DEMONSTRATIO MATHEMATICA
Vol. XVII No 2 1984

Konstanty Radziszewski

ON SOME MAPPINGS OF A PRINCIPAL BUNDLE INTO ITS STRUCTURAL GROUP

Introduction

The great part of geometrical structures on differentiable manifolds is given with the aid of a non-degenerate tensor field of type (0,2) or (2,0) or (1,1). For instance, the Riemannian, pseudo-Riemannian, almost Hamiltonian, almost complex, conformal structures are of such types. The values of such tensor field on a linear frame determine a mapping of the bundle of linear frames into linear group $Gl(n)$. For instance, if $u = (x, u_1, \ldots, u_n)$, $x \in M$, $u_i \in T_xM$, is a frame (u_i - vectors tangent to an n-dimensional manifold M at $x \in M$), $\Theta = (x, \Theta^1, \ldots, \Theta^n)$, $\Theta^i \in T^*_xM$, $\Theta^i(u_j) = \delta^i_j$, Θ- dual frame of u and f is non-degenerate tensor field of type (1,1) on M then the matrix $(f(u_i, \Theta^j))$, $i,j = 1,\ldots,n$, is an element of $Gl(n)$. Thus we got a mapping of the principal bundle $\mathcal{F}(M, Gl(n), p)$ of linear frames into $Gl(n)$.

From historical point of view, such structures were investigated separately. Treating such structures as given by a mapping $\pi : \mathcal{F} \to Gl(n)$ we can present a theory which contains their common properties and allows us to consider these structures from more general point of view. Obviously, such mappings must satisfy some conditions.

In this paper we shall generalize this theory by introducing some mappings of a principal fiber bundle $E(M,G,p)$ into its structural Lie group G. This makes possible to investigate analogous structures on general principal fiber bundles.

As the main tool of investigations we use the notion of covariant derivation, i.e. we assume that a connection on a considered principal fiber bundle is given.

All the considered here functions are assumed to be of class $C^{(\infty)}$. Connections and connection forms on E will be denoted by the same symbol ω.

The tangent vector space to a manifold M at $x \in M$ will be denoted by T_xM, $\bigcup_{x\in M} T_xM = TM$. A manifold is considered as Hausdorff with countable basis.

If $f:M \to G$ is a mapping of M into a group G then the mapping $x \to (f(x))^{-1}$, where $(f(x))^{-1}$ is converse to $f(x)$ in G, will be denoted by f^{-1}, i.e. $f^{-1}(x) = (f(x))^{-1}$. The identity mapping will be denoted by I, $I(x) = x$.

1. Π-mapping and π-forms

Let E(M,G,p) be a principal fiber bundle where M is an n-dimensional manifold, G is an r-dimensional Lie group and $p:E \to M$ is the projection mapping.

D e f i n i t i o n 1. A mapping $\pi:E \to G$ is called Π-mapping of type (ϱ,τ) if

$$(1) \qquad \pi(zg) = \varrho(g)\pi(z)\tau(g) \quad \text{for all} \quad z \in E,\ g \in G,$$

where $\varrho:G \to G$ and $\tau:G \to G$ are respectively right and left homomorphisms, i.e.

$\varrho(ab) = \varrho(b)\varrho(a)$ right homomorphism

$\tau(ab) = \tau(a)\tau(b)$ left homomorphism

for all $a,b \in G$.

R e m a r k . This mapping was introduced and called principal object by J.Gancarzewicz in [1].

Now, if $U \subset M$ is a coordinate neighborhood and $z_U:U \to E$ a cross section over U then for each $z = z_U(x)g$ we have

$$(2) \qquad \pi(z) = \varrho(g)\pi_U(x)\tau(g), \quad \pi_U = \pi(z_U).$$

Let us consider the diffeomorphism $\phi_U: U\times G \longrightarrow p^{-1}(U) \subset E$ given by the formula

$$\phi_U(x,g) = z_U(x)g.$$

This allows us to introduce a coordinate system in a neighborhood of $z_U(U) \subset E$. Namely, if x^i are coordinates of $x \in U$ and g^α of $g \in W_e \subset G$ in a coordinate neighborhood W_e containing the unity e of G then we assign to the point $z = z_U(x)g = \phi_U(x,g)$ the coordinates $(x^1,\dots,x^n,g_1,\dots,g^r)$.

Suppose ω is a connection form given on E. Using ϕ_U we want to give a formula for ω by means of the local connections form $\omega_U = \omega(dz_U)$. Each vector X_z, $z = z_U(x)g$, tangent to E at z, may be written in the form $X_z = dz_U(v)g + z_U(x)dg(w)$, $v \in T_xM$, $w \in T_gG$. Thus, $\omega(X_z) = g^{-1}\omega_U(v)g + g^{-1}dg(w)$ for $X_z = d\phi_U(v,w)$. We identify a neighborhood of $z_U(U)$, $V(z_U(U))$, with $\phi_U^{-1}(V(z_U(U))$ and ω with $\omega(d\phi_U)$ there. Thus we can write

$$\omega = g^{-1}(dg + \omega_U g) \tag{3}$$

in $\phi_U^{-1}(V(z_U(U))$.

Now, we shall define a linear form $F: TE \longrightarrow T_eG$ which is the main notion in this paper.

D e f i n i t i o n 2. If E(M,G,p) is a principal fiber bundle with a given connection ω and a Π-mapping of type (ϱ,τ) then

$$F: TE \longrightarrow T_eG$$

defined by the formula

$$F(X) = \pi^{-1}d\pi(HX), \tag{4}$$

where HX is the horizontal part of X, is called π-form (or tensorial 1-form of type (π,ω)).

For us it is more convenient to present F in terms of π and ω.

From (3) the horizontal vectors satisfy

$$dg = -\omega_U g.$$

Thus we shall obtain from (2) and (4)

$$d\pi(dz) = d\varrho(dg)\pi_U\tau + \varrho\, d\pi_U\tau + \varrho\pi_U d\tau(dg)$$

$$F = \tau^{-1}\pi_U^{-1}\tau^{-1}(d\varrho(-\omega_U g)\pi_U\tau + \varrho d\pi_U\tau + \varrho\pi_U d\tau(-\omega_U g) \quad \text{at} \quad z = z_U(x)g.$$

Since $d\varrho(-\omega_U g) = -\varrho(g)d\varrho(\omega_U)$, $d\tau(-\omega_U g) = -d\tau(\omega_U)\tau(g)$ we get finally in $\phi_U^{-1}(V(z_U(U))$

$$(5) \qquad F = \tau^{-1}(\pi_U^{-1}d\pi_U - \pi_U^{-1}d\varrho(\omega_U)\pi_Z - d\tau(\omega_U))\tau.$$

It is easy to verify by substituting

$$z = z_U g, \quad dz = dz_U g + z_U dg, \quad d\pi = d\varrho\pi_U\tau + \varrho d\pi_U\tau + \varrho\pi_U d\tau,$$

$$\pi = \varrho(g)\pi_U\tau(g), \quad \omega = g^{-1}dg + g^{-1}\omega_U g, \quad d\varrho(g^{-1}dg) = d\varrho(dg)\varrho^{-1}(g),$$

$$d\tau(g^{-1}dg) = \tau^{-1}(g)d\tau(dg)$$

into the formula

$$\pi^{-1}d\pi - \pi^{-1}d\varrho(\omega)\pi - d\tau(\omega)$$

that we shall obtain the formula (5), (after long calculations). In virtue of (5) and this remark we are in a position to formulate

T h e o r e m 1. If the Π-mapping is of type (ϱ,τ) then π-form F is tensorial 1-form of type $ad\tau^{-1}$ and is given by the formula

$$(6) \qquad F = \pi^{-1}d\pi - \pi^{-1}d\varrho(\omega)\pi - d\tau(\omega).$$

Thus we have $F(Xg) = \tau^{-1}(g)F(X)\tau(g)$, (or $F(Xg) = \tau(g^{-1})F(x)\tau(g))$. If $\tau = I$ is the identity automorphism then

$$\omega_\pi = \pi^{-1}d\pi - \pi^{-1}d\varrho(\omega)\pi \tag{7}$$

is a connection form called π-conjugate with ω.

P r o p o s i t i o n 2. If Π-mapping is of type (ϱ, I) then π-form F can be written as

$$F = \omega_\pi - \omega \tag{6_1}$$

where ω_π is π-conjugate with the connection form ω.

The π-form exists if and only if the Π-mapping exists, the next theorem clears up some aspects of this problem.

T h e o r e m 3. Let $E(M,G,p)$ be a principal fiber bundle, ϱ and τ respectively right and left homomorphisms of G, $H_{g_0} = \{g, \varrho(g)g_0\tau(g) = g_0$ and $g \in G\}$ the isotropy subgroup of G at $g_0 \in G$ with respect to the transformations $\psi_a : g \longrightarrow \varrho(a)g\tau(a)$ and $O_{g_0} = \{g;\ \exists a \in G,\ g = \varrho(a)g_0\tau(a)\}$ the orbit of g_0 with respect to ψ_a.

A Π-mapping of type (ϱ, τ) taking its values in O_{g_0}, $\pi(E) \subset O_{g_0}$, exists if and only if E is reducible to the bundle $E_0(M, H_{g_0}, p)$.

P r o o f . Suppose that $\pi : E \longrightarrow G$ exists and $\pi(E) \subset O_{g_0}$. If z_0 is such that $\pi(z_0) = g_0$ then for $a \in H_{g_0}$ we have

$$\pi(z_0 a) = \varrho(a)\pi(z_0)\tau(a) = \varrho(a)g_0\tau(a) = g_0.$$

Thus $E_0 = \{z_0; \pi(z_0) = g_0\}$ is reduced subbundle of E.

Conversely, if $E_0(M, H_{g_0}, p)$ is reduced bundle of E then putting $\pi(z_0) = g_0$ for all $z_0 \in E_0$ we get the mapping $\pi(z_0 g) = \varrho(g)g_0\tau(g)$ with values in O_{g_0}.

R e m a r k . In the case $g_0 = e$, $H^{\varrho} = \{ g; \varrho^{-1}(g) = g\}$; $\tau = I$. J.Gancarzewicz proved a one-to-one correspondence between the set of $E_0(M,H^{\varrho},p)$ and such Π-mappings (see [1]).

E x a m p l e 1. Let $E(M,G,p)$ be a principal fiber bundle reducible to $E_0(M,H_{g_0},p)$ where $H_{g_0} = \{ g; g^{-1}g_0g = g_0$ and $g \in G\}$, and $g_0g_0 = c$, $c \neq e$, $cc = e$, $cg = gc$ for all $g \in G$. Putting $\pi(z_0) = g_0$, $\pi(z_0g) = g^{-1}g_0g$ for all $z_0 \in E_0$ and $g \in G$ we obtain a Π-mapping. This mapping has the following properties

$$\pi(z)\pi(z) = c, \quad \varrho(g) = g^{-1}, \quad \tau(g) = g.$$

A principal fiber bundle E with given such Π-mapping may be called generalized almost complex bundle.

If z_U is a cross section over $U \subset M$ then

$$F = g^{-1}(\pi_U^{-1}d\pi_U + \pi_U^{-1}\omega_U\pi_U - \omega_U)g \quad \text{at} \quad z = z_Ug,$$

and for $z_U(U) \subset E_0$

$$F(dz_U) = g_0^{-1}\omega_Ug_0 - \omega_U, \qquad (\omega_\pi)_U = g_0^{-1}\omega_Ug_0,$$

where $(\omega_\pi)_U = \omega_\pi(dz_U)$.

2. Exterior and covariant differentials of π-form

In order to calculate the exterior derivative dF of a π-form F we need to recall some notions and formulas.

Let φ and ψ be 1-forms with values in the Lie algebra T_eG of a Lie group G. By $[\varphi\wedge\psi]$ we denote the 2-forms given by the formula

$$[\varphi\wedge\psi] = \varphi^\alpha \wedge \psi^\beta [\varepsilon_\alpha, \varepsilon_\beta]$$

where ε_α are basis vectors of T_eG, $[\varepsilon_\alpha, \varepsilon_\beta]$ is the product in Lie algebra T_eG, $\varphi^\alpha \wedge \psi^\beta$ usual exterior product of 1-forms φ^α and ψ^β, $\varphi = \varphi^\beta\varepsilon_\beta$, $\psi = \psi^\beta\varepsilon_\beta$, (see [4]).

In these notations the equations of structure of the Lie group G have the form (see [4])

$$d\Theta = -\frac{1}{2}[\Theta \wedge \Theta]$$

where Θ is the fundamental left invariant form of G, $\Theta = g^{-1}dg$. The fundamental form Θ assigns to each $\xi_g \in T_g G$ the vector $g^{-1}\xi_g \in T_e G$ and the form $\pi^{-1}d\pi$ assigns to each $\xi_{\pi(z)} = d\pi(X_z)$ the vector $(\pi(z))^{-1}\xi_{\pi(z)}$. Thus $\pi^*\Theta = \pi^{-1}d\pi$ and

$$d(\pi^{-1}d\pi) = -\frac{1}{2}\left[\pi^{-1}d\pi \wedge \pi^{-1}d\pi\right].$$

We shall recall one more formula. If $t \longrightarrow g(t)$, $g(0) = e$, is a curve in a Lie group G and $X \in T_e G$ then (see [3])

$$\frac{d}{dt}(g^{-1}Xg)_{t=0} = -\left[\frac{dg(0)}{dt}, X\right].$$

Now, we shall calculate the exterior derivative of the form F given by (6).

If $\varepsilon_1, \ldots, \varepsilon_r$ is a base of $T_e G$ and $z_0 \in E$ then in virtue of (6)

$$F = \pi^{-1}d\pi - \pi^{-1}\pi(z_0)\pi^{-1}(z_0)d\varrho(\varepsilon_\alpha)\pi(z_0)\pi^{-1}(z_0)\pi\omega^\alpha - d\tau(\varepsilon_\alpha)\omega^\alpha.$$

Using the formulas given above we shall calculate dF at $z_0 \in E$.

$$dF = d(\pi^{-1}d\pi) + \left[\pi^{-1}d\pi, \pi^{-1}d\varrho(\varepsilon_\alpha)\pi\right]\wedge\omega^\alpha - \pi^{-1}d\varrho(\varepsilon_\alpha)\pi d\omega^\alpha - d\tau(\varepsilon_\alpha)d\omega^\alpha$$

or

$$dF = -\frac{1}{2}\left[\pi^{-1}d\pi \wedge \pi^{-1}d\pi\right] + \left[\pi^{-1}d\pi \wedge \pi^{-1}d\varrho(\omega)\pi\right] - \pi^{-1}d\varrho(d\omega)\pi - d\tau(d\omega).$$

We substitute here $\pi^{-1}d\pi$ taken from (6) and $d\omega = -\frac{1}{2}[\omega \wedge \omega] + \Omega$, where Ω is curvature form of ω.

$$dF = -\frac{1}{2}[F\wedge F] - [F\wedge d\tau(\omega)] + \frac{1}{2}[\pi^{-1}d\varrho(\omega)\pi\wedge\pi^{-1}d\varrho(\omega)\pi] +$$

$$+\frac{1}{2}\pi^{-1}d\varrho([\omega\wedge\omega])\pi - \pi^{-1}d\varrho(\Omega)\pi - d\tau(\Omega).$$

But $\varphi:G \longrightarrow G$ given by $\varphi = \varrho^{-1}$ is left homomorphism and we have $d\varphi = -d\varrho$ at unity $e \in G$. Therefore,

$$d\varrho([\omega\wedge\omega]) = -d\varphi([\omega\wedge\omega]) = -[d\varphi(\omega)\wedge d\varphi(\omega)] = -[-d\varrho(\omega)\wedge(-d\varrho(\omega))] =$$

$$= -[d\varrho(\omega)\wedge d\varrho(\omega)].$$

This implies the disappearance of two terms in above formula and we can formulate

T h e o r e m 4. If a Π-mapping is of type (ϱ,τ) and ω is a connection form on $E(M,G,p)$ then the exterior derivative dF of corresponding π-form F is expressed by the formula

$$(8) \qquad dF = -\pi^{-1}d\varrho(\Omega)\pi - d\tau(\Omega) - [F\wedge d\tau(\omega)] - \frac{1}{2}[F\wedge F].$$

The covariant derivative of F is given by the formula

$$(9) \qquad DF = dF + [F\wedge d\tau(\omega)].$$

Now, let π be a mapping of type (ϱ, I). Then π-conjugate connection form ω_π can be written as $\omega_\pi = F+\omega$ and we have

$$d\omega_\pi = dF + d\omega.$$

Substituting here F from (6_1) and $d\omega = -\frac{1}{2}[\omega\wedge\omega] + \Omega$ we get

P r o p o s i t i o n 5. The structure equations of π-conjugate connection form on $E(M,G,p)$ has the form

$$(10) \qquad d\omega_\pi = -\frac{1}{2}[\omega_\pi\wedge\omega_\pi] - \pi^{-1}d\varrho(\Omega)\pi.$$

Thus the curvature form Ω_π of π-conjugate connection is given by the formula

(11) $$\Omega_\pi = -\pi^{-1}d\varrho(\Omega)\pi,$$

where Ω is curvature form of ω.

For the covariant derivative $D\omega_\pi$ of ω_π (with respect to ω) we get the formula)

(12) $$D\omega_\pi = -\frac{1}{2}\left[F\wedge F\right] - \pi^{-1}d\varrho(\Omega)\pi.$$

R e m a r k . The formula (9) was proved for general tensorial 1-form of type AdG in [4].

3. Some special Π-mappings

The theory constructed here is analogous to that defined on a manifold by means of the tensors of type (r,s), r+s = 2. This analogy enables us to introduce some special types of Π-mappings and special types of connections depending on a given Π-mapping.

D e f i n i t i o n 3. A Π-mapping is called parallel relatively to a connection ω, in short ω-parallel, if the corresponding π-form F is equal to zero.

In virtue of (6) we get the following condition for a ω-parallel Π-mapping

(13) $$\pi^{-1}d\pi - \pi^{-1}d\varrho(\omega)\pi - d\tau(\omega) = 0.$$

We consider a Π-mapping taking its values in the orbit $O_{g_0} = \{ g; g=\varrho(a)g_0\tau(a),\ a\in G,\ \pi(E)\subset O_{g_0}$, and the reduced subbundle $E_0(M,H_{g_0},p)$ of a bundle $E(M,G,p)$, where $H_{g_0} = \{ g; \varrho(g)g_0\tau(g)=g_0\}$ and $\pi(z_0) = g_0$ for all $z_0\in E_0$. Let $z^0_U: U \to E_0$ be a cross section (local) in E_0. Then $\pi_U = \pi(z^0_U) = g_0$ and (13) takes the form

a) $$d\varrho(\omega_U)g_0 + g_0 d\tau(\omega_U) = 0,\quad \omega_U = \omega(dz^0_U).$$

On the other hand, the Lie algebra $T_eH_{g_0}$ of the group H_{g_0} is determined by the condition

b) $d\varrho(dg)g_0+g_0d\tau(dg) = 0$, $dg(X) = X$ for $X\in T_eG$.

Thus a) means that $\omega(X_0)\in T_eH_{g_0}$ for all $X_0\in TE_0$.

Conversely, if $\omega(X_0)\in T_eH_{g_0}$ for $X_0\in TE_0$ then a) is satisfied and $F(dz_U^0) = 0$ for $z_U^0(U)\subset E_0$. But for $z_V = z_U^0g$ we have $F(dz_V) = F(dz_U^0g+z_U^0dg) = \tau^{-1}(g)F(dz_U^0)\tau(g) = 0$. Thus we proved

Theorem 6. If a Π-mapping takes its values in the orbit O_{g_0}, $\pi(E)\subset O_{g_0}$, $\pi(E_0) = g_0$, then it is ω-parallel if and only if $\omega(X_0)\in T_eH_{g_0}$ for all $X_0\in TE_0$ tangent to the reduced bundle $E_0(M,H_{g_0},p)$ of principal fiber bundle $E(M,G,p)$.

From (8) we get

$$\pi^{-1}d\varrho(\Omega)\pi + d\tau(\Omega) = 0 \tag{14}$$

for ω-parallel Π-mappings. Thus for $z_0\in E_0$, $\pi(z_0) = g_0$, we get from b) and (14):

Theorem 7. If a Π-mapping is ω-parallel and takes its values in O_{g_0}, $\pi(E)\subset O_{g_0}$, then

$$\Omega(X_{z_0},Y_{z_0})\in T_eH_{g_0} \quad \text{for all} \quad X_{z_0},Y_{z_0}\in T_{z_0}E, \quad z_0\in E_0,$$

where $E_0 = \{z_0;\pi(z_0) = g_0\}$ and Ω is curvature form of ω.

This means, Ω takes its values in $T_eH_{g_0}$ on the vectors tangent to E at the points z_0 of reduced bundle $E_0=\{z_0;\pi(z_0)=g_0\}$.

Example 2. In Example 1 the subgroup H_{g_0} consists of the elements $g\in G$ which commute with g_0, $gg_0 = g_0g$. π is a ω-parallel if and only if $\omega(X)\in T_eH_{g_0}$ for $X\in TE_0$, $E_0 = \{z_0;\pi(z_0)=g_0\}$ i.e. if $\omega(X)g_0 = g_0\omega(X)$ for $X\in TE_0$, If z_M is a global cross section of E and π is defined by $\pi(z_M) = g_0$ then this π is ω-parallel with respect to the connection ω given by $\omega(dz_M) = 0$.

The characteristic property of ω-parallel Π-mappings is such that their equations (13) along a curve L depend only on the projection p(L) of L on M. An analogous case we obtain if, a π-form F takes its values in the central ideal of structure group G. Thus, if n_i is a basis of the central ideal $q_e \subset T_eG$ of the Lie algebra T_eG and χ^i are some 1-forms on M then this Π-mapping is called ω-(q_e)parallel if $F(X) = \chi^i(dp(X))n_i$. The principal fiber bundle with given such Π-mapping may be considered as analogous to well known Weyl space. Such Π-mappings will be not considered in this paper.

REFERENCES

[1] J. Gancarzewicz: Mixed covariant derivative and conjugate connections, Ann. Pol. Math., 32 (1976) 235-271.

[2] S. Sternberg: Lectures on differential geometry. New Jersey 1964.

[3] L.S. Pontriagin: Niepreryvnyje grupy, Moskva 1964.

[4] S. Kobayashi, K. Nomizu: Foundations of differential geometry. New York-London 1963.

[5] K. Radziszewski: Conjugate connection on a principal fiber bundle; Coll. Math. Soc. Janos Bolyai, 31, Diff. Geometry. Budapest 1979.

INSTITUTE OF MATHEMATICS, M.CURIE-SKŁODOWSKA UNIVERSITY,
20-031 LUBLIN, POLAND
Received March 5, 1981.