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PARTITION OF SETS OF THREE DIMENSIONAL EUCLIDEAN SPACE INTO SUBSETS OF TWO TIMES LESS DIAMETERS

In connection with the known Borsuk partition problem [1] (see the survey paper [5] of Grünbaum) Lenz [8] showed that any plane set of diameter $\leqslant 1$ can be partitioned into 7 subsets, each of diameter $\leqslant 1/2$ and that 7 is the smallest possible number.

The analogous question for Euclidean 3-space E^3 is more difficult. Borsuk [2] proved that any set $A \subset E^3$ of diameter $\leqslant 1$ can be partitioned into 48 subsets of diameters $\leqslant 1/2$. In the present note we improve the last estimation showing the possibility of a partition into 31 subsets.

Let $H_1, H_2 \subset E^2$ be half-lines with a common vertex and the angle α such that $0^\circ < \alpha < 180^\circ$. The half-lines will be called supporting a closed set $G \subset E^2$ if they have non-empty intersections with G and if the convex cone between them contains G. Remember that a line L is called a supporting line of G if $L \cap G \neq \emptyset$ and if G lies in one of closed half-planes bounded by L.

L e m m a 1. If two half-lines H_1, H_2 with a common vertex v and a constant angle α $(0^\circ < \alpha < 180^\circ)$ are turned around a bounded closed set $F \subset E^2$ permanently supporting it, then v moves continuously.

P r o o f . Let $\varepsilon > 0$. In the disk with the center v and radius ε we take a rhombus R with the center v and the pairs of sides parallel to H_1, H_2, respectively (see Fig.1). Denote by r the vertex of R lying in the convex cone between

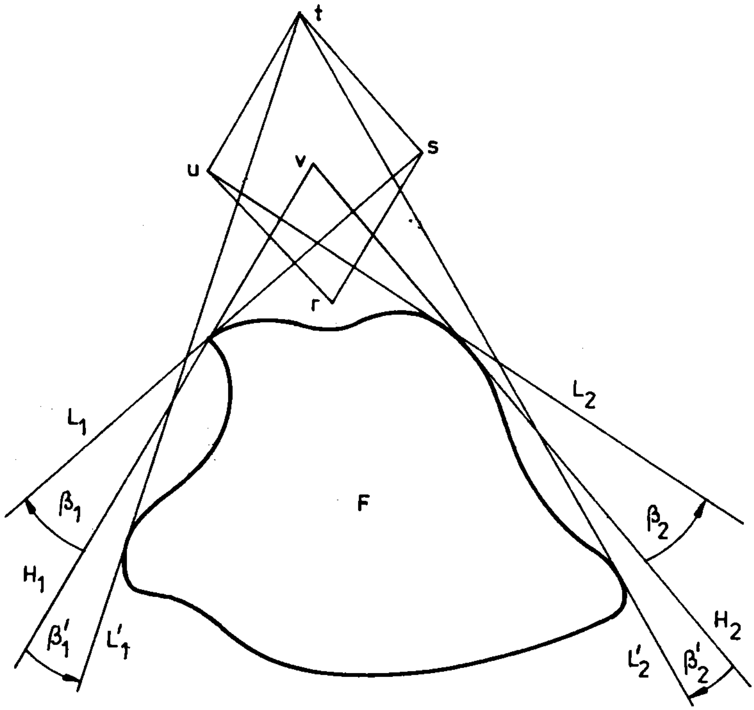

Fig.1

H_1, H_2, by t the symmetric vertex and by s, u the other ones. Note that there exist supporting lines L_1, L_1', L_2, L_2' of F such that $s \in L_1$, $t \in L_1'$, $u \in L_2$, $t \in L_2'$ and that L_1, L_1' cut the segment ru and L_2, L_2' cut the segment rs. Let β_i denote the angle between H_i, L_i and β_i' the angle between H_i, L_i', i=1,2. Put $\beta = \min\{|\beta_1|, |\beta_1'|, |\beta_2|, |\beta_2'|\}$. Obviously, $\beta > 0$. Now, if we turn our half-lines H_1, H_2 on any angle γ, where $|\gamma| < \beta$, then the new vertex w of them lies in the quadrilateral Q between L_1, L_2', L_2, L_2'. Since $Q \subset R$, we have $|vw| \leqslant \varepsilon$ which ends the proof.

A set $C \subset E^n$ such that any set S of diameter $\leqslant 1$ is congruent to a subset of C is called a universal cover for sets of diameter $\leqslant 1$.

An equivalent property to the case $\alpha = 120^\circ$ of Lemma 1 was used by Pál [9] to prove that the regular hexagon whose parallel sides are in the distance 1 is a universal cover for sets of diameter $\leqslant 1$. Now, we show a similar one:

L e m m a 2. The heptagon D obtained from the square, whose parallel sides are in the distance 1, by cutting off three triangles by lines perpendicular to the diagonals at distance 1/2 from the center is a universal cover for plane sets of diameter $\leqslant 1$.

P r o o f . Let $S \subset E^2$ be a set of diameter $\leqslant 1$. We cover S by a set T of constant width 1 (see e.g. [4], p.126). Consider three pairs of parallel supporting lines of T, the angles of which to a fixed axis are ψ, $\psi + 45^\circ$, $\psi + 90^\circ$, respectively. Denote by $H(\psi)$ the hexagon abcdef lying between the lines (Fig.2). From Lemma 1 it follows that the lenghts

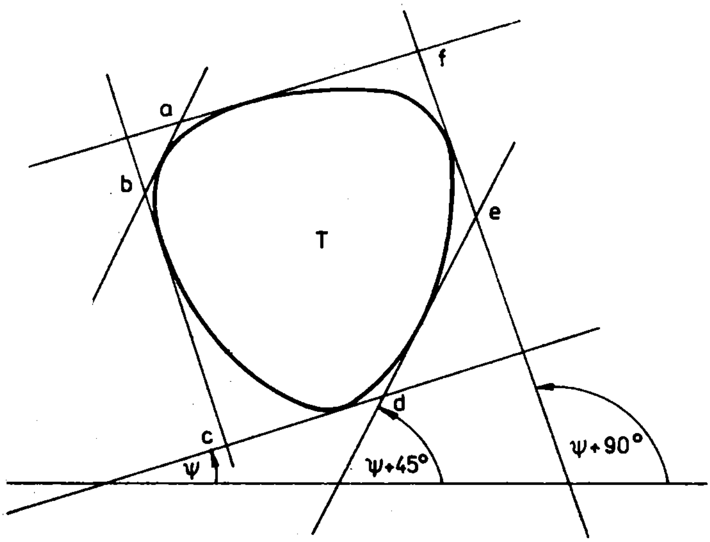

Fig.2

of the sides of $H(\psi)$ are continuous functions of ψ. Consequently, there exists ψ' such that in the hexagon $H(\psi') = a'b'c'd'e'f'$ we have $|a'b'| = |d'e'|$. Denote by G_1, G_2 the closed half-planes bounded by lines perpendicular to the segment $c'f'$ at the distance 1/2 from the center of $c'f'$ and containing the center. Since T is of constant width 1, we have $T \subset G$, where $G = G_1$ or $G = G_2$. Obviously, $G \cap H(\psi')$ is the searched heptagon D.

L e m m a 3. The heptagon D (and, consequently, any plane set of diameter $\leqslant 1$) can be covered by 10 sets, each of diameter $\delta = (\sqrt{14} - \sqrt{2})/6 \approx 0.38791$.

P r o o f . Denote by o the point lying in equal distances from all sides of D. Let D_1 be the disk of radius $\delta/2$ with the center o (Fig.3). Lines through o perpendicular to

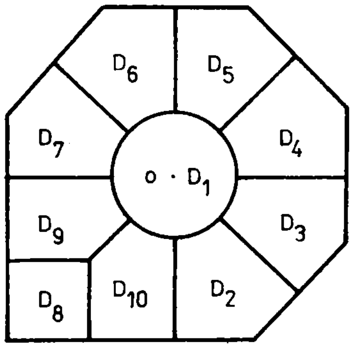

Fig.3

the sides of D determine a covering of $D \setminus D_1$ by 6 congruent sets $D_2, \ldots, D_7$ and an additional set V. As in Figure 3, we cover V by a square D_8 of diameter δ and two congruent sets D_9, D_{10}. Thus $D = D_1 \cup \ldots \cup D_{10}$. Obviously, D_1 and D_8 are of diameter δ. An elementary calculation shows that also $D_2, \ldots, D_7$ and D_9, D_{10} are of diameter δ.

T h e o r e m . Any set of E^3 of diameter $\leqslant 1$ can be partitioned into 31 subsets, each of diameter $\leqslant 1/2$.

P r o o f . Any subset of diameter $\leqslant 1$ lies in a set of constant width 1 ([4], p.126). Thus it is sufficient to show that any set A of constant width 1 can be covered by 31 sets of diameters $\leqslant 1/2$.

In virtue of the classical theorem of Jung [6], A is a subset of a closed ball of radius $\sqrt{6}/4$. Hence there exists a ball J of radius $\sqrt{6}/4$ containing A and having a point p of A in its boundary. Obviously, A is a subset of the closed ball K of radius 1 with the center p. Thus $A \subset J \cap K$ (i.e. $J \cap K$ is a universal cover [7]).

Let P_1, P_2, P_3, P_4 be planes perpendicular to the axis of symmetry of $J \cap K$ cutting $J \cap K$ in the distances, respectively, $\lambda + 3\omega$, $\lambda + 2\omega$, $\lambda + \omega$ and λ from p, where $\lambda = (\sqrt{6} - \sqrt{5})/4 \approx \approx 0.05336$, $\omega = \sqrt{4\sqrt{7} - 7}/6 \approx 0.31548$.

An elementary calculation shows that P_4 cuts off a part of diameter 1/2 from J, and so it cuts off a part of diameter $\leqslant 1/2$ from A.

The projection of A onto P_1 is a plane set Z of diameter 1. We cover Z by the heptagon D, as in Lemma 2, and cover D by sets $D_1, \ldots, D_{10}$ of diameter δ, as in Lemma 3. Consequently, the part of A lying between P_1 and P_4 can be covered by 30 cylindrical sets, each of diameter $\sqrt{\delta^2 + \omega^2} = 1/2$.

Thus $A \setminus W$ is covered by 31 sets of diameters $\leqslant 1/2$, where W denotes the small part of A cut off by the plane P_1. To end the proof, we show below that $M \cup W$ is of diameter 1/2, where M is the cylindrical set between P_1 and P_2 with the base D_1.

Since A is of constant width 1 in E^3, Z is a plane set of constant width 1. Moreover, Z is covered simultaneously by the heptagon D and by the disk which is the projection of J onto P_1. Hence the center u of the disk is in distances $\leqslant \sqrt{6}/4$ from the sides of the heptagon D. Therefore the distance between u and the center o of D_1 is not greater than $\tau = = (\sqrt{6}/4 - 1/2)\sqrt{2} \approx 0.15892$.

Note that W is contained in the segment of the ball K cut off by the plane P_1. The height of the segment equals $\mu = = 1 - \lambda - 3\omega \approx 0.00020$ and the radius of its base is equal to $\varrho = \sqrt{2\mu - \mu^2} \approx 0.01999$.

Since $\sqrt{(\omega+\mu)^2 + (\delta/2 + \tau + \varrho)^2} \approx 0.48855 < 1/2$ and since M is of diameter 1/2, we get that $M \cup W$ is of diameter 1/2. The proof is complete.

Note that if we slightly decrease λ and ω in the proof, our 31 parts are of diameters $< 1/2$.

It may be very difficult to find the smallest number of subsets of diameters $\leqslant 1/2$ on which can be partitioned any set $A \subset E^3$ of diameter $\leqslant 1$. It seems that more simple may be the question about the smallest number of subsets of diameters $\leqslant 1/2$ on which can be partitioned the ball (or, equivalently, the sphere) of diameter 1. From the paper [3] of Danzer it results that this number is not greater than 20. On the other hand, the number is not smaller than 12 because the vertices of the icosahedron inscribed in the ball are in the distances exceeding 1/2. Next, it appears the question concerning the spherical analogue of a known plane property: is the area of any subset of a sphere not greater than the area of the spherical segment of identical diameter? If this is true for spherical subsets of diameter equal to the radius of the sphere, then the lower estimation can be improved from 12 to 15.

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