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ON SOME BOUNDARY VALUE PROBLEM FOR THE EQUATION $\Delta u(X)-c^2(X)=0$

1. Consider the equation

$$\Delta u(X) - c^2u(X) = 0, \tag{1}$$

$X = (x_1,x_2,\dots,x_n)$, c - positive constant, in the half-space

$$D = \{ X = (x_1,x_2,\dots,x_n),\ x_n > 0\} \quad (n \geqslant 2),$$

and let $f(X') = f(x_1,x_2,\dots,x_{n-1})$, $X' = (x_1,x_2,\dots,x_{n-1}) \in R^{n-1}$, be a continuous and bounded function on the space R^{n-1}, $|f(X')| \leqslant M$ for $X' \in R^{n-1}$.

In this paper we give the particular solution $u(X) \in C^2(D)$ of the equation (1) with the boundary condition

$$\lim_{X\to X_0} \left[D_{x_n} u(X) - hu(X)\right] = f(X_0'), \tag{2}$$

where $X_0 = (x_1^0,x_2^0,\dots,x_{n-1}^0,0)$ is a fixed point, $X_0' = (x_1^0,x_2^0,\dots,x_{n-1}^0)$ and h is some positive constant.

In this problem we will use properties of Mac Donald functions, modified Bessel functions $K_p(x)$ (see [4] §52). These functions satisfy the following equations

$$K_{p+1}(x) - K_{p-1}(x) = \frac{2p}{x}\cdot K_p(x), \tag{3}$$

$$\frac{d}{dx}\left(x^{-p}K_p(x)\right) = -x^{-p}K_{p+1}(x). \tag{4}$$

2. Let

$$(5)\qquad g(r) = \frac{K_p(cr)}{r^p}, \qquad r^2 = \sum_{i=1}^{n} (x_i-a_i)^2, \quad X \neq A(a_1,a_2,\dots,a_n).$$

L e m m a 1. The function (5) of index $p \neq 0$ satisfies equation (1) on $R^n \setminus A$ only if $p = \frac{n-2}{2}$.

P r o o f . For the function $g(r)$ we have

$$\Delta g(r) = g''(r) + \frac{n-1}{r} g'(r).$$

From properties (3), (4) of the functions $K_p(x)$, we infer that

$$g'(r) = -\frac{K_{p+1}(cr)}{r^p} \cdot c = -\frac{c}{r^p}\left[\frac{2p}{cr} \cdot K_p(cr) + K_{p-1}(cr)\right] =$$

$$= -\frac{K_p(cr)}{r^p} \cdot \frac{2p}{r} - \frac{K_{p-1}(cr)}{r^{p-1}} \cdot \frac{c}{r},$$

$$g''(r) = \frac{K_p(cr)}{r^{p-1}} \cdot \frac{c^2}{r} + \frac{K_{p-1}(cr)}{r^{p-1}} \cdot \frac{c}{r^2} + \frac{K_{p+1}(cr)}{r^p} \cdot \frac{2pc}{r} + \frac{K_p(cr)}{r^p} \cdot \frac{2p}{r} =$$

$$= g(r)c^2 + \frac{K_{p-1}(cr)}{r^{p-1}} \cdot \frac{c}{r^2} + \frac{2pc}{r^{p+1}}\left[\frac{2p}{cr} \cdot K_p(cr) + K_{p-1}(cr)\right] +$$

$$+ \frac{K_p(cr)}{r^p} \cdot \frac{2p}{r^2} = g(r)c^2 + \frac{K_{p-1}(cr)}{r^{p-1}} \cdot \frac{2pc+c}{r^2} + \frac{K_p(cr)}{r^p} \cdot \frac{4p^2+2p}{r^2}.$$

Next, we have

$$\Delta g(r) = g(r)c^2 + \frac{K_{p-1}(cr)}{r^{p-1}} \cdot \frac{2pc+c}{r^2} + \frac{K_p(cr)}{r^p} \cdot \frac{4p^2+2p}{r^2} +$$

$$+ \frac{n-1}{r} \cdot \left[-\frac{K_p(cr)}{r^p} \cdot \frac{2p}{r} - \frac{K_{p-1}(cr)}{r^{p-1}} \cdot \frac{c}{r}\right] =$$

$$= g(r)c^2 + \frac{K_{p-1}(cr)}{r^{p-1}} \cdot \frac{2pc+c-c(n-1)}{r^2} + \frac{K_p(cr)}{r^p} \cdot \frac{4p^2+2p-2p(n-1)}{r^2}.$$

Note, that the functions $\frac{K_{p-1}(cr)}{r^{p-1}}$ and $\frac{K_p(cr)}{r^p}$ are linearly independent and $g(r) = g(r)c^2$ only when $2pc+c-c(n-1) = 0$ and $4p^2+2p-2p(n-1) = 0$ or $2p+2-n = 0$.

3. Let us take a function $g(r)$ of index $p = \frac{n-2}{2}$, $n = 3,5,7,\ldots$. Using (see [3] form. 223) the formula

$$K_{\frac{2v+1}{2}}(x) = \sqrt{\frac{\pi}{2x}} \cdot e^{-x} \sum_{k=0}^{v} \frac{(v+k)!}{k!(v-k)!} \cdot \left(\frac{1}{2x}\right)^k, \quad v=0,1,2,\ldots,$$

we have

L e m m a 2. If $q = \frac{n}{2}$, $n = 1,3,5,\ldots$, then

$$\frac{K_q(cr)}{r^q} = Q_n\left(\frac{1}{r}\right)\cdot e^{-cr}, \tag{6}$$

where

$$Q_n\left(\frac{1}{r}\right) = a_n\cdot\left(\frac{1}{r}\right)^n + a_{n-1}\cdot\left(\frac{1}{r}\right)^{n-1} + \ldots + a_{\frac{n+1}{2}}\cdot\left(\frac{1}{r}\right)^{\frac{n+1}{2}} \tag{7}$$

is a polynomial with coefficients

$$a_n = \frac{(n-2)!}{2^{\frac{n-3}{2}}\left(\frac{n-3}{2}\right)!\,c^{\frac{n}{2}}}\cdot\sqrt{\frac{\pi}{2}}, \quad a_{\frac{n+1}{2}} = \sqrt{\frac{\pi}{2c}} \quad (n \geqslant 3).$$

From (6), (7) for $p = \frac{n-2}{2}$, $n = 3,5,7,\ldots$, we have

$$g(r) = Q_{n-2}\left(\frac{1}{r}\right)\cdot e^{-cr}.$$

Similarly using (3), (4), we obtain formulas for partial derivatives up to the second order (of upper index p) for the

$$D_{x_i}g(r) = -r\cdot\frac{K_{p+1}(cr)}{r^{p+1}}\cdot cD_{x_i}r = -r\, Q_n\left(\frac{1}{r}\right)e^{-cr}\cdot cD_{x_i}r,$$

$$i = 1,2,\ldots,n,$$

$$D_{x_ix_j}g(r) = \left[-\frac{K_{p+1}(cr)}{r^{p+1}}\cdot c + r\cdot\frac{K_{p+2}(cr)}{r^{p+1}}\cdot c^2\right]D_{x_i}r\cdot D_{x_j}r -$$

$$- r\cdot\frac{K_{p+1}(cr)}{r^{p+1}}\cdot cD_{x_ix_j}r =$$

$$= \left[-\frac{K_{p+1}(cr)}{r^{p+1}}\cdot c + \frac{c^2}{r^p}\left(\frac{2(p+1)}{cr}\cdot K_{p+1}(cr) + K_p(cr)\right)\right] D_{x_i}r\cdot D_{x_j}r -$$

$$- r\cdot\frac{K_{p+1}(cr)}{r^{p+1}}\cdot cD_{x_ix_j}r =$$

$$= \left[-\frac{K_{p+1}(cr)}{r^{p+1}}\cdot c + cn\frac{K_{p+1}(cr)}{r^{p+1}} + c^2\cdot\frac{K_p(cr)}{r^p}\right]D_{x_i}r\cdot D_{x_j}r -$$

$$- r\cdot\frac{K_{p+1}(cr)}{r^{p+1}}\cdot cD_{x_ix_j}r =$$

$$= \left[c(n-1)Q_n\left(\frac{1}{r}\right) + c^2Q_{n-2}\left(\frac{1}{r}\right)\right] e^{-cr}\cdot D_{x_i}r\cdot D_{x_j}r -$$

$$- crQ_n\left(\frac{1}{r}\right)e^{-cr}\cdot D_{x_ix_j}r, \qquad i,j = 1,2,\dots,n.$$

In that case we have the estimates:

$$\text{(8a)} \qquad \left|D_{x_i}g(r)\right| \leqslant crQ_n\left(\frac{1}{r}\right)e^{-cr}, \quad i = 1,2,\dots,n,$$

$$\text{(8b)} \quad \left|D_{x_ix_j}g(r)\right| \leqslant \left[c^2Q_{n-2}\left(\frac{1}{r}\right) + cnQ_n\left(\frac{1}{r}\right)\right] e^{-cr}, \quad i,j = 1,2,\dots,n.$$

4. Using (5) we construct the following function of points $Y,X \in D$

$$\text{(9)} \quad G(Y:X) = g(r_1) + g(r_2) - 2h\cdot\int_{y_n+x_n}^{+\infty} e^{-h(t-y_n-x_n)}\cdot g(r_3)dt, \quad Y\neq X,$$

$p = \frac{n-2}{2}$, n - odd number, $n>3$, where h is a positive constant and

$$r_1^2 = \sum_{i=1}^{n} (y_i-x_i)^2, \quad r_2^2 = \sum_{i=1}^{n-1} (y_i-x_i)^2 + (y_n+x_n)^2,$$

$$r_3^2 = \sum_{i=1}^{n-1} (y_i-x_i)^2 + t^2.$$

Letting $\tau = t - y_n - x_n$ in the improper integral above, we have

$$\text{(9a)} \qquad G(Y:X) = g(r_1) + g(r_2) - 2h\cdot \int_0^{+\infty} e^{-h\tau} g(\bar{r}_3)d\tau,$$

where

$$\bar{r}_3^2 = \sum_{i=1}^{n-1} (y_i-x_i)^2 + (\tau+y_n+x_n)^2.$$

The function (9) has the following properties:

1. It is symmetric with respect to $Y,X \in D$, i.e. $G(Y:X) = G(X:Y)$.

2. It is of class $C^2(D)$ with respect to Y and X, because by (8a), (8b) the improper integral in (9a) and the corresponding integrals of partial derivatives are uniformly convergent in any bounded and closed domain $E \subset D$ with respect to X and Y, hence they are almost uniformly convergent with respect to X,Y, $X \neq Y$, in the domain D.

3. It satisfies the identity

$$\text{(10)} \quad D_{y_n} G(Y',0:X) - hG(Y',0:X) = 0, \; Y' = (y_1,y_2,\ldots,y_{n-1}), \; X \in D.$$

Indeed, with $X \in D$, $Y \in \bar{D}$, $Y \neq X$

$$D_{y_n} G(Y:X) = \frac{-K_{p+1}(cr_1)}{r_1^p} c \cdot \frac{y_n-x_n}{r_1} - \frac{K_{p+1}(cr_2)}{r_2^p} c \cdot \frac{y_n+x_n}{r_2} + 2hg(r_2) -$$

$$- 2h^2 \cdot \int_{y_n+x_n}^{+\infty} e^{-h(t-y_n-x_n)} g(r_3)dt.$$

In that case

$$D_{y_n}G(Y',0:X) = 2hg(r) - 2h^2\cdot\int_{x_n}^{+\infty} e^{-h(t-x_n)}g(r_3)dt,\ r^2 = \sum_{i=1}^{n-1}(y_i-x_i)^2 + x_n^2.$$

Similarly,

$$G(Y',0:X) = 2g(r) - 2h\cdot\int_{x_n}^{+\infty} e^{-h(t-x_n)}g(r_3)dt,$$

which implies (10).

Moreover, by Lemma 1 the function (9) satisfies equation (1) with respect to X for every $Y \neq X$.

(11) $u(X) = \frac{1}{2\omega_n}\cdot\int_{R^{n-1}} f(Y')G(Y',0:X)dY'$, $X \in D$, ω_n - a number.

T h e o r e m 1. The function (11) is of class $C^2(D)$ and satisfies equation (1) in D.

P r o o f . Having

$$\int_{R^{n-1}} f(Y')G(Y',0:X)dY' =$$

$$= 2\cdot\int_{R^{n-1}} f(Y')\left[g(r) - h\cdot\int_{x_n}^{+\infty} e^{-h(t-x_n)}g(r_3)dt\right]dY',$$

by (6), (8a), (8b) and $|f(Y')| \leqslant M$ in R^{n-1}, we obtain the inequalities

$$\int_{R^{n-1}} |f(Y')G(Y',0:X)|\,dY' \leqslant 4MQ_{n-1}\left(\frac{1}{x_n}\right)\cdot\int_{R^{n-1}} e^{-cr}dY',$$

$$\int_{R^{n-1}} \left| f(Y')D_{x_i}G(Y',0:X)\right| dY' \leqslant 4cM\bar{Q}_{n-1}\left(\frac{1}{x_n}\right)\cdot \int_{R^{n-1}} e^{-cr}dY',$$

where $\bar{Q}_{n-1}\left(\frac{1}{r}\right) = rQ_n\left(\frac{1}{r}\right)$, $i = 1,2,\dots,n-1$,

$$\int_{R^{n-1}} \left| f(Y')D_{x_ix_j}G(Y',0:X)\right| dY' \leqslant$$

$$\leqslant 4M\left[c^2Q_{n-2}\left(\frac{1}{x_n}\right) + cnQ_n\left(\frac{1}{x_n}\right)\right] \int_{R^{n-1}} e^{-cr}dY',$$

$i,j = 1,2,\dots,n-1$, and similarly, differentiating that integral with respect to its parameter, we get

$$\int_{R^{n-1}} \left| f(Y')D_{x_n}G(Y',0:X)\right| dY' \leqslant$$

$$\leqslant 2M\left[c\bar{Q}_{n-1}\left(\frac{1}{x_n}\right) + 2hQ_{n-2}\left(\frac{1}{x_n}\right)\right] \int_{R^{n-1}} e^{-cr}dY',$$

$$\int_{R^{n-1}} \left| f(Y')D_{x_ix_n}G(Y',0:X)\right| dY' \leqslant$$

$$\leqslant 2M\left[c^2Q_{n-2}\left(\frac{1}{x_n}\right) + cnQ_n\left(\frac{1}{x_n}\right) + 2ch\bar{Q}_{n-1}\left(\frac{1}{x_n}\right)\right] \int_{R^{n-1}} e^{-cr}dY',$$

$i = 1,2,\dots,n-1$,

$$\int_{R^{n-1}} \left| f(Y')D_{x_nx_n}G(Y',0:X)\right| dY' \leqslant$$

$$\leqslant 2M\left[(c^2+2h^2)Q_{n-2}\left(\frac{1}{x_n}\right) + cnQ_n\left(\frac{1}{x_n}\right) + ch\bar{Q}_{n-1}\left(\frac{1}{x_n}\right)\right] \int_{R^{n-1}} d^{-cr}dY'.$$

This implies for the above integrals the uniform convergence in any bounded and closed set $E \subset D$. Hence $u(X) \in C^2(D)$.

The second part of the theorem results from the properties of the function $G(Y:X)$ and from Lemma 1.

5. Let us take numbers $R_0 > 0$ and $\alpha \in (1 - \frac{1}{n}, 1)$ and for some point $X \in D$ the sphere of centre $X' = (x_1, x_2, \dots, x_{n-1})$ and of radius $R = R_0 x_n^{\alpha}$,

$$U_R = \left\{ (y_1, y_2, \dots, y_{n-1}) : \sum_{i=1}^{n-1} (y_i - x_i)^2 < R^2 \right\},$$

and let

$$(12) \qquad H_m(X) = \int_{U_R} e^{-cr} \cdot \frac{x_n}{r^m}\, dY', \qquad r^2 = \sum_{i=1}^{n-1} (y_i - x_i)^2 + x_n^2,$$
$$m = 0, 1, 2, \dots, n.$$

L e m m a 3. The function (12) has a limit

$$(13) \quad \lim_{X \to X_0} H_m(X) = \begin{cases} 0 & \text{for } m = 0, 1, 2, \dots, n-1, \\ \dfrac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} & \text{for } m = n, \text{ where } X_0 = (x_1^0, x_2^0, \dots, x_{n-1}^0, 0). \end{cases}$$

P r o o f . It is obvious from the proof of Lemma 3 in [2].

The partial derivative of the function (11) with respect to x_n is equal to

$$D_{x_n} u(X) = \frac{1}{\omega_n} \cdot \int_{R^{n-1}} f(Y') \left[- \frac{K_{p+1}(cr)}{r^{p+1}}\, c x_n - \right.$$
$$\left. - h^2 \int_{x_n}^{+\infty} e^{-h(t - x_n)} g(r_3)\, dt + h g(r) \right] dY',$$

hence we have

$$(14)\quad D_{x_n}u(X) - hu(X) = -\frac{1}{\omega_n}\cdot\int\limits_{R^{n-1}} f(Y')\cdot\frac{K_{p+1}(cr)}{r^{p+1}}\,cx_n dY' \qquad X\in D.$$

T h e o r e m 2. If a function f satisfies the assumptions of Section 1 and

$$(15)\qquad \omega_n = -\pi\left(\frac{2\pi}{c}\right)^{\frac{n-2}{2}},$$

then $u(X)$ is a solution of problem (2).

P r o o f . For some point $X\in D$ take sphere $U_R\subset R^{n-1}$. Then

$$\int\limits_{R^{n-1}} f(Y')\cdot\frac{K_{p+1}(cr)}{r^{p+1}}\cdot cx_n dY' =$$

$$= \int\limits_{U_R}\left[f(Y') - f(X_0')\right]\frac{K_{p+1}(cr)}{r^{p+1}}\,cx_n dY' + f(X_0')\cdot\int\limits_{U_R}\frac{K_{p+1}(cr)}{r^{p+1}}\,cx_n dY' +$$

$$+\int\limits_{CU_R} f(Y')\cdot\frac{K_{p+1}(cr)}{r^{p+1}}\,cx_n dY',$$

where $X_0' = (x_1^0, x_2^0, \ldots, x_{n-1}^0)$ is a fixed point. Using formula (6) $(p+1 = \frac{n}{2})$, with $|f(Y')|\leqslant M$, $M>0$, we have

$$\int\limits_{U_R}\left|f(Y') - f(X_0')\right|\frac{K_{p+1}(cr)}{r^{p+1}}\,cx_n dY' \leqslant c\cdot\mu(X_0')\cdot\int\limits_{U_R} Q_n\left(\frac{1}{r}\right)e^{-cr}x_n dY',$$

where $\mu(X_0') = \max|f(Y') - f(X_0')|$ in closed sphere $\overline{U}_R$ and

$$\int\limits_{U_R}\frac{K_{p+1}(cr)}{r^{p+1}}\,cx_n dY' = c\cdot\int\limits_{U_R} Q_n\left(\frac{1}{r}\right)e^{-cr}x_n dY',$$

$$\int\limits_{CU_R} |f(Y')| \frac{K_{p+1}(cr)}{r^{p+1}} cx_n dY' \leqslant cx_n M \int\limits_{CU_R} \frac{1}{r} \bar{Q}_{n-1}\left(\frac{1}{r}\right) e^{-cr} dY' \leqslant$$

$$\leqslant c \frac{x_n}{R} M \int\limits_{CU_R} \bar{Q}_{n-1}\left(\frac{1}{r}\right) e^{-cr} dY',$$

$\left(\bar{Q}_{n-1}\left(\frac{1}{r}\right) = rQ_n\left(\frac{1}{r}\right)\right.$ and for any point $Y' \in CU_R$ is $\left. r = XY' > R\right)$.

From Lemma 3 and the formula for the coefficient a_n of the polynomial $Q_n\left(\frac{1}{r}\right)$ in (7) we have

(16a)
$$\lim_{X \to X_o} \int\limits_{U_R} \frac{K_{p+1}(cr)}{r^{p+1}} cx_n dY' =$$

$$= ca_n \cdot \frac{\pi^{\frac{n}{2}}}{\sqrt{\left(\frac{n}{2}\right)}} = ca_n \cdot \frac{2^{n-2}\left(\frac{n-3}{2}\right)!}{(n-2)!} \cdot \frac{\pi^{\frac{n}{2}}}{\sqrt{\left(\frac{1}{2}\right)}} = \pi\left(\frac{2\pi}{c}\right)^{\frac{n-2}{2}}.$$

Similarly, from the continuity of $f(Y')$ in R^{n-1} we obtain

(16b)
$$\lim_{X \to X_o} c\mu(X_o') \int\limits_{U_R} Q_n\left(\frac{1}{r}\right) e^{-cr} x_n dY' = 0.$$

Also

(16c)
$$\lim_{X \to X_o} \frac{x_n}{R} \int\limits_{CU_R} \bar{Q}_{n-1}\left(\frac{1}{r}\right) e^{-cr} dY' = 0.$$

Indeed, in the polar coordinates on R^{n-1} the integral (16c) becomes

$$\frac{x_n}{R} \cdot \Omega \int\limits_{R}^{+\infty} \bar{Q}_{n-1}\left(\frac{1}{r}\right) e^{-cr} \cdot \varrho^{n-2} d\varrho,$$

where Ω is the coefficient of the polar transformation and $r = \sqrt{\varrho^2 + x_n^2}$. For $\frac{x_n}{R} = \frac{1}{R_o} \cdot x_n^{1-\alpha}$, $1 - \frac{1}{n} < \alpha < 1$, with $\frac{\varrho}{\sqrt{\varrho^2 + x_n^2}} \leqslant 1$, we have

$$\int_R^{+\infty} \bar{Q}_{n-1}\left(\frac{1}{r}\right) e^{-cr} \cdot \varrho^{n-2} d\varrho \leqslant$$

$$\leqslant \int_R^{+\infty} \left(a_n \cdot \frac{1}{r} + a_{n-1} + a_{n-2}\, \varrho + \dots + a_{\frac{n+1}{2}} \cdot \varrho^{\frac{n-3}{2}} \right) e^{-c\varrho} d\varrho .$$

On the other side, with $a > R$,

$$\int_R^{+\infty} \frac{e^{-cr}}{r}\, d\varrho \leqslant \int_0^a \frac{1}{r}\, d\varrho + \int_a^{+\infty} \frac{e^{-c}}{r}\, d\varrho \leqslant \ln \frac{a + \sqrt{a^2 + x_n^2}}{x_n} + \frac{1}{ac} \cdot e^{-ac} .$$

In that case we have the estimation

$$\frac{x_n}{R} \int_{CU_R} \bar{Q}_{n-1}\left(\frac{1}{r}\right) e^{-cr} dY' \leqslant$$

$$\leqslant \frac{x_n^{1-\alpha}}{R_0} \left[a_n \left(\ln \frac{a + \sqrt{a^2 + x_n^2}}{x_n} + \frac{1}{ac} \cdot e^{-ac} \right) + \int_0^{+\infty} \left(\sum_{v=1}^{m} a_{n-v}\, \varrho^{v-1} \right) e^{-c\varrho} d\varrho \right]$$

where $m = \frac{n-1}{2}$, which implies (16c).

From (16a), (16b), (16c) the theorem follows.

6. Let the dimension n of the space R^n be even, $n = 2v$, $v \in N$. In this case we use Mac Donald's functions

$$(17) \qquad K_v(x) = \frac{1}{2} \int_0^{+\infty} \exp\left[- \frac{x}{2}\, (t + \frac{1}{t}) \right] t^{v-1} dt ,$$

$$x > 0, \ v = 0, 1, 2, \dots \text{ (see [1], [4])}$$

$$(18a) \qquad K_0(x) = -I_0(x)(\ln \frac{x}{2} + C) + \frac{1}{2} \sum_{m=1}^{+\infty} \frac{\left(\frac{x}{2}\right)^{2m}}{(m!)^2} \sum_{k=1}^{m} \frac{1}{k} , \quad x > 0,$$

(18b)
$$K_\nu(x) = (-1)^{\nu+1} I_\nu(x)\left(\ln\frac{x}{2} + C\right) + \frac{1}{2}\cdot\sum_{m=0}^{\nu-1}(-1)^m\cdot\frac{(\nu-m-1)!}{m!}\left(\frac{x}{2}\right)^{2m-\nu} + \left(-\frac{x}{2}\right)^\nu\cdot\frac{1}{\nu!}\sum_{k=1}^{\nu}\frac{1}{k} + \frac{(-1)^\nu}{2}\sum_{m=1}^{+\infty}\frac{\left(\frac{x}{2}\right)^{2m+\nu}}{m!(m+\nu)!}\left(\sum_{k=1}^{m}\frac{1}{k} + \sum_{k=1}^{m+\nu}\frac{1}{k}\right),$$

$x>0$, $\nu\in N$,

where $I_p(x) = \sum_{m=0}^{+\infty}\frac{\left(\frac{x}{2}\right)^{2m+p}}{m!\Gamma(m+p+1)}$, C is the Euler constant.

L e m m a 4. If $\nu = 1,2,3,\ldots$, that

(19)
$$K_\nu(x) \leqslant \frac{1}{2}\left[1 + \frac{2}{x} + \left(\frac{2}{x}\right)^2(\nu-1) + \left(\frac{2}{x}\right)^3(\nu-1)(\nu-2) + \ldots + \left(\frac{2}{x}\right)^\nu(\nu-1)!\right]e^{\frac{-x}{2}}.$$

P r o o f . From (17) (with $x>0$)

$$K_\nu(x) \leqslant \frac{1}{2}\left(\int_0^1 e^{-\frac{x}{2}}dt + \int_1^{+\infty} e^{-\frac{x}{2}}\cdot t^{\nu-1}dt\right).$$

Integrating the second integral by parts we have (19).

L e m m a 5. If $\nu = 1,2,3,\ldots$, that

(20)
$$\sum_{m=1}^{+\infty}\frac{\left(\frac{x}{2}\right)^{2m}}{m!(m+\nu)!}\left(\sum_{k=1}^{m}\frac{1}{k} + \sum_{k=1}^{m+\nu}\frac{1}{k}\right) \leqslant \frac{x^2}{2}\cdot I_0(x).$$

P r o o f .

$$\sum_{m=1}^{+\infty} \frac{\left(\frac{x}{2}\right)^{2m}}{m!(m+v)!}\left(\sum_{k=1}^{m} \frac{1}{k} + \sum_{k=1}^{m+v} \frac{1}{k}\right) \leqslant \sum_{m=1}^{+\infty} \frac{\left(\frac{x}{2}\right)^{2m}}{m!(m+v)!}\, 2(m+v) =$$

$$= 2 \sum_{m=1}^{+\infty} \frac{\left(\frac{x}{2}\right)^{2m}}{m!(m+v-1)!}\ 2 \sum_{m=0}^{+\infty} \frac{\left(\frac{x}{2}\right)^{2(m+1)}}{(m+1)!(m+v)!} =$$

$$= 2\left(\frac{x}{2}\right)^2 \sum_{m=0}^{+\infty} \frac{\left(\frac{x}{2}\right)^{2m}}{(m+1)!(m+v)!} \leqslant \frac{x^2}{2} \sum_{m=0}^{+\infty} \frac{\left(\frac{x}{2}\right)^{2m}}{(m!)^2} = \frac{x^2}{2} \cdot I_0(x).$$

By Lemma 1, the function

$$(21) \quad k(r) = \frac{K_{v-1}(cr)}{r^{v-1}}, \quad r^2 = \sum_{i=1}^{n} (x_i - a_i)^2, \quad X \neq A(a_1, a_2, \dots, a_n),$$

satisfies equation (1) in $R^n \setminus A$. Using formulas (18a), (18b) we have

$$k(r) = -I_0(cr)\left(\ln \frac{cr}{2} + C\right) + \sum_{m=1}^{+\infty} \frac{\left(\frac{cr}{2}\right)^{2m}}{(m!)^2} \cdot \sum_{i=1}^{m} \frac{1}{i} \quad \text{for} \quad v = 1,$$

$$k(r) = (-1)^v \cdot \frac{I_{v-1}(cr)}{r^{v-1}} \left(\ln \frac{cr}{2} + C\right) +$$

$$+ \frac{1}{2}\left(\frac{c}{2}\right)^{v-1} \sum_{m=0}^{v-2} (-1)^m \cdot \frac{(v-m-2)!}{m!} \left(\frac{cr}{2}\right)^{2m-n+2} + \frac{\left(-\frac{c}{2}\right)^{v-1}}{(v-1)!} \cdot \sum_{i=1}^{v-1} \frac{1}{i} +$$

$$+ \frac{1}{2}\left(\frac{c}{2}\right)^{v-1} \sum_{m=1}^{+\infty} \frac{\left(\frac{cr}{2}\right)^{2m}}{m!(m+v-1)!}\left(\sum_{i=1}^{m} \frac{1}{i} + \sum_{i=1}^{m+v-1} \frac{1}{i}\right) \quad \text{for} \quad v \geqslant 2.$$

Similarly, we conclude from Lemma 4 that for $n = 2v$, $v \in N$, there exists a polynomial $W_n\left(\frac{1}{r}\right)$ of degree n with positive coefficients and the free term equal to 0 such that

$$\frac{K_v(cr)}{r^v} \leqslant W_n\left(\frac{1}{r}\right)\cdot e^{-\frac{cr}{2}}, \quad n = 2v. \tag{22}$$

In that case, similarly in Section 3, we have estimations of partial derivatives of function k(r) up to the second order

$$\left|D_{x_i}k(r)\right| \leqslant crW_n\left(\frac{1}{r}\right)\cdot e^{-\frac{cr}{2}}, \quad i = 1,2,\dots,n,$$

$$\left|D_{x_ix_j}k(r)\right| \leqslant \left[c^2W_{n-2}\left(\frac{1}{r}\right) + cnW_n\left(\frac{1}{r}\right)\right]e^{-\frac{cr}{2}}, \quad i,j = 1,2,\dots,n.$$

With $n = 2v$, $v \in N$, the functions (9), (11) have an analogous form

$$G(Y:X) = k(r_1) + k(r_2) - 2h\cdot\int_{y_n+x_n}^{+\infty} e^{-h(t-y_n-x_n)}k(r_3)dt, \quad Y\neq X\in D,$$

where r_1, r_2, r_3 are like in (9)

$$u(X) = \frac{1}{2\bar{\omega}_n}\cdot\int_{R^{n-1}} f(Y')G(Y',0;X)dY',$$

$\bar{\omega}_n$ - proper coefficient, $X \in D$.

These functions are of class $C^2(D)$ and satisfy equation (1) in D because of (22) and estimations of partial derivatives of k(r). With condition (2) we have, similarly to (14), formula

$$D_{x_n}u(X) - hu(X) = -\frac{1}{\bar{\omega}_n}\cdot\int_{R^{n-1}} f(Y')\cdot\frac{K_v(cr)}{r^v}cx_n dY', \tag{23}$$

$$n = 2v, \ v \in N, \ X \in D,$$

where $r^2 = \sum_{i=1}^{n-1}(y_i-x_i)^2 + x_n^2$.

We shall prove, that $u(X)$ satisfies condition (2) in any point $X_0' \in R^{n-1}$ with proper coefficient $\bar{\omega}_n$.

If $v = 1$, or $n = 2$, then formula (23) has the form

$$D_{x_2}u(X) - hu(X) = -\frac{1}{\bar{\omega}_2}\cdot\int_{-\infty}^{+\infty} f(y_1)\cdot\frac{K_1(cr)}{r}\, cx_2dy_1,\quad r^2=(y_1-x_1)^2+x_2^2,$$

where $X = (x_1,x_2)$, $Y = (y_1,0)$, while (from (18b))

$$\frac{K_1(cr)}{r} = \frac{c}{2}\cdot\sum_{m=0}^{+\infty}\frac{\left(\frac{cr}{2}\right)^{2m}}{m!(m+1)!}\left(\ln\frac{cr}{2}+C\right)+$$

$$+\frac{1}{cr^2}-\frac{c}{2}-\frac{c}{4}\cdot\sum_{m=1}^{+\infty}\frac{\left(\frac{cr}{2}\right)^{2m}}{m!(m+1)!}\left(2\sum_{i=1}^{m}\frac{1}{i}+\frac{1}{m+1}\right).$$

In this integral we put $y_1 = x_1 + t$ and next we take in sum of integrals

$$\int_{-\infty}^{+\infty} f(x_1+t)\cdot\frac{K_1(c\bar{r})}{r}\,cx_2dt = \int_{-\infty}^{-a}+\int_{-a}^{a}+\int_{a}^{+\infty},\quad \bar{r}^2=t^2+x_2^2,\ a=x_2^{\alpha},$$

with any α, $0<\alpha<\frac{1}{2}$. Because of (22), with $n = 2$, and $|f(x_1 \pm t)| \leqslant M$,

$$\left|\int_{-\infty}^{-a}+\int_{a}^{+\infty}\right| \leqslant \int_{a}^{+\infty}\left|f(x_1+t)+f(x_1-t)\right|\frac{K_1(c\bar{r})}{\bar{r}}\,cx_2dt\leqslant$$

$$\leqslant 2cx_2M\int_{a}^{+\infty}W_2\left(\frac{1}{\bar{r}}\right)e^{-\frac{c\bar{r}}{2}}dt\leqslant 2cMx_2W_2\left(\frac{1}{a}\right)\int_{a}^{+\infty}e^{-\frac{ct}{2}}dt = 4Mx_2W_2\left(\frac{1}{a}\right)\cdot e^{-\frac{ac}{2}},$$

and using $\lim_{x_2\to 0} x_2W_2\left(\frac{1}{a}\right) = 0$, the sum of integrals $\int_{-\infty}^{-a}+\int_{a}^{+\infty}$ tends to 0, when $X \longrightarrow X_0' = (x_1^0,0)$. The integral $\int_{-a}^{a}$ is equal to the sum

$$\frac{c}{2}\cdot\int_{-a}^{a} f(t+x_1)\sum_{m=0}^{+\infty}\frac{\left(\frac{c\bar{r}}{2}\right)^{2m}}{m!(m+1)!}\left(\ln\frac{c\bar{r}}{2}+C\right)cx_2dt+$$

$$+\int_{-a}^{a}\left[f(t+x_1)-f(x_1^0)\right]\frac{x_2}{\bar{r}^2}dt+f(x_1^0)\cdot\int_{-a}^{a}\frac{x_2}{\bar{r}^2}dt-$$

$$-\frac{c}{2}\cdot\int_{-a}^{a} f(t+x_1)\left[1+\frac{1}{2}\cdot\sum_{m=1}^{+\infty}\frac{\left(\frac{c\bar{r}}{2}\right)^{2m}}{m!(m+1)!}\left(\sum_{i=1}^{m}\frac{1}{i}+\sum_{i=1}^{m+1}\frac{1}{i}\right)\right]cx_2dt=$$

$$=A_1+A_2+A_2+A_4.$$

With small $x_2>0$, $\bar{r}<1$ for any $t\in(-a,a)$ we have estimations:

$$|A_1|\leqslant\frac{c^2}{2}M\cdot I_0(c)\cdot\int_{-a}^{a}\left(|\ln\bar{r}|+\left|\ln\frac{c}{2}+C\right|\right)dt\cdot x_2,$$

$$|A_2|\leqslant\max_{\langle -a,a\rangle}\left|f(t+x_1)-f(x_1^0)\right|\cdot\int_{-a}^{a}\frac{x_2}{\bar{r}^2}dt,$$

$$|A_4|\leqslant\frac{c^2}{2}M\cdot\int_{-a}^{a}\left[1+\frac{c^2r^2}{4}\cdot I_0(c)\right]dt\cdot x_2,\quad ((20),\ v=1).$$

Because of $\int_{-a}^{a}\frac{x_2}{r^2}dt=\int_{-x_2^{\alpha-1}}^{x_2^{-1}}\frac{dt}{1+t^2}=2\operatorname{arctg}(x_2^{\alpha-1})$, A_3 has limit $\lim A_3=f(x_1^0)\pi$, where $(x_1,x_2)\to(x_1^0,0)$. Obviously, $\lim A_1=0$, $\lim A_4=0$ and, by the continuity of the function f, $\lim A_2=0$ when $(x_1,x_2)\to(x_1^0,0)$. Hence the theorem in case $\overline{\omega}_2=-\pi$ is proved.

Now let $n = 2v$, $v \in N$, $v \geqslant 2$. Like in Section 5 for any $X \in D$ we take the sphere U_R, $R = R_o x_n^\alpha$, $\alpha \in (1 - \frac{1}{n}, 1)$ and we split integral (23) into the sum of integrals

$$\int\limits_{R^{n-1}} f(Y') \cdot \frac{K_v(cr)}{r^v} \cdot cx_n dY' = \int\limits_{CU_R} + \int\limits_{U_R}.$$

Because of (22), with $|f(Y')| \leqslant M$ we have

$$\left| \int\limits_{CU_R} f(Y') \cdot \frac{K_v(cr)}{r^v} cx_n dY' \right| \leqslant cMx_n \int\limits_{CU_R} W_n\left(\frac{1}{r}\right) e^{-\frac{cr}{2}} dY' \leqslant$$

$$\leqslant \frac{cM}{R_o} \cdot x_n^{1-\alpha} \cdot \int\limits_{CU_R} W_{n-1}\left(\frac{1}{r}\right) e^{-\frac{cr}{2}} dY',$$

where $W_{n-1}\left(\frac{1}{r}\right) = rW_n\left(\frac{1}{r}\right)$. Similarly in the proof of (16c) we state

$$\lim_{X \to X_o} \int\limits_{CU_R} f(Y') \cdot \frac{K_v(cr)}{r^v} cx_n dY' = 0.$$

According to (18b) we have

$$\frac{K_v(cr)}{r^v} = (-1)^{v+1} \cdot \frac{I_v(cr)}{r^v} \left(\ln \frac{cr}{2} + C \right) +$$

$$+ \frac{1}{2} \cdot \sum_{m=0}^{v-1} (-1)^m \cdot \frac{(v-m-1)!}{m!} \left(\frac{c}{2}\right)^{2m-v} \left(\frac{1}{r}\right)^{n-2m} +$$

$$+ (-1)^v \left(\frac{c}{2}\right)^v \frac{1}{v!} \cdot \sum_{i=1}^{v} \frac{1}{i} +$$

$$+ \frac{(-1)^v}{2} \left(\frac{c}{2}\right)^v \sum_{m=1}^{+} \frac{\left(\frac{cr}{2}\right)^{2m}}{m!(m+v)!} \left(\sum_{i=1}^{m} \frac{1}{i} + \sum_{i=1}^{m+v} \frac{1}{i} \right).$$

In that case the integral $\int\limits_{U_R} f(Y') \cdot \frac{K_v(cr)}{r^v} cx_n dY'$ is equal to the sum of integrals:

$$J_1 = (-1)^{v+1} cx_n \int\limits_{U_R} f(Y') \cdot \frac{I_v(cr)}{r^v} \left(\ln \frac{cr}{2} + C\right) dY',$$

$$J_2 = \frac{c}{2} \int\limits_{U_R} \left[f(Y') - f(X_o')\right] \sum_{m=0}^{v-1} (-1)^m \cdot \frac{(v-m-1)!}{m!} \left(\frac{c}{2}\right)^{2m-v} \cdot \frac{x_n}{r^{n-2m}} dY',$$

$$J_3 = f(X_o') \frac{c}{2} \cdot \sum_{m=0}^{v-1} (-1)^m \cdot \frac{(v-m-1)!}{m!} \left(\frac{c}{2}\right)^{2m-v} \cdot \int\limits_{U_R} \frac{x_n}{r^{n-2m}} dY',$$

$$J_4 = \left(-\frac{c}{2}\right)^v \int\limits_{U_R} f(Y') \left[\frac{1}{v!} \cdot \sum_{i=1}^{v} \frac{1}{i} + \frac{1}{2} \cdot \sum_{m=1}^{+\infty} \frac{\left(\frac{cr}{2}\right)^{2m}}{m!(m+v)!} \left(\sum_{i=1}^{m} \frac{1}{i} + \sum_{i=1}^{m+v} \frac{1}{i}\right)\right] dx_n dY'.$$

For $Y' \in U_R$ we have $r^2 \leqslant R_o^2 x_n^{2\alpha} + x_n^2$ and so we can put $\delta > 0$ such that from condition $x_n < \delta$ we have $r < 1$ for every $Y' \in U_R$. Then, letting $|U_R|$ be the volume of U_R, we have the estimations

$$|J_1| \leqslant c\left(\frac{c}{2}\right)^v Mx_n(|\ln x_n| + |\ln \frac{c}{2} + C|) I_o(c) \cdot |U_R|,$$

and from (20)

$$|J_4| \leqslant c\left(\frac{c}{2}\right)^v Mx_n\left[1 + \frac{c^2}{4} \cdot I_o(c)\right] |U_R|.$$

From that $\lim\limits_{X \to X_o} J_1 = 0$, $\lim\limits_{X \to X_o} J_4 = 0$.

By Lemma 3 in [2], we have

$$\lim_{X\to X_0}\sum_{m=0}^{v-1}(-1)^m\cdot\frac{(v-m-1)!}{m!}\left(\frac{c}{2}\right)^{2m-v+1}\cdot\int_{U_R}\frac{x_n}{r^{n-2m}}\,dY'=\pi\cdot\left(\frac{2\pi}{c}\right)^{v-1}$$

$(X_0=(x_1^0,x_2^0,\ldots,x_{n-1}^0,0),\ v=\frac{n}{2})$, and thus the continuity of f implies the equalities $\lim\limits_{X\to X_0} J_2=0$, $\lim\limits_{X\to X_0} J_3=f(X_0')\cdot\pi\left(\frac{2\pi}{c}\right)^{v-1}$.

Thus we conclude that the boundary problem (2) is satisfied when

$$\bar{\omega}_n=-\pi\left(\frac{2\pi}{c}\right)^{\frac{n-2}{2}}.$$

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