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ON BARELY CONTINUOUS AND CLIQUISH MAPS

1. Let Y be a uniform space with a uniformity $\mathcal{U}$. Simultaneously we will consider Y as a topological space with the topology induced by $\mathcal{U}$. For any $y \in Y$, $A \subset Y$ and $V \in \mathcal{U}$ we denote $B(y,V) = \{x \in Y: (x,y) \in V\}$ and $B(A,V) = \bigcup\{B(y,V): y \in A\}$. Let X be a topological space. A subset A of a space X is said to be semi-open, if there exists an open set $U \subset X$ such that $U \subset A \subset \bar{U}$ (cf. [1,4]).

A map $f : X \to Y$ is called:

- quasi-continuous at a point $x_0 \in X$, if for every neighbourhood G of $f(x_0)$ there exists a semi-open set $A \subset X$ satisfying the conditions: $x_0 \in A$ and $f(A) \subset G$ (cf. [1,6,9]);

- cliquish at a point x_0, if for every neighbourhood U of x_0 and for every $V \in \mathcal{U}$ there exists an open non-empty set $U_1 \subset U$ such that $(f(x'),f(x'')) \in V$ for any $x',x'' \in U_1$ (cf. [2]). A map f is quasi-continuous (cliquish), if it has this property at every point.

If a uniformity $\mathcal{U}$ is given by a metric on Y, then the above definition coincides with the well-known definition of the cliquishness (cf. [6]).

Evidently every continuous map is quasi-continuous and every quasi-continuous map is cliquish but these classes of maps are different.

A map $f : X \to Y$ is barely continuous, if for every non-empty closed set $M \subset X$ the restriction $f_{/M}$ has at least one point of the continuity (cf. [5]).

T h e o r e m 1.1. Any barely continuous map $f:X \to Y$ is cliquish.

P r o o f . Let $x_0 \in X$ and let U be a neighbourhood of x_0. By x_1 we denote a point of the continuity of $f_{/\bar{U}}$. For arbitrary $V \in \mathcal{U}$ we choose $W \in \mathcal{U}$ such that $W = W^{-1}$ and $W^2 \subset V$ (cf. [8]). Then there exists a neighbourhood U_1 of x_1 in X such that $f(x) \in B(f(x_0),W)$ for each $x \in U_1 \cap \bar{U}$. Hence $(f(x'),f(x'')) \in W^2 \subset V$ for every $x',x'' \in U_1 \cap U$ and f is cliquish at a point x_0.

Note that the quasi-continuity and the barely continuity are independent properties; moreover the class of cliquish maps is greater then the class of barely continuous maps, as shown by the following examples.

E x a m p l e 1.2. Let us consider the set $X = [0,\infty)$ with the topology $T = \{\emptyset,X\} \cup \{(r,\infty): r>0\}$ and let R be the space of real numbers with the natural metric. By Q we denote the set of rational numbers. The function $f:X \to R$ given by

$$f(x) = \begin{cases} \frac{1}{n} & x \in [n-1,n) \cap Q \\ \\ \frac{1}{n+1} & x \in [n-1,n) \setminus Q, \quad n = 1,2,\dots \end{cases}$$

is cliquish. The set $M = [0,1]$ is closed in X, the function $f_{/M}$ has no continuity points, so f is not barely continuous.

E x a m p l e 1.3. We take $X = (-\infty,\infty)$ with the topology $T = \{\emptyset, [0,1), [1,2], X\} \cup \{[0,2]\}$. Semi-open sets in this space are of the form: $[0,1) \cup A$, $[1,2] \cup B$, $[0,2] \cup C$ where A, B, C are arbitrary sets. Let R be the space of real numbers with natural metric and let Q be the set of rational numbers. The function $f:X \to R$ given by the formula

$$f(x) = \begin{cases} 0 & x \in [0,1) \cup (R \setminus [0,2]) \cap Q \\ \\ 1 & x \in [1,2] \cup (R \setminus [0,2]) \cap (R \setminus Q) \end{cases}$$

is quasi-continuous. Let us put $M = (-\infty,0)\cup(2,\infty)$, then $f_{/M}$ has no continuity points; so f is not barely continuous.

Example 1.4. The function $f:R \to R$ given by $f(x) = 1$ if x is the natural number and $f(x) = 0$ in the other case is barely continuous but is not quasi-continuous.

In the sequel for a map $f:X \to Y$, we will denote by $C(f)$ the set of all continuity points of f.

Lemma 1.5. If $C(f)$ is a dense set, then f is a cliquish map.

Lemma 1.6. If a uniformity $\mathcal{U}$ on Y has a countable base, then for each cliquish map $f:X \to Y$ the set $X\setminus C(f)$ is of the first category.

These Lemmas follow by [2, Theorem 5] and [2, Theorem 9] respectively.

A topological space X is said to be a Baire space in the narrow sense, if every closed subspace of X is a Baire space (cf. [3]).

Theorem 1.7. Let X be a Baire space in the narrow sense and let a uniformity $\mathcal{U}$ on Y have a countable base. If $f:X \to Y$ is a barely continuous map, then for any non-empty closed set $M\subset X$ the set $C(f_{/M})$ is dense in M.

Proof. Let M be any closed non-empty subset of X. Since $f_{/M}:M \to Y$ is barely continuous Theorem 1.1 and Lemma 1.6 imply that $M\setminus C(f_{/M})$ is of the first category in M and the proof is completed.

Corollary 1.8. [5]. Let X be a Baire space in the narrow sense and let Y be a metric space. If $f:X \to Y$ is barely continuous, then for each closed non-empty set $M\subset X$ the set $C(f_{/M})$ is dense in M.

Let $f:X \to Y$ be a cliquish map and let $M\subset X$. If M is semi-open or dense, then $f_{/M}$ is cliquish. If M is a closed set, then $f_{/M}$ need not be cliquish, as shown by Example 1.3.

Theorem 1.9. Let X be a Baire space in the narrow sense and let a uniformity on Y have a countable base. A map $f:X \to Y$ is barely continuous if and only if for every non-empty closed set $M\subset X$ a map $f_{/M}:M \to Y$ is cliquish.

P r o o f . If f is barely continuous and $M \subset X$ is a closed non-empty set, then by Theorem 1.7 we have $C(f_{/M})$ is dense in M. So Lemma 1.5 implies that $f_{/M}$ is cliquish.

Now let $f_{/M}$ be cliquish for any closed non-empty set $M \subset X$. Then by Lemma 1.6 we obtain $C(f_{/M}) \neq \emptyset$, this f is barely continuous.

A sequence $\{f_n : n=1,2,\ldots\}$ of maps $f_n : X \longrightarrow Y$ is convergent to a map f at a point $x \in X$, if for every $V \in \mathcal{U}$ there exists n_o such that $(f_n(x), f(x)) \in V$ for $n > n_o$. A map f is said to be the limit of a sequence $\{f_n : n = 1,2,\ldots\}$, if that sequence converges to f at every point; then we write $f = \lim_{n \to \infty} f_n$.

Simple examples show that the limit of a sequence of barely continuous maps need not be cliquish.

T h e o r e m 1.10. Let X be a Baire space in the narrow sense and let a uniformity on Y have a countable base. If $f_n : X \longrightarrow Y$ is barely continuous map for $n = 1,2,\ldots$ and $f = \lim_{n \to \infty} f_n$, then the following conditions are equivalent:

a) f is barely continuous,

b) for every $V \in \mathcal{U}$ and for every closed set $M \subset X$ and open set $U \subset X$ such that $M \cap U \neq \emptyset$ there exist a number m and an open set $U' \subset U$ such that $M \cap U' \neq \emptyset$ and $(f_m(x), f(x)) \in V$ for any $x \in M \cap U'$.

P r o o f . We assume that f is barely continuous. Let $M \subset X$ be closed and let $U \subset X$ be an open set for which $M \cap U \neq \emptyset$. For any $V \in \mathcal{U}$ let us take $W \in \mathcal{U}$ such that $W = W^{-1}$ and $W^3 \subset V$. The sets $C(f_{/\overline{M \cap U}})$ and $C(f_{n/\overline{M \cap U}})$ for $n = 1,2,\ldots$ are dense G_δ in $\overline{M \cap U}$, thus the set $C(f_{/\overline{M \cap U}}) \cap \bigcap_{n=1}^{\infty} C(f_{n/\overline{M \cap U}})$ in dense in $M \cap U$. Let $x_1 \in C(f_{/\overline{M \cap U}}) \cap \bigcap_{n=1}^{\infty} C(f_{n/\overline{M \cap U}})$. There exists a number m such that $(f_m(x_1), f(x_1)) \in W$. Moreover, there exists a neighbourhood U_o of x_1 such that $(f_m(x), f_m(x_1)) \in$ $\in W$ and $(f(x), f(x_1)) \in W$ for $x \in U_o \cap \overline{M \cap U}$. We denote $U' =$

$= U_0 \cap U$; hence we have $U' \subset U$, $U' \cap M \neq \emptyset$ and $(f_m(x), f(x)) \in W^3 \subset V$ for any $x \in U' \cap M$.

Conversely, we assume that the condition (b) holds. By $\{V_n: n=1,2,\dots\}$ we denote a base for the uniformity $\mathcal{U}$ on Y such that $V_n = V_n^{-1}$ for $n = 1,2,\dots$ Let M be closed and let U be an open set such that $M \cap U \neq \emptyset$. According to assumptions we can choose a decreasing sequence $\{U_n: n=1,2,\dots\}$ of open sets and a sequence $\{m_n: n=1,2,\dots\}$ of natural numbers such that $U_n \cap M \neq \emptyset$ for $n=1,2,\dots$ and $(f_{m_n}(x), f(x)) \in V_n$ for $x \in U_n \cap M$. Take any $V \in \mathcal{U}$ and V_n such that $V_n^3 \subset B$. Then there exists m_n such that $(f_{m_n}(x), f(x)) \in V_n$ for $x \in U_n \cap M$. Since the sets $C(f_{k/M})$ for $k = 1,2,\dots$ are dense G_δ in M their intersection is dense in M. Let $x_0 \in U_n \cap \bigcap_{k=1}^{\infty} C(f_{k/M})$, then we have $(f_{m_n}(x_0), f(x_0)) \in V_n$. Furthermore, there exists a neighbourhood U_0 of x_0 such that $(f_{m_n}(x_0), f_{m_n}(x)) \in V_n$ for any $x \in U_0 \cap M$. Hence $(f(x_0), f(x)) \in V_n^3 \subset V$ for $x \subset U_0 \cap U_n \cap M$, so $x_0 \in C(f_{/M})$ and f is barely continuous.

C o r o l l a r y 1.11. Let X be a Baire space in the narrow sense and let a uniformity $\mathcal{U}$ on Y have a countable base. If a sequence $\{f_n: n=1,2,\dots\}$ of barely continuous maps $f_n: X \to Y$ is uniformly convergent to a map f, then f is barely continuous.

2. Let X and Y be topological spaces and let $\mathcal{U}$ be a uniformity on Z. For any map $f: X \times Y \to Z$ we will denote by f_x and f^y maps given by: $f_x(y) = f(x,y)$ and $f^y(x) = f(x,y)$ for $x \in X$, $y \in Y$.

T h e o r e m 2.1. Let X be a Baire space and let Y be locally second countable. If $f: X \times Y \to Z$ is such that f_x is cliquish for any $x \in X$ and f^y is quasi-continuous for any $y \in Y$, then f is cliquish.

P r o o f . Let $(x_0, y_0) \in X \times Y$ and let $U \times V$ be any neighbourhood of it point. We can assume that V has a countable base $\{G_n: n=1,2,\ldots\}$. For arbitrary $W_1 \in \mathcal{U}$ we take $W \in \mathcal{U}$ which satisfies $W = W^{-1}$ and $W^8 \subset W_1$. We denote

$$H_n = \{x \in U: (f(x,y'),f(x,y'')) \in W \text{ for } y',y'' \in G_n\}.$$

It is easy to see that $U = \bigcup_{n=1}^{\infty} H_n$. Since U is of the second category, $\operatorname{Int} \overline{H}_n \neq \emptyset$ for certain n. Let $U_1 = U \cap \operatorname{Int} \overline{H}_n$ and $b \in G_n$. By the quasi-continuity of f^b there exists an open non-empty set $U_2 \subset U_1$ such that

(1) $(f(x',b),f(x'',b)) \in W$ for every $x',x'' \in U_2$.

Let (x_1,y_1), $(x_2,y_2) \in (U_2 \cap H_n) \times G_n$. From the condition $x_1,x_2 \in H_n$ it follows that $(f(x_1,y_1),(f(x_1,b)) \in W$ and $(f(x_2,b),f(x_2,y_2)) \in W$. Applying (1) we have

(2) $(f(x_1,y_1),f(x_2,y_2)) \in W^3$ for every $(x_1,y_1),(x_2,y_2) \in (U_2 \cap H_n) \times G_n$.

Now we take $(x_1,y_1) \in (U_2 \cap H_n) \times G_n$ and $(x_2,y_2) \in (U_2 \setminus H_n) \times G_n$. Since f^{y_2} is quasi-continuous at a point x_2 there exists a non-empty open set $U_3 \subset U_2$ such that $(f(x_2,y_2),f(x',y_2)) \in W$ for $x' \in U_3$. Hence

(3) for each $(x_2,y_2) \in (U_2 \setminus H_2) \times G_n$ there exists a point $(x_2',y_2) \in (U_2 \cap H_n) \times G_n$ such that $(f(x_2,y_2),f(x_2',y_2)) \in W$.

The condition $x_1,x_2' \in H_n$ implies $(f(x_1,y_1),f(x_1,b)) \in W$ and $(f(x_2',y_2),f(x_2',b)) \in W$. Applying (1) and (3) we have

(4) $(f(x_1,y_1),f(x_2,y_2)) \in W^4$ for every $(x_1,y_1) \in (U_2 \cap H_n) \times G_n$ and $(x_2,y_2) \in (U_2 \setminus H_n) \times G_n$.

Finally let (x_1,y_1), $(x_2,y_2) \in (U_2 \setminus H_n) \times G_n$. By (4) it follows that

(5) $$(f(x_1,y_1),f(x_2,y_2)) \in W^8.$$

Thus by (2), (4) and (5) we have $(f(x_1,y_1),f(x_2,y_2)) \in W_1$ for all points (x_1,y_1), $(x_2,y_2) \in U_2 \times G_n \subset U \times V$ and the proof is completed.

3. Let X, Y be topological vector spaces and let $\mathcal{W}$ be the neighbourhood filter of $0 \in Y$. The cliquishness of a map $f: X \to Y$ at a point $x_0 \in X$ in this case means that for every neighbourhood U of x_0 and for every $V \in \mathcal{W}$ there exists a non-empty open set $U' \subset U$ such that $f(x') - f(x'') \in V$ for any $x', x'' \in U'$.

Theorem 3.1. Let X and Y be topological vector spaces. For any linear map $f: X \to Y$ the following properties are equivalent:

a) f is continuous,

b) f is quasi-continuous,

c) f is barely continuous,

d) f is cliquish.

Proof. It is sufficient to show that every cliquish linear map is continuous. Let $V \in \mathcal{W}$. Since f is cliquish at 0 there exists an open non-empty set $G \subset X$ such that $f(x') - f(x'') \in V$ for each $x', x'' \in G$. We take any point $x_1 \in G$; there exists a neighbourhood U of $0 \in X$ such that $x_1 + U \subset G$. Then for every $x \in U$ we have $f(x_1+x) - f(x_1) \in V$, i.e. $f(U) \subset V$ so f is continuous.

Theorem 3.2. Let both topological vector spaces X and Y have a countable neighbourhood filter of 0. Moreover, let X, Y be Baire spaces and let Z be topological vector space. If $f: X \times Y \to Z$ is a bilinear map such that f_x and f^y are cliquish for every $x \in X$, $y \in Y$, then f is continuous.

Proof. By Theorem 3.1 it follows that f_x and f^y are continuous for each $x \in X$ and $y \in Y$. Thus according to [7, p.88] f is continuous.

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