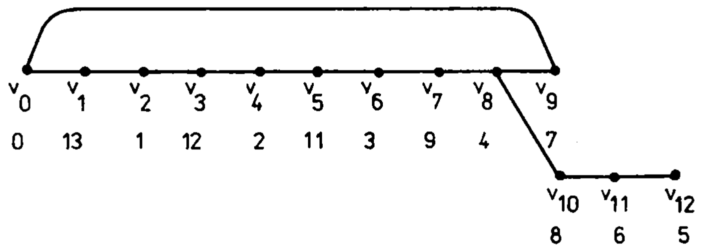

Figure 9

E x a m p l e 5. $p = 1$, $k = 6$, $m = 2$, $b_1 = 6$, $b_2 = 7$, $c_1 = 8$, $c_i = 7-i$, $2 \leqslant i \leqslant 5$.

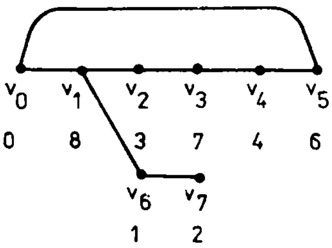

Figure 10

We conclude the paper with the conjecture, supported by the variety of graceful unicyclic graphs exibited here, that all unicyclic graphs except cycles C_n, $n \equiv 1,2 \pmod 4$, are graceful. There are also another examples supporting this conjecture, e.g. cycles with an edge leading to a vertex of degree 1, attached to each of their vertices. These unicyclic graphs were shown to be graceful by Frucht [4].

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Received November 3, 1982.