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INCOMPRESSIBILITY OF SURFACES WITH FOUR BOUNDARY COMPONENTS AFTER DEHN SURGERY

Let F be an unknotted surface with four boundary components embedded in a surface-bundle over S^1. Let each component of ∂F cut each fiber exactly once. We study whether incompressibility of F is preserved after Dehn surgery.

In this paper we generalize Theorem 1.4 of [P] to surfaces with four boundary components. Because of Example 1.15b) of [P] we use some additional assumptions. Namely we prove the following:

T h e o r e m 1. Let M be a bundle over S^1 with a compact surface as a fiber and $\partial M = T^2$. Let F be a properly embedded, 2-sided surface in M. Suppose that the following three conditions are satisfied:

1. F has four boundary components and each of them cuts each fiber exactly once.

2. F disconnects M into two handlebodies H_n and H_n' $(n \geq 2)$.

3. Denote by A_1 and A_2 (resp. A_1' and A_2') two annuli of $H_n \cap \partial M$ (resp. $H_n' \cap \partial M$). We assume that F can be isotoped mod ∂F in such a way that the natural projection $p: M \longrightarrow S^1$ restricted to F is a Morse function and there exists points $t_1, t_2, t_1', t_2' \in S^1$ such that some component K_i (resp. K_i') of $p^{-1}(t_i) \cap H_n$ (resp. $p^{-1}(t') \cap H_n'$) is a disk and $K_i \cap A_i \neq \emptyset$ (resp. $K_i' \cap A_i' \neq \emptyset$) i = 1 or 2.

Then F is incompressible in M iff $F^\wedge$ is incompressible in $M^{\partial F}$.

1. Preliminaries

We work in PL-category. We follow the terminology of [H], [P] and [L].

Definition 1.1. a) Let M be a 3-manifold and F a surface which is either properly embedded in M or contained in ∂M. We say that F is compressible in M if one of the following conditions is satisfied:

(i) F is a 2-sphere which bounds a 3-cell in M;

(ii) F is a 2-cell and either $F \subset \partial M$ or there is a 3-cell $X \subset M$ with $\partial X \subset F \cup \partial M$;

(iii) there is a 2-cell $D \subset M$ with $D \cap F = \partial D$ and with ∂D not contractible in F.

We say that F is incompressible if it is not compressible.

b) Let F be a submanifold of a manifold M. We say that F is π_1-injective in M if the inclusion-induced homomorphism from $\pi_1(F)$ to $\pi_1(M)$ is an injection.

c) Let F be a surface properly embedded in a compact 3-manifold M. We say that F is ∂-incompressible in M if there is no a 2-disk $D \subset M$ such that $D \cap F = \alpha$ is an arc in ∂D, $D \cap \partial M = \beta$ is an arc in ∂D, with $\alpha \cap \beta = \partial\alpha = \partial\beta$ and $\alpha \cup \beta = \partial D$, and α is not parallel to ∂F in F.

Definition 1.2. Let M be a 3-manifold and λ a simple closed, 2-sided curve on ∂M. We define a new 3-manifold M_λ to be M with a 2-handle glued along λ. That is: let A_λ be a regular neighbourhood of λ in ∂M. Let (D^3, A) be a 3-disk with an annulus on the boundary and Φ a homeomorphism $A_\lambda \to A$, then $M_\lambda = (M, A_\lambda) \cup_\Phi (D^3, A)$. If $\{\lambda_i\}_{i=1}^n$ is a finite collection of disjoint, 2-sided, simple, closed curves on ∂M then $M_{\{\lambda_i\}} \overset{def}{=} (\ldots((M_{\lambda_1})_{\lambda_2})\ldots)_{\lambda_n}$. The definition does not depend on the ordering of the λ_i. If $\partial_1 M = T^2$ is a boundary component of M, and λ is a nontrivial, simple, closed curve on $\partial_1 M$, then by the Dehn surgery on M along λ, we mean the manifold, M^λ, obtained from M_λ by capping off the new boundary component of M_λ, which equals S^2. If $\{\lambda_i\}_{i=1}^n$ is a finite collection of nontrivial, disjoint, parallel, simple, closed curves on $\partial_1 M = T^2$, then by $M^{\{\lambda_i\}}$ we mean M^{λ_1}.

R e m a r k 1.3. The manifold M^λ is obtained from M by operation which is in fact only the second part of the original Dehn surgery (which consists of drilling and filling) and, perhaps, should be called Dehn filling.

D e f i n i t i o n 1.4. Let $(F, \partial F) \to (M, \partial M)$ be a surface properly embedded in a 3-manifold M. We say that F is <u>unknotted in M</u> iff M-int V_F is a collection of handlebodies, where V_F is a regular neighborhood of F in M.

D e f i n i t i o n 1.5. 3-manifolds M_1 and M_2 are congruent iff there exist homotopy spheres Σ_1^3 and Σ_2^3 such that $M_1 \# \Sigma_1^3 = M_2 \# \Sigma_2^3$.

D e f i n i t i o n 1.6. Let $W \subset F$ be a set of words (or cyclic words) in the basis X of a free group F. The incidence graph J(W) is the graph whose vertices are in 1-1 correspondence with the non-trivial words in W with an edge joining vertices w_1 and w_2 if there exists $x \in X$ such that x or x^{-1} lies in w_1 and x or x^{-1} lies in w_2. W is connected with respect to the basis X if J(W) is connected, and is <u>connected</u> if it is connected with respect to each basis of F. If the set W of elements (or cyclic elements) is not contained in any proper free factor of F and if W is connected we say W <u>binds</u> F.

2. P r o o f of Theorem 1. Cut open M along F to obtain H_n and H_n'. $\partial M \cap H_n$ consists of two annuli A_1 and A_2 which classes in $\pi_1(H_n)$, for an appropriate basis of $\pi_1(H_n)$, are either

(i) $[A_1] = x_1$, $[A_2] = x_2$ in which case neither side of the conclusion of Theorem 1 is satisfied, or

(ii) $[A_1] = x_1$, $[A_2] = x_1 w'$ where w' is written using letters different than x_1.

To see this, consider a system of n properly embedded in H_n 2-disks which cut H_n into a 3-cell. We have uniquely associated with such a system the basis of $\pi_1(H_n)$. If we can assume that $t_1 = t_2$ and $K_1 = K_2$ in the assumption 3 of Theorem 1 then we choose the basis which is associated with a system of disks which contains $K_1 = K_2$. Then we deal with the case (ii). If the previous situation cannot be achieved then $K_1 \neq K_2$ and $K_1 \cup K_2$ does not disconnect H_n. Then we choose the basis which is associated with a system of disks which contains K_1 and K_2. We deal with the case (i). We get the conditions analogous to (i) and (ii) for H_n' and $\partial M \cap H_n' = A_1' \cup A_2'$. To conclude Theorem 1 it suffices to prove the following proposition (compare [H; Chapter 6]):

P r o p o s i t i o n 2.1. Consider a handlebody H_n $(n \geqslant 2)$ with a family $\{\gamma_i\}_{i=1}^{s}$ of disjoint, 2-sided, simple closed curves in H_n. Assume that for some basis $x_1,\dots,x_n$ of $F_n = \pi_1(H_n)$ we have

$$[\gamma_1] = x_1,\ [\gamma_2] = x_2,\dots,[\gamma_{s-1}] = x_{s-1},\ [\gamma_s] = x_1^{a_1}\cdots x_{s-1}^{a_{s-1}}w'$$

where $w' \subset F_{n-s+1} = \{x_s, x_{s+1},\dots,x_n\}$.

Then

$\partial H_n - \bigcup_{i=1}^{s} \gamma_i$ is incompressible iff $\partial(H_n)_{\{\gamma_i\}_{i=1}^{s}}$ is incompressible.

P r o o f . We have the following equivalences:

$H_n - \bigcup_{i=1}^{s} \gamma_i$ is incompressible iff ([L]; see Lemma 2.2 below)

$[\gamma_1],\dots,[\gamma_s]$ bind F_n iff (see Lemma 2.3 below)

w' binds F_{k-s+1} and $w' \neq x_s$ iff ([Sh])

$G = \{F_{k-s+1} : w'\}$ cannot be decomposed into a nontrivial free product and $G \neq Z, 1$ iff [H; Chapter 7]

$\partial(H_n)_{\{\gamma_i\}_{i=1}^{s}}$ is incompressible and $(H_n)_{\{\gamma_i\}_{i=1}^{s}}$ is congruent to an irreducible manifold or is equal to a lens space with hole iff (see Lemma 2.4 below) $\partial(H_n)_{\{\gamma_i\}_{i=1}^{s}}$ is incompressible.

Lemma 2.2 [L]. Let $\{\gamma_i\}_{i=1}^{s} \subset \partial H_n$ be a collection of pairwise disjoint, 2-sided, simple closed curves. Then $\partial H_n - \{\gamma_i\}_{i=1}^{s}$ is incompressible iff $\{[\gamma_i]\}_{i=1}^{s}$ bind $F_n = \pi_1(H_n)$ and no γ_i is contractible in ∂H_n.

Lemma 2.2 is proved in [L] for an orientable handlebody but the proof can be extended to a nonorientable handlebody as well (using the fact that each automorphism of a free group $\pi_1(H_n)$ can be got from a homeomorphism of H_n).

Lemma 2.3. Let $F_n = \langle x_1,\dots,x_n\rangle$ be a free group of rank $n \geq 2$. Assume $w_1 = x_1^{a_1}$, $w_2 = x_1^{b_2}w'$ and $w_3,w_4,\dots,w_s$, $w' \in F_{n-1} = \langle x_2,\dots,x_n\rangle$. Then cyclic words $w_1,\dots,w_s$ bind F_n iff cyclic words $w',w_3,\dots,w_s$ bind F_{n-1} and $w' \neq x_2$ for $s = 2$.

Proof. $\Longrightarrow$ If cyclic words $w',w_3,\dots,w_s$ does not bind F_{n-1} then there exists an automorphism φ of F_{n-1} such that either cyclic words $\varphi(w'),\varphi(w_3),\dots,\varphi(w_s)$ do not use all letters $(x_2,\dots,x_n)$ or the incidence graph of cyclic words $\varphi(w'),\varphi(w_3),\dots,\varphi(w_s)$ is not connected with respect to the basis $(x_2,\dots,x_n)$. Let $\varphi(w') = w_0^{-1}w_0'w_0$ where w_0' is cyclically reduced word, then $\bar\varphi(w_2) = x_1^{b_1}w_0^{-1}w_0'w_0$ (by $\bar\varphi$ we understand the automorphism of F_n which is the unique extension of $\varphi: F_{n-1} \longrightarrow F_{n-1}$ given by $\bar\varphi(x_1) = x_1$). Let ψ be an automorphism of F_n defined as follows: $\psi(x_1) = w_0^{-1}x_1w_0$ and $\psi(x_i) = x_i$ for $i > 1$. Then $\psi\bar\varphi(w_1)$ and w_1 are equal as cyclic words, $\psi\bar\varphi(w_2)$ and $x_1^{b_1}w_0'$ are equal as cyclic words, and $\psi\bar\varphi(w_i) = \bar\varphi(w_i)$ for $i > 2$. Thus either the cyclic words $\psi\bar\varphi(w_1),\dots,\psi\bar\varphi(w_s)$ do not use all letters $x_1,x_2,\dots,x_n$ or the incidence graph of cyclic words $\psi\bar\varphi(w_1),\dots,\psi\bar\varphi(w_s)$ is not connected with respect to basis $(x_1,\dots,x_n)$; so $w_1,\dots,w_s$ do not bind F_n. If $s = 2$ and $w' = x_2$ then cyclic words $x_1^{a_1}$ and $x_1^{b_1}x_2$ obviously do not bind F_n.

$\Longleftarrow$ We can assume that for some automorphism φ of F_{n-1}, cyclic words $(\varphi(w'),\varphi(w_3),\dots,\varphi(w_s))$ have minimal length under all automorphisms of F_{n-1}. $\varphi(w') = w_0^{-1}w_0'w_0$ where w_0' is a cyclically reduced word. Let ψ be the automorphism of F_n defined by $\psi(x_1) = w_0^{-1}x_1w_0$ and $\psi(x_i) = \varphi(x_i)$ for $i > 1$. Now we apply Whitehead automorphisms of F_n to show that cyclic words $(\psi(w_1),\dots,\psi(w_s))$ have minimal length under all automorphisms of F_n ([L-S]). Therefore, by [L], $w_1,\dots,w_s$ bind F_n.

Lemma 2.4. Let γ be a two-sided, simple closed curve in ∂H_n which is not homotopically trivial in H_n. If $\partial(H_n)_\gamma$ is incompressible in $(H_n)_\gamma$ then $(H_n)_\gamma$ is congruent to an irreducible manifold or is equal to $L(p,q)\#D^3$, $p\neq 0,1$ ($L(p,q)$ denotes a lens space).

Proof. If $n = 1$ then $(H_n)_\gamma$ is equal to $L(p,q)\#D^3$, $p\neq 0$ (or $P^2\times I$ if H_n is a solid Klein bottle) so the conclusion of Lemma 2.4 is satisfied. Let $n\geq 2$. Assume that $(H_n)_\gamma$ is not congruent to an irreducible manifold. Because no boundary component of $(H_n)_\gamma$ is a 2-sphere so the group $\pi_1((H_n)_\gamma) = \{F_n : [\gamma]\}$ can be decomposed into non-trivial free product. Therefore γ does not bind F_n, so $\partial H_n - \gamma$ is compressible. Let D_0^2 be a disk of compression of $\partial H_n - \gamma$. We have three possibilities:

(i) ∂D_0^2 is parallel to γ in ∂H_n. This contradicts the assumption that γ is homotopically nontrivial in H_n.

(ii) ∂D_0^2 and ∂A_γ (A_γ is a regular neighborhood of γ in ∂H_n) bound a pair of pants on ∂H_n. In this case, a disk D_1^2 can be easily find which is a compressing disk of $\partial H_n - \gamma$ but which does not satisfy (i) and (ii).

(iii) If D_0^2 does not satisfy (i) and (ii), then D_0^2 is a compressing disk of $\partial(H_n)_\gamma$ in $(H_n)_\gamma$. This contradicts the assumption that $\partial(H_n)_\gamma$ is incompressible.

3. Final remarks

The following corollary is an important application of Theorem 1.

Corollary 3.1. [C-J-R], [P]. Let M be a punctured-torus bundle over S^1 with hyperbolic monodromy map.

If F is an incompressible, ∂-incompressible and π_1-injective surface in M, then $F^\wedge$ is incompressible, π_1-injective in $M^{\partial F}$.

P r o o f . F has 1,2 or 4 boundary components [H-F]. If F has four boundary components then Corollary 3.1 follows from Theorem 1 (by [H-F] F satisfies all assumptions of Theorem 1). If F has one or two boundary components then we use Theorem 1.4 [P] or reduce these cases to the case of four boundary components (as in Corollary 2.2 [P]) and then use Theorem 1.

R e m a r k 3.2. Theorem 1 remains valid if we allow ∂M to consist of two tori. Then we have to understand by $M^{\partial F}$ appropriate Dehn surgeries along both boundary components of M.

C o n j e c t u r e 3.3. Theorem 1 remains valid without assuming the condition 3.

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